Math 113 Final Exam Solutions

Jon Lee
`jlee@math`

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1. Given $T \in \mathcal{L}(V)$, there exists a subspace $W \subseteq V$ such that we have the isomorphism:

   $$T|_W : W \cong \operatorname{im} T\,.$$

   (This can be shown using a basis argument, which would amount to what most of you wrote anyway.) Thus, given $S \in \mathcal{L}(V)$ such that $\operatorname{im} S \subseteq \operatorname{im} T$, we can define $R = (T|_W)^{-1}S$, from which it follows $TR = S$.

   Conversely, if $S = TR$, then since linear operators preserve inclusion and $R(V) \subseteq V$, we have $S(V) = (TR)(V) \subseteq T(V)$; that is, $\operatorname{im} S \subseteq \operatorname{im} T$.

2. Let T be a linear operator on V with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. For each $1 \leq i \leq n$, define the *normalized* Lagrange interpolating polynomial:

   $$P_k(x) = \prod_{j \neq i} \frac{x - \lambda_j}{\lambda_i - \lambda_j} \quad \text{so that} \quad P_k(\lambda_j) = \delta_{kj} \quad \text{and} \quad P_k(T)v_j = \delta_{kj} v_j\,.$$

   (Note that this definition of P_k differs from the unnormalized version used in class; this is to ease notation a bit.) Then if $w = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, we have that $P_k(T)w = \alpha_k v_k$ for all k. Hence, if $w = 0$, then $\alpha_k v_k = P_k(T)w = 0$ for all k; thus, the v_k are linearly independent.

   (a) Let S be a linear operator on V commuting with T. For each $1 \leq k \leq n$, take scalars $\alpha_{k1}, \ldots, \alpha_{kn}$ such that

   $$S(v_k) = \alpha_{k1} v_1 + \cdots + \alpha_{kn} v_n\,.$$

   Then,

   $$\begin{aligned} (ST)(v_k) &= (\lambda_k \alpha_{k1}) v_1 + \cdots + (\lambda_k \alpha_{kn}) v_n \quad \text{whereas} \\ (TS)(v_k) &= (\lambda_1 \alpha_{k1}) v_1 + \cdots + (\lambda_n \alpha_{kn}) v_n\,. \end{aligned}$$

   Since S, T commute, these expressions are the same; comparing coefficients, we deduce that for $j \neq k$, $\alpha_{kj} = 0$. It follows that for each k,

   $$S(v_k) = \alpha_{kk} v_k = (\alpha_{11} P_1(T) + \cdots + \alpha_{nn} P_n(T)) v_k\,;$$

   by linearity, this means

   $$S = \alpha_{11} P_1(T) + \cdots + \alpha_{nn} P_n(T)\,.$$

   (b) Consider the vector $v = a_1 v_1 + \cdots + a_n v_n$ for scalars $a_i \in \mathbb{C}$.

Suppose that $a_j \neq 0$ for all j. Then in particular, each a_j is invertible, so given any vector $w = b_1 v_1 + \cdots + b_n v_n$, we have that

$$\begin{aligned} w &= b_1 v_1 + \cdots + b_n v_n \\ &= (b_1/a_1) a_1 v_1 + \cdots + (b_n/a_n) a_n v_n \\ &= \left(\frac{b_1}{a_1} P_1(T) + \cdots + \frac{b_n}{a_n} P_n(T) \right) v \,; \end{aligned}$$

that is, v is a cyclic vector for T.

Conversely, suppose there exists some j such that $a_j = 0$. Then, v is contained in the proper subspace W spanned by the λ_k-eigenspaces for $k \neq j$; since the eigenspaces are T-invariant, so is W. Thus, the span of v and T is contained in W, so v is not a cyclic vector for T.

3. Let A be an $n \times n$ complex matrix — we first show that A is similar to an upper triangular matrix. Write T for the linear operator on the inner product space $\mathbb{C}^n$ that is represented by the matrix A in the standard basis; we will construct an increasing chain of T-invariant subspaces

$$\{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$$

where $\dim_{\mathbb{C}} W_i = i$ for all i.

