Student Algebraic Geometry Seminar Talk: The Octahedral Axiom in Triangulated Categories

Jason Lo

Thursday, November 13, 2008

Abstract
The octahedral axiom for triangulated categories is often misunderstood and considered irrelevant. I will give two examples illustrating why it is actually useful.

1 Seven Ways to Write Down the Octahedral Axiom

1. p.11 of Alex Dimca’s 'Sheaves in Topology' ([ST]). There is no diagram here...

2. p.36 of Kashiwara and Shapira’s 'Sheaves on Manifolds' ([KS-SM]). This formulation of the octahedral axiomatic is very useful for checking which morphisms commute, although the repetition of certain objects in the diagram make it somewhat hard to understand exactly what is going on.

3. p.38 of Kashiwara and Shapira. Here we can actually see the shape of an octahedron.

4. p.240 of Gelfand and Manin’s 'Methods of Homological Algebra' ([GM]). The octahedron is broken into two halves.

5. p.375 of Weibel’s 'An Introduction to Homological Algebra' ([Weibel]). Here the shape of an octahedron can also be seen.

6. p.7 of Theo Bülker’s notes 'An Introduction to the Derived Category' ([TB]). In this formulation, the input for the octahedral axiom is the commutative triangle in the middle. The output is the rest of the diagram.

7. The formulation as I learned it from Arend Bayer, which also appears in Weibel’s book. See next section.
2 Applications of the Octahedral axiom

This is my favourite way of stating the octahedral axiom, and one that I find
the most enlightening. I learned it at a lecture given by Arend Bayer: in
a triangulated category, given any composition of morphisms, i.e. given any
commutative triangle

\[
\begin{array}{ccc}
A & \xrightarrow{v \circ u} & C \\
\uparrow{u} & & \downarrow{v} \\
B & \xleftarrow{v} & \downarrow{u} \\
\end{array}
\]  

the octahedral axiom gives as the output the following commutative diagram,
in which all straight lines are exact triangles:

\[
\begin{array}{ccc}
\text{cone}(u) & \xrightarrow{v \circ u \circ v} & \text{cone}(v) \\
\uparrow{u} & & \downarrow{v} \\
A & \xrightarrow{v \circ u} & C \\
\end{array}
\]

Example 1: long exact sequences  Let \( R = k[x] \), the polynomial ring
over a field \( k \). Consider the commutative triangle in the category of \( R \)-mod
of \( R \)-modules (which is a full abelian subcategory of \( D(\text{R - mod}) \)), where all
morphisms are injections

\[
\begin{array}{ccc}
(x^{m-1})R & \xrightarrow{v} & R \\
\uparrow{u} & & \downarrow{v} \\
(x^m)R & \xrightarrow{v \circ u} & \text{cone}(v) \\
\end{array}
\]
We can now take cokernels of each of these maps, thus completing each side to an exact triangle in $D(R - \text{mod})$

\[
\begin{array}{ccc}
R/x^m & \overset{v}{\rightarrow} & R/x^m - 1 \\
\downarrow & & \downarrow \\
R & \overset{\gamma}{\rightarrow} & R/x^m
\end{array}
\]

(Aside: Given an abelian category $A$, we have a full embedding $A \subset D(A)$, under which any short exact sequence $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ in $A$ is sent to an exact triangle in $D(A)$.)

The octahedral axiom then tells us that the vertical line is an exact triangle. This is not surprising, since this is just the third isomorphism axiom for an abelian category. So far, I have not told you anything new.

Now, let me:

1. apply $- \otimes_R R/x^m$ (a right exact functor from $R$-mod to $R/x^m$-mod) to the entire diagram, and then

2. take kernels where we can (or, in other words, computing the Tor groups listed below)

Then we end up with the following commutative diagram

\[
\begin{array}{ccc}
R\otimes_R k & \overset{\epsilon}{\rightarrow} & R/x^m_{m,1} \\
\downarrow & & \downarrow \\
R/x^m & \overset{\beta'}{\rightarrow} & R/x^m_{m,1} - 1
\end{array}
\]

where $\beta'$ is induced by $\beta$, $\gamma$ is just the composition of the other two morphisms in the triangle, and $\epsilon$ is induced by the commutativity of the triangle in the centre of the diagram.
Above,

- \( \text{Tor}_1^R(R/x^{m-1}, R/x^m) = H^{-1}(R/x^{m-1} \xrightarrow{x^m} R/x^m) = R/x^{m-1} \)
- \( \text{Tor}_1^R(R/x^m, R/x^m) = R/x^m \)
- \( \text{Tor}_1^R(k, R/x^m) = H^{-1}([R \xrightarrow{x^m} R] \otimes_R k) = H^{-1}(k \xrightarrow{x^m} k) = k \)

In the above diagram (the commutativity can be easily checked using the formulation of the octahedral axiom in, say, [KS-SM], or [NeemanTC]), the three straight lines that go through the middle triangle are all exact sequences, while the six terms that wind around the triangle also form an exact sequence. All these follow immediately from the statement of the octahedral axiom!

