Problem Set 3

April 17, 2008

More problems will be added throughout the week. Problem 5 should be typed using LaTeX or another mathematical typesetting program and turned in separately.

1. Let $P_n = \{c_0 + c_1x + c_2x^2 + \ldots + c_nx^n | c_i \in \mathbb{R}\}$ be the set of polynomials of degree $n$ with real coefficients. Show that $P_n$ is a vector space over the field of real numbers.

2. Let $\mathcal{B}$ be the set of all bounded real-valued functions from $[0,1]$ to $\mathbb{R}$. Define the following function on $\mathcal{B}$:

$$||f||_\infty = \text{sup}\{|f(x)| \mid x \in [0,1]\}.$$ 

(a) Verify that $|| \cdot ||_\infty$ is a norm on this vector space. (The scalar field is $\mathbb{R}$.)
(b) Let $d_{sup}(f,g) = ||f - g||_\infty$ be the metric on $\mathcal{B}$ induced by the norm from 2a. Show that $(\mathcal{B},d_{sup})$ is a complete metric space.
(c) Let $T : \mathcal{B} \to \mathcal{B}$ be defined by $Tf(x) = \frac{1}{2}f(x)$. Show that $T$ is a contraction and find the fixed point. Prove your answer.

3. (a) Show that the following function is a norm on $\mathbb{R}^n$:

$$||x||_\infty = \text{max}_{1 \leq i \leq n}|x_i|$$

This norm induces the infinity metric encountered previously. (Note that there should be no confusion between this norm and the one introduced in Problem 2 because the underlying spaces are different.)
(b) Show that $A \subset \mathbb{R}^n$ is open in $(\mathbb{R}^n,d_\infty)$ if and only if it is open in $(\mathbb{R}^n,d_E)$. This proves that these two metrics induce the same topology on $\mathbb{R}^n$.

4. (Exercise 3 in M & H) Let $U$ be an open set in a metric space $M$ and $U \subset A$. Show that $U \subset \text{int}(A)$. What is the corresponding statement for closed sets?

5. (Exercise 12 in M & H) Prove the following properties for subsets $A$ and $B$ of a metric space:

(a) $\text{int}(\text{int}(A)) = \text{int}(A)$
(b) $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$
(c) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$

Please TeX this problem and turn it in separately.
6. (Exercise 20 in M & H) Let \((M, d)\) be a metric space with \(A \subset M\). For \(x \in M\), let
\[
d(x, A) = \inf \{d(x, y) \mid y \in A\},
\]
and for \(\epsilon > 0\), let \(D(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}\).

(a) Show that \(D(A, \epsilon)\) is open.

(b) Let \(A \subset M\) and \(N_\epsilon = \{x \in M \mid d(x, A) \leq \epsilon\}\) where \(\epsilon > 0\). Show that \(N_\epsilon\) is closed and that \(A\) is closed if and only if \(A = \bigcap_{\epsilon > 0} N_\epsilon\).

7. (Exercise 36 in M & H) Let \(A, B \subset \mathbb{R}^n\) be closed sets. Does \(A + B = \{x + y \mid x \in A \text{ and } y \in B\}\) have to be closed?