Today's topics:

- limits of functions \( f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \)
- \((any\ lingering\ questions\ from\ linear\ algebra?)\)

What does it mean for a function \( f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) to have a limit at \( x = a \)? When we write \( \lim_{x \to a} f(x) = v \), what we mean intuitively is that, as \( x \) gets "closer and closer" to \( a \) in \( \mathbb{R}^n \), \( f(x) \) gets "closer and closer" to \( v \).

Draw the graph of a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) with discontinuity at \( f(1,1) = 2 \), with \( \lim_{(x,y) \to (1,1)} f(x,y) = -2 \) but \( f(1,1) = 2 \).

Example 1 (Ex 2.2.21, Ex 2.2.17). Evaluate the limit, or explain why it does not exist:

(a) \( \lim_{(x,y,z) \to (0,0,0)} \frac{xy - xz + yz}{x^2 + y^2 + z^2} \)

(b) \( \lim_{(x,y) \to (0,0), x \neq y} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \).

Solution. (a) Put \( f(x,y,z) = \frac{xy - xz + yz}{x^2 + y^2 + z^2} \). We can let \((x,y,z)\) get closer to \((0,0,0)\) along two different paths:

\[
x = y = z
\]

and \( x = y = \frac{1}{2} z \)

(Draw these two lines in \( \mathbb{R}^3 \); to make it easier to draw the picture, make the positive \( x \)-axis point to the right and the positive \( y \)-axis point into the blackboard.)

When \((x,y,z)\) approaches the origin along the first path, \( f(x) \) is always equal to \( 1/3 \). Along the second path, however, \( f(x) \) is always equal to \( 1/6 \). So the limit does not exist.

(b) Note

\[
\frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \frac{x(x - y)}{\sqrt{x} - \sqrt{y}} = \frac{x(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})}{\sqrt{x} - \sqrt{y}} = x(\sqrt{x} + \sqrt{y}),
\]

so as \((x,y) \to (0,0)\), the limit is \( 0(0 + 0) = 0 \).

What does it mean for a function \( f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) to be continuous? Taking the cue from single-variable, scalar-valued functions, this should mean that there is no "break" in the graph of the function. For instance, the picture I drew at the beginning of this section is the graph of a function that is not continuous at \((x,y) = (1,1)\), even though the limit exists at \((x,y) = (1,1)\).
The rigorous definition of continuity is:

\[ f \text{ is continuous at } a \overset{\text{def}}{=} \lim_{x \to a} f(x) = f(a) \]

Note that there are two requirements for \( f \) to be continuous at \( a \):

1. the function should be defined at \( x = a \)
2. the function should have a limit at \( x = a \)

**Example 2 (Ex 2.2.42).** Determine the value of the constant \( c \) so that

\[
g(x, y) = \begin{cases} 
  \frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
  c & \text{if } (x, y) = (0, 0)
\end{cases}
\]

is continuous.

**Solution.** By the definition of continuity above, if we can prove that the limit exists at \((0,0)\), then we can just set \( c = \lim_{(x,y)\to(0,0)} g(x,y) \) and then \( g(0,0) \) would be continuous at \((0,0)\), hence everywhere (why so?).

To prove that \( \lim_{(x,y)\to(0,0)} g(x,y) \) exists, it helps to simplify the function a little:

\[
g(x, y) = \frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} = \frac{(x + 2)(x^2 + y^2)}{x^2 + y^2} = x + 2 \text{ when } (x, y) \neq (0, 0)
\]

From this we immediately see that the limit must exist and \( \lim_{(x,y)\to(0,0)} g(x,y) = 0 + 2 = 2 \).

**Example 3 (Continuation of Example 2).** If we consider the two lines \( x = 0 \) and \( y = 1 \), we have:

1. along \( x = 0 \), \( g(x, y) = 2 \) always
2. along \( y = 1 \), \( g(x, y) = \frac{x^3 + 2x^2 + x + 2}{x^2 + 1} \), so \( g(x, y) \to 2 \) as \( x \to 0 \)

We might be tempted to say these two paths can be used to show that the limit does not exist at \((0,0)\) for \( g \). What is wrong with this argument?