So the moral is: the octahedral axiom allows us to get a ”tangle” of long exact sequences. So if we know some of the cohomology involved already, we can get some information about the cohomology that we do not already know.

The general idea. More generally, we could take any commutative triangle in a triangulated category such as Figure 3 as the input, and then the octahedral axiom would give as the output a commutative diagram such as Figure 4. Then, if we have a t-structure on the triangulated category, we could get a ”tangle” of four long exact sequences of cohomology such as Figure 5.

Example 2: extending t-structures  I was wondering if a t-structure on a bounded derived category \( D^b(\mathcal{A}) \) always extends to a t-structure on the corresponding unbounded derived category \( D(\mathcal{A}) \) (where \( \mathcal{A} \) is an abelian category).

An easier question is: given a t-structure on \( D^b(\mathcal{A}) \) obtained by tilting the standard t-structure, does it always extend to a t-structure on \( D(\mathcal{A}) \)? I think the answer is yes.

Why do I care?

- having a t-structure means that we can talk about cohomology
- extending t-structures to \( D \) (from \( D^b \)) means that I can talk about cohomology without constantly having to make sure I am inside a bounded derived category. This is especially true when we are using derived functors to pass between derived categories of (coherent sheaves of) not necessarily smooth schemes.

Given an abelian category \( \mathcal{A} \) and a torsion pair \((\mathcal{T}, \mathcal{F})\) in it (e.g. \( \mathcal{A} = \text{Coh}(X), \mathcal{T} = \{\text{torsion sheaves}\}, \mathcal{F} = \{\text{torsion-free sheaves}\} \)), we know from [?] that the following pair define a t-structure on \( D^b(\mathcal{A}) \)

\[
D^{\leq 0, b} := \{ E \in D^b(\mathcal{A}) : H^0(E) \in \mathcal{T}, H^i(E) = 0, \forall i > 0 \} \\
D^{\geq 0, b} := \{ E \in D^b(\mathcal{A}) : H^{-1}(E) \in \mathcal{F}, H^i(E) = 0, \forall i < -1 \}
\]

Aside: recall ([TACQA]) that a t-structure on a triangulated category \( \mathcal{C} \) with a translation functor \([\cdot]\) is a pair of full subcategories \((\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})\) satisfied for \( \mathcal{C}^{\geq n} := \mathcal{C}^{\geq 0}[\cdot - n], \mathcal{C}^{\leq n} := \mathcal{C}^{\leq 0}[\cdot - n] \) and \( n \in \mathbb{Z}_0^+ \).
1. \( \text{Hom}(X, Y) = 0 \) for all \( X \in \mathcal{C}^{\leq 0} \) and \( Y \in \mathcal{C}^{\geq 1} \)
2. \( \mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 1} \) and \( \mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0} \)
3. for all \( X \in \mathcal{C} \) there is a triangle \( X' \to X \to X'' \to X'[1] \) such that \( X' \in \mathcal{C}^{\leq 0} \) and \( X'' \in \mathcal{C}^{\geq 1} \)

**Lemma 2.1.** The following pair defines a t-structure on the unbounded derived category

\[
\begin{align*}
D^{\leq 0} &:= \{ E \in D(A) : H^0(E) \in T, H^i(E) = 0 \forall i > 0 \} \\
D^{\geq 0} &:= \{ E \in D(A) : H^{-1}(E) \in \mathcal{F}, H^i(E) = 0 \forall i < -1 \}
\end{align*}
\]

**Proof.** Axiom 1: Take any \( X \in D^{\leq 0}, Y \in D^{\geq 1} \). Let \( \tau^{\geq 0}, \tau^{\leq 0} \) denote the truncation functors with respect to the standard t-structure. Choose any \( i \ll 0 \). Then we have an exact triangle

\[
\tau^{\leq i} X \to X \to \tau^{\geq i+1} X \to \tau^{\leq i} X[1]
\]

Since \( \text{Hom}(\tau^{\leq i} X, Y) = 0 = \text{Hom}(\tau^{\leq i} X[1], Y) \), we have that

\[
\text{Hom}(X, Y) \cong \text{Hom}(\tau^{\leq i} X, Y)
\]

Similarly,

\[
\text{Hom}(\tau^{\geq i+1} X, Y) \cong \text{Hom}(\tau^{\geq i+1} X, \tau^{\leq -i} Y)
\]

But now \( \tau^{\geq i+1} X, \tau^{\leq -i} Y \in D^b(A) \), and in fact they lie in \( D^{\leq 0, b}, D^{\geq 0, b} \), respectively. So from axiom 1 for the bounded case, we have \( \text{Hom}(X, Y) = 0 \).

