Practice Exam

This is a closed-book, no notes exam. Unless otherwise indicated, you should prove each of your answers. You may use results proved in the text or in class, but you must clearly state the result before applying it to your problem.

1. Define *invariant subspace*.

2. Show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent list of vectors if and only if $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{v}\}$ is a linearly independent list of vectors.

3. Indicate whether each of the statements below is always **true** or sometimes **false**. You do not need to prove your answers.

   (a) Every proper subspace of an infinite-dimensional vector space is finite-dimensional.

   (b) The null space of an isomorphism $T : V \to W$ is a subspace of V.

   (c) If V is finite-dimensional, every surjective linear map $S : V \to V$ is injective.

   (d) Every polynomial of degree k with real coefficeints factors into k linear terms with real coefficients.

   (e) If $T : V \to W$ is an injective linear map, then $V \cong range(T)$.

   (f) If $T : \mathbb{R}^{2n} \to \mathbb{R}^n$ is a linear map with $dim(null(T)) = n$, then $null(T) \cong range(T)$.

   (g) Let $T : V \to V$ be a linear transformation. If $T(\mathbf{v}) = 2\mathbf{v}$ for some $\mathbf{v} \in V$, then 2 is an eigenvalue for T.

4. Let $f : \mathcal{P}(1) \to \mathcal{P}(2)$ be the linear map defined by

$$f(2+x) = x^2$$
$$f(1+2x) = 1+x.$$

   (Recall that $\mathcal{P}(n) = \{a_0 + a_1x + ... + a_nx^n \mid a_i \in \mathbb{R}\}$.)

   (a) Compute $f(x-1)$.

   (b) Find a basis for $\mathcal{L}(\mathcal{P}(1), \mathcal{P}(2))$ which includes the function f.

5. Let M denote the vector space of $n \times n$ matrices, and let X denote the subspace of skew-symmetric matrices. (You do not need to show that X is a subspace.) Find a subspace $Y \subset M$ such that $M = X \oplus Y$.