Indeed, define the subspace $W_0 = \{0\}$ of $\mathbb{C}^n$ — note that this is T-invariant. In general, for each i in order from 1 to n, W_{i-1} is T-invariant, so we may consider the linear operator $T_{\mathbb{C}^n/W_{i-1}}$. Since $\mathbb{C}$ is algebraically closed, $T_{\mathbb{C}^n/W_{i-1}}$ has an eigenvector, say $\beta_i \in \mathbb{C}^n/W_{i-1}$. Let α_i be a representative of β_i in $\mathbb{C}^n$ — then defining $W_i = \operatorname{span}[W_{i-1}, \alpha_i]$, we have that W_i is a T-invariant subspace of dimension i. Thus, $\{\alpha_1, \ldots, \alpha_n\}$ forms a basis of $\mathbb{C}^n$; since $T(\alpha_i) \subseteq W_i$, the matrix of T with respect to this basis is upper triangular.

We have thus shown A is similar to an upper triangular matrix; that is, there exists an invertible matrix P such that $P^{-1}AP$ is upper triangular. (Note that the columns of P are the α_i represented in the standard basis.) Applying Gram-Schmidt orthogonalization to the columns of the matrix P, we obtain an upper triangular matrix Q such that PQ is orthonormal. By definition, A is unitarily equivalent to $(PQ)^{-1}A(PQ)$; on the other hand, since $P^{-1}AP$ and Q are upper triangular, so is $Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ)$.

4. Given a matrix $A = [a_{ij}]_{1 \le i,j \le n}$, we can define $\operatorname{tr} A$ to be

$$\sum_{i=1}^{n} a_{ii}$$

or $(-1)^{n-1}$ times the coefficient of x^{n-1} in χ_A, the characteristic polynomial of A.

(a) If $A, B \in \mathcal{M}(n)$ are similar, then by the similarity invariance of characteristic polynomials, we have that $\chi_A = \chi_B$. In particular, the coefficients of $(-x)^{n-1}$ in χ_A and χ_B are the same, so $\operatorname{tr} A = \operatorname{tr} B$.

Alternate proof: if $A = P^{-1}BP$, then by the general result that $\operatorname{tr} M_1 M_2 = \operatorname{tr} M_2 M_1$, we have that

$$\operatorname{tr} A = \operatorname{tr} P^{-1}BP = \operatorname{tr} BPP^{-1} = \operatorname{tr} B \,.$$

(b) Given matrices $A = [a_{ij}]_{1 \le i,j \le n}$ and $B = [b_{ij}]_{1 \le i,j \le n}$, a simple computation gives

$$\operatorname{tr}(A+B) = \operatorname{tr}[a_{ij} + b_{ij}]_{1 \le i,j \le n} = \sum_{i=1}^{n} a_{ii} + b_{ii} = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{tr} A + \operatorname{tr} B \,.$$

5. (a) Note that $V(a_1, \ldots, a_n, x)$ is a polynomial of degree n in x. Since det is an alternating function, we may assume the a_i are distinct; otherwise, we are done. By the same reasoning,

$$V(a_1, \ldots, a_n, a_j) = 0$$

for $j = 1, \ldots, n$; thus, a_j is a root of $V(a_1, \ldots, a_n, x)$ for each j. It follows that

$$V(a_1, \ldots, a_n, x) = \alpha \prod_{j=1}^{n} (x - a_j)$$

for some scalar α; evaluating at $x = 0$ shows that

$$\alpha = V(a_1, \ldots, a_n)\,.$$

(b) By the previous part,

$$\begin{aligned} V(a_1, \ldots, a_n, a_{n+1}) &= V(a_1, \ldots, a_n) \prod_{1 \le j \le n} (a_{n+1} - a_j) \\ &= \prod_{1 \le i < j \le n} (a_j - a_i) \cdot \prod_{1 \le j \le n} (a_{n+1} - a_j) \\ &= \prod_{1 \le i < j \le n+1} (a_j - a_i)\,. \end{aligned}$$

