# Physics problems and work

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Recall the following physics formulae:

$$\begin{aligned} F &= m \cdot a \quad (\text{force} = \text{mass} \times \text{acceleration}) \\ W &= F \cdot d \quad (\text{work} = \text{force} \times \text{distance}) \\ m &= \rho \cdot V \quad (\text{mass} = \text{density} \times \text{volume})\,. \end{aligned}$$

6.5.1 *A particle is moved along the $x$-axis by a force that measures $10/(1+x)^2$ pounds at a point $x$ feet from the origin. Find the work done in moving the particle from the origin to a distance of $9$ ft.*

To set up the integral, divide the *path* of the particle into segments, and for each segment, let $x$ denote its distance from the origin and $\Delta x$ denote its length. For each such segment, the work done to move the particle across it is approximately

$$\begin{aligned} \text{work across a segment} &\approx \underbrace{10/(1+x^2)\ \text{lb}}_{\text{force}} \cdot \underbrace{\Delta x\ \text{ft}}_{\text{distance}} \quad (W = F \cdot d) \\ &= 10/(1+x)^2 \cdot \Delta x\ \text{lb·ft}\,. \end{aligned}$$

Summing over all our segments from $x = 0$ to $x = 9$, this is

$$\text{total work across path} \approx \sum 10/(1+x)^2\ \text{lb·ft}\,.$$

As our segments become smaller, we obtain the integral

$$\begin{aligned} \text{total work across path} &= \int_0^9 10/(1+x^2) \cdot \Delta x\, dx\ \text{lb·ft} \\ &= 10 \int_1^{10} 1/u^2\, du\ \text{lb·ft} \\ &= 10 \cdot (-1/u)\big|_{u=1}^{1} 0\ \text{lb·ft} \\ &= 10(1 - 1/10)\ \text{lb·ft} \\ &= 9\ \text{lb·ft}\,. \end{aligned}$$

6.5.11 *A cable that weighs* 2 *lb/ft is used to lift* 800 *lb of coal up a mineshaft* 500 *ft deep. Find the work done.*

First, we find the work done on the *cable* itself. To set up the integral, divide the cable into segments, and for each segment, let $x$ denote its distance from the top of the mineshaft and $\Delta x$ denote its length. For such a segment, the work done to move it to the top is approximately

$$\begin{aligned} \text{work on a cable segment} &\approx \underbrace{\overbrace{(2\text{ lb/ft})}^{\text{force density}}\overbrace{(\Delta x\text{ ft})}^{\text{length}}}_{\text{force}}\cdot\underbrace{x\text{ ft}}_{\text{distance}} \qquad (W = F\cdot d) \\ &= 2\cdot x\cdot\Delta x\text{ lb}\cdot\text{ft}\,. \end{aligned}$$

Summing over all our segments from $x = 0$ to $x = 500$, this is

$$\text{total work on the cable} \approx \sum 2\cdot x\cdot\Delta x\text{ lb}\cdot\text{ft}\,.$$

As our segments become smaller, we obtain the integral

$$\begin{aligned} \text{total work on the cable} &= \int_0^{500} 2x\,dx\text{ lb}\cdot\text{ft} \\ &= x^2\Big|_{x=0}^{500}\text{ lb}\cdot\text{ft} \\ &= 250000\text{ lb}\cdot\text{ft}\,. \end{aligned}$$

It remains to calculate the work done on the *coal.* Since all the coal is concentrated at the end of the rope, there's no need to approximate our integral using slices. Thus, we just have

$$\begin{aligned} \text{work done on coal} &= \underbrace{800\text{ lb}}_{\text{force}}\cdot\underbrace{500\text{ ft}}_{\text{distance}} \qquad (W = F\cdot d) \\ &= 400000\text{ lb}\cdot\text{ft}\,. \end{aligned}$$

Thus, our total work done is

$$\begin{aligned} \text{total work} &= \underbrace{250000\text{ lb}\cdot\text{ft}}_{\text{on cable}} + \underbrace{400000\text{ lb}\cdot\text{ft}}_{\text{on coal}} \\ &= 650000\text{ lb}\cdot\text{ft}\,. \end{aligned}$$

6.5.15 *An aquarium* 2 *m long,* 1 *m wide, and* 1 *m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is* 1000 *kg/m*$^3$*.)*

To set up the integral, we divide the *upper* half of the aquarium into horizontal slices, and for each slice, let $x$ denote its distance from the top of the tank and $\Delta x$ denote its thickness. (We choose horizontal slices because we want each drop of water in a given slice to be the same distance from the top of the tank.) Using the formulae at the beginning of this handout, we see that the work taken to pump such a slice out of the tank is

$$\begin{aligned}\text{work for a slice} &= W\\ &= F \cdot d\\ &= (m \cdot a) \cdot d\\ &= (\rho \cdot V) \cdot a \cdot d\,.\end{aligned}$$

Since the length, width and thickness of the slice are given by 2 m, 1 m and $\Delta x$ m, respectively, its volume is $2 \cdot 1 \cdot \Delta x \text{ m}^3 = 2\Delta x \text{ m}^3$. Thus, the equation above becomes

$$\begin{aligned}\text{work for a slice} &\approx \overbrace{\overbrace{\underbrace{(1000 \text{ kg/m}^3)}_{\text{density}}\,\underbrace{(2\Delta x \text{ m}^3)}_{\text{volume}}}^{\text{mass}}\,\underbrace{(9.8 \text{ m/s}^2)}_{\text{gravity}}}^{\text{force}}\,\underbrace{(x \text{ m})}_{\text{distance}}\\ &= (1000)(9.8)(2)x \cdot \Delta x\ (\text{kg} \cdot \text{m/s}^2) \cdot \text{m}\\ &= (1000)(9.8)(2)x \cdot \Delta x\ \text{N} \cdot \text{m}\\ &= (1000)(9.8)(2)x \cdot \Delta x\ \text{J}\,.\end{aligned}$$

Summing over our slices, this is

$$\text{total work for top half of aquarium} \approx \sum (1000)(9.8)(2)x \cdot \Delta x\ \text{J}\,,$$

where the sum is over the slices in the top half of the aquarium; that is, from distance $x = 0$ to $x = 1/2$. As we refine our slices, this becomes the integral

$$\begin{aligned}\text{total work} &= \int_0^{1/2} (1000)(9.8)(2)x\,dx\ \text{J}\\ &= (1000)(9.8)(2)\int_0^{1/2} x\,dx\ \text{J}\\ &= (1000)(9.8)(2)(1/8)\ \text{J}\\ &= 2450\ \text{J}\,.\end{aligned}$$