Homotopy colimits and the space of square-zero upper-triangular matrices.

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Conjecture (Halperin-Carlsson, topological)

Let $G = \mathbb{Z}/(p)$. If G^n acts freely on a finite CW-complex X, then

$$r = \sum_i \operatorname{rank}_G H_i(X; G) \geq 2^n .$$

Conjecture (Halperin-Carlsson, topological)

Let $G = S^1$. If G^n acts freely on a finite CW-complex X, then

$$r = \sum_i \operatorname{rank}_{\mathbb{Q}} H_i(X;\mathbb{Q}) \geq 2^n .$$

This arises out of an effort to generalize the notion of *torus rank*.

Conjecture (Halperin-Carlsson, algebraic)

Let k be a field and $R = k[x_1, \ldots, x_n]$ be a graded polynomial ring whose indeterminates are assigned grading -1.
Given a free, finitely-generated differential-graded R-module (M, d) (where the differential has grading -1), if $H_*(M)$ is a non-zero finite-dimensional k-vector space, then $r = \mathrm{rank}_R(M) \geq 2^n$.

Two key results:

- such an M is solvable — there exists a basis $\mathcal{B}$ such that d, written with respect to $\mathcal{B}$, takes the form of an upper-triangular matrix whose entries are homogeneous polynomials in R
- let $\vec{p} \in k^n$ be a point other than the origin; evaluating the polynomials at $\vec{p}$ results in a half-rank matrix

Problem can be recast in the language of algebraic geometry

- maps betweens varieties: polynomials $f \in R = k[x_1, \ldots, x_n]$ correspond to maps of varieties

$$\mathbb{A}^n_k \longrightarrow \mathbb{A}^1_k$$

- by the previous two key results, the differential-graded R-module can be interpreted as a k^*-equivariant map of varieties

$$\mathbb{A}^n_k \setminus 0 \longrightarrow X\,,$$

where X denotes the space of upper-triangular square-zero half-rank matrices

Reformulating the problem within the context of classical homotopy theory amounts to setting $k = \mathbb{C}$.

The inner product on $\mathbb{C}$ turns out to be a big advantage:

- canonical vector space complements exist
- the singular value decomposition holds

Let V denote the space of upper-triangular $r \times r$ square matrices that have rank $r/2$ and square to zero. Moreover, suppose V has an S^1-action corresponding to the homogeneous degrees of polynomials.

The problem becomes:
Given r, for which values of n does there exist an S^1-equivariant map

$$f : \mathbb{C}^n \setminus 0 \to V \,?$$

S^1-equivariant cohomology provides a means of establishing bounds. A purported map

$$f : \mathbb{C}^n \setminus 0 \to V$$

would induce a map of $H^*(BS^1)$-modules:

$$H^*_{S^1}(f) : H^*_{S^1}(V) \to H^*_{S^1}(\mathbb{C}^n \setminus 0) \,.$$

Knowledge of the $H^*(BS^1)$-module structure of $H^*_{S^1}(V)$ would be helpful!

Consider the following commutative diagram:

$$\begin{array}{ccc} H^*_{S^1}(V) & \xrightarrow{\quad H^*_{S^1}(f) \quad} & \mathbb{Z}[c_2]/(c_2^n) \\ & \nwarrow \qquad \nearrow & \\ & \mathbb{Z}[c_2]\,. & \end{array}$$

Thus, knowing the vanishing order of the first Chern class provides an obstruction: if $c_2^m \cdot H^*_{S^1}(V) = 0$, then $m \geq n$.

In fact, it suffices to understand the $H^*(BS^1)$-module structure of $H^*_{S^1}(Y)$ for irreducible components Y of V.

Reasoning: $\mathbb{C}^n \setminus 0$ is irreducible, and thus can only map into an irreducible component of V.

Rothbach has not only identified the irreducible components of V, but gives a stratification of V into orbits of partial permutation matrices under the Borel group of upper-triangular invertible matrices.

Two additional very nice properties:

- each orbit is an affine bundle over a torus $T = (S^1)^\ell$ — easy calculations?
- there is a partial ordering on the strata:

$$\{\text{Zariski-closure of orbits}\} \leftrightarrow$$

$$\{\text{combinatorial rank conditions of partial permutation matrices}\}$$

One big disadvantage:

- There are too many orbits!

How to fix this?

My approach: consider orbits under the parabolic subgroup P_Y, where the shape of P_Y depends on the irreducible component Y.

- there are fewer orbits — patching together cohomology should be easier
- each orbit is a quotient of P_Y
- each orbit is invariant under the S^1-action

One of the first non-trivial irreducible components Y to study is when the parabolic group P_Y has diagonal blocks of sizes $n, n+k, m+k, m$ for $n, m, k > 0$.

There is a nice totally-ordered stratification in this case. Let $N = \min(n, m)$. Then, Y stratifies as

$$Y = Y_0 \cup Y_1 \cup \cdots \cup Y_N \, .$$

Theorem

The stratum Y_i is homotopy-equivalent to G/H, where

$$G = \mathrm{U}(n) \times \mathrm{U}(n+k) \times \mathrm{U}(m+k) \times \mathrm{U}(m)\,,$$

$$H = \mathrm{U}(n-i) \times \mathrm{U}(i) \times \mathrm{U}(k) \times \mathrm{U}(i) \times \mathrm{U}(m-i)\,,$$

and H embeds as a subgroup of G via diagonal blocks.

The singular value theorem states that any $\mathbb{C}$-valued matrix can be factored as

$$U \Sigma V^* ,$$

where U and V are unitary matrices, and Σ is the diagonal matrix of real, non-negative *singular values*

This allows for two important operations:

- least squares projections, or "rank reduction of matrices", used for homotoping between orbits
- homotoping matrices to unitary ones