6. Let T, S be commuting self-adjoint operators on a finite-dimensional unitary space $\mathcal{H}$.

(a) Given a T-invariant subspace $W \subseteq \mathcal{H}$,

$$\begin{aligned} \left\langle W, TW^{\perp} \right\rangle &= \left\langle TW, W^{\perp} \right\rangle && \text{by self-adjointness of } T \\ &\subseteq \left\langle W, W^{\perp} \right\rangle && \text{by } T\text{-invariance of } W \\ &= \{0\} && \text{by definition,} \end{aligned}$$

so $W^{\perp}$ is T-invariant.

(b) Take W as above. By Gram-Schmidt orthogonalization, $\mathcal{H} = W \oplus W^{\perp}$; since the above shows that $W^{\perp}$ is T-invariant, W reduces T.

(c) If $Tv = \lambda v$ for non-zero v, then note that $\langle v, v \rangle = ||v||^2 \in \mathbb{R} \setminus \{0\}$, so that

$$\begin{aligned} \lambda &= \frac{\langle Tv, v \rangle}{\langle v, v \rangle} \\ &= \frac{\langle v, Tv \rangle}{\langle v, v \rangle} && \text{by self-adjointness of } T \\ &= \frac{\overline{\langle Tv, v \rangle}}{\langle v, v \rangle} && \text{by anti-symmetry of the inner product} \\ &= \overline{\lambda} \end{aligned}$$

is real.

(d) Given non-zero v_1, v_2 such that $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$, then

$$\begin{aligned} (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle &= \langle Tv_1, v_2 \rangle - \langle v_1, Tv_2 \rangle && \text{by conjugate linearity} \\ &= \langle Tv_1, v_2 \rangle - \langle Tv_1, v_2 \rangle && \text{by self-adjointness of } T \\ &= 0\,; \end{aligned}$$

thus, $\lambda_1 - \lambda_2 \neq 0$ implies $\langle v_1, v_2 \rangle = 0$.

(e) If $\langle Tw_1, w_2\rangle = \langle w_1, Tw_2\rangle$ holds for all $w_1, w_2 \in \mathcal{H}$, then it does for all $w_1, w_2 \in W$ since $W \subseteq \mathcal{H}$.

(f) If $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, then the Spectral Mapping Theorem for self-adjoint operators applied to $\mathcal{H}$ in terms of T yields the decomposition:

$$V = \bigoplus_{j=1}^{m} \ker(T - \lambda_j)\,.$$

For each j, that S, T commute imply that

$$(T - \lambda_j) S \ker(T - \lambda_j) = S(T - \lambda_j) \ker(T - \lambda_j) = S\{0\} = \{0\}\,;$$

that is, $\ker(T - \lambda_j)$ is S-invariant. If $\gamma_1, \ldots, \gamma_{m'}$ denote the eigenvalues of S (so that the eigenvalues of $S|_{\ker(T-\lambda_j)}$ are contained in this set), then applying the Spectral Mapping Theorem to each $\ker(T - \lambda_j)$ in terms of S then yields the decomposition

$$\begin{aligned} V &= \bigoplus_{j=1}^{m} \ker(T - \lambda_j) \\ &= \bigoplus_{j=1}^{m} \left(\bigoplus_{k=1}^{m'} \ker(S|_{\ker(T-\lambda_j)} - \gamma_k) \right) \\ &= \bigoplus_{j,k} \left(\ker(T - \lambda_j) \cap \ker(S - \gamma_k) \right)\,. \end{aligned}$$

By Gram-Schmidt orthogonalization, each $\ker(T - \lambda_j) \cap \ker(S - \gamma_k)$ has an orthonormal basis; by the direct sum decomposition of V into orthogonal subspaces, the union of these bases is an orthonormal basis for V.