Today’s topics:

- Inverses
- Determinants

## 1 Inverses

Consider the linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m : x \mapsto Ax$ where $A$ is an $m \times n$ matrix. Then Prop 16.4, 16.5 say:

- $T$ is onto $\iff \text{im} \ T = \mathbb{R}^m$
- $\iff C(A) = \mathbb{R}^m$
- $\iff$ each row of $A$ has a pivot
- $\iff \text{rank} \ (A) = m$

and

- $T$ is 1-1 $\iff \ker T = 0$
- $\iff N(A) = 0$
- $\iff$ each column of $A$ has a pivot
- $\iff \text{rank} \ (A) = n$

### Example 1

Let $A = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 0 & -7 \end{bmatrix}$. Is the linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2 : x \mapsto Ax$ 1-1? Onto?

**Solution.** Since $\text{rank} \ (A) = \dim C(A) = 2$ (look at first two columns), $T$ is onto and not 1-1. Note that $A$ is already in rref.

To find the inverse $A^{-1}$ of an $n \times n$ square matrix $A$, use Proposition 16.7:

$$\text{rref}[A|I_n] = [I_n|A^{-1}]$$

### Example 2

Find the inverse of $B = \begin{bmatrix} 1 & 10 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 1000 \end{bmatrix}$. 

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Solution. Use the above formula to get

$$B^{-1} = \begin{bmatrix} 1 & -10 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{100} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It’s useful to remember the formula for the inverse of a 2 by 2 matrix: if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(note: here $1/(ad-bc)$ is det $A$).

Note that in Example 2 above, the upper left (resp. lower right) corner of $B^{-1}$ is exactly the inverse for the upper left (resp. lower right) corner of $B$.

2 Determinants

See p.115 of Levandosky for how to find the determinant of a square matrix.

Proposition 17.2 might seem a little weird at first. Here is an example that illustrates its use:

Example 3. Find $\det B$ where $B = \begin{bmatrix} 4 & 0 & 7 \\ 0 & 1 & 0 \\ 11 & 0 & -13 \end{bmatrix}$.

Solution

$$\begin{vmatrix} 4 & 0 & 7 \\ 0 & 1 & 0 \\ 11 & 0 & -13 \end{vmatrix} = 4 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 11 & 0 & -13 \end{vmatrix} + 7 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 11 & 0 & -13 \end{vmatrix}$$

by Prop 17.2, parts 2(a) and (b)

$$= 4 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 11 & 0 & -13 \end{vmatrix} + (-1)7 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 11 & 0 & -13 \end{vmatrix}$$

by Prop 17.2 part 1

$$= 4 \cdot (-13) + (-7) \cdot 11$$

since the determinant of an upper/lower triangular matrix is just the product of the diagonal entries

$$= -52 - 77 = -129$$

Other important properties of the determinant: Prop 17.3, 17.4, 17.5
Extra Example 1. Find the determinant of the matrix \[
\begin{bmatrix}
e_1 & e_2 & e_3 \\
v_1 & v_2 & v_3 \\
w_1 & w_2 & w_3
\end{bmatrix}.
\] Explain how this is related to the cross product
\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} \times \begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix}
\]

Extra Example 2. Area and determinants (Prop 17.5, 17.6) Ex 17.15.