Axiom 2: is obvious.

Axiom 3: this is where the octahedral axiom comes in. The idea is that, given any \( X \in D(A) \), since it is not bounded, we truncate it first, then use what we know about bounded t-structures to break it apart. In the end, the octahedral axiom assures us that we can glue everything back together.

(Insert an illustrative diagram in the future?)

First, support that we want to prove this axiom for \( E \in D^-(A) \). Choose an \( k \gg 0 \). As above, let \( \tau^{\leq 0}, \tau^{\geq 0} \) denote the truncation functors with respect to the standard t-structures on \( D(A) \), and \( \tau^{\leq 0}, \tau^{\geq 0} \) denote the truncation functors with respect to the tilted t-structure on \( D^b(A) \). Then we have two exact triangles:
in which $Y \in D^b(A)$ and $\tau^{\geq 0}Y \in D^{\geq 0,b} \subseteq D^{\geq 0}$.

Intuitively, we would like to be able to place an object from $D^{\leq -1}$ at the upper left corner of the diagram, so that the three outer most vertices together form an exact triangle, while the dotted lines also form the edges of an exact triangle. This intuition (or 'guess', rather) can be realised by the use of the octahedral axiom.

Take $\alpha, \beta$ and $\beta \circ \alpha$ as the input for the octahedral axiom. That is, we start with the commutative triangle

\[
\begin{array}{ccc}
Y & \xrightarrow{\beta} & \bar{\tau}^{\geq 0}Y \\
\alpha & \downarrow & \beta \\
E & \xrightarrow{\beta \circ \alpha} & \bar{\tau}^{\geq 0}Y
\end{array}
\]

The octahedral axiom then gives

\[
\begin{array}{ccc}
\tau^{\leq k-1}E[1] & \xrightarrow{\bar{\tau}^{\geq 0}Y} & Z[1] \\
\bar{\tau}^{\geq 0}Y & \xrightarrow{\beta \circ \alpha} & \bar{\tau}^{\geq 0}Y \\
E & \xrightarrow{\beta \circ \alpha} & \bar{\tau}^{\geq 0}Y
\end{array}
\]

where $Z := \text{cone}(\beta \circ \alpha)$.

Since $k \ll 0$, with respect to the standard t-structure we have $H^n(Z[1]) \cong H^n(\bar{\tau}^{\leq -1}Y[1])$ for $n$ close to 0, and so $H^n(Z) \cong H^n(\bar{\tau}^{\leq -1}Y)$. Hence $Z \in D^-(A) \cap D^{\leq -1}$, and we have the exact triangle

\[
Z \to E \to \bar{\tau}^{\geq 0}Y \to Z[1]
\]

thus proving axiom 3 for the case $D^-(A)$.

Now take any $E \in D(A)$. Choose some $k \gg 0$, so that we have exact triangles

\[
\begin{array}{ccc}
\bar{\tau}^{\leq -1}Y & \xrightarrow{\alpha} & \tau^{\leq k}E \\
\bar{\tau}^{\geq 0}Y & \xrightarrow{\beta} & \tau^{\geq k+1}E \\
\bar{\tau}^{\geq 0}Y & \xrightarrow{\tau^{\geq k+1}E} & \bar{\tau}^{\geq 0}Y
\end{array}
\]
where $Y := \tau^{\leq k}E \in D^-(A)$. Define $Z := \text{cone}(\beta \circ \alpha)$. As we are used to by now, the octahedral axiom gives

\[
\begin{array}{c}
\tau^{\geq 0}Y \\
\downarrow \\
Y \\
\downarrow \\
\tau^{\leq -1}Y \\
\downarrow \beta \circ \alpha \\
E \\
\downarrow \\
\tau^{\geq k+1}E \\
\end{array}
\]

from which we see that $\tau^{\geq 0}Y$ and $Z$ have the same cohomology near degree 0. So $Z \in D^{\geq 0}$ as wanted. And we have proved axiom 3 for the unbounded derived categogy $D(A)$. 

\[\square\]

References


