Today’s topics:

- dimension, rank, nullity
- more on basis

**Example 1.** Take \( A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \\ -1 & -3 & 1 & 1 \end{bmatrix} \).

1. Find a basis for \( C(A) \).

**Solution** First we find \( rref(A) \):

\[
\begin{bmatrix}
1 & 0 & 7/5 & 0 \\
0 & 1 & -4/5 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Since columns 1, 2 and 4 of \( rref(A) \) are the ones with pivots, columns 1, 2 and 4 of \( A \) itself form a basis for \( C(A) \). (And so dim \( C(A) = 3 \).)

2. What is dim \( N(A) \)?

**Solution.** It should be 1 (since dim \( C(A) + \) dim \( N(A) = 4 \), the number of columns of \( A \). See Proposition 12.4.)

3. Find a basis for \( N(A) \).

**Solution.** From \( rref(A) \), we see that

\[
\begin{align*}
\mathbf{x} \in N(A) & \iff \\
x_1 + \frac{7}{5}x_3 & = 0 \\
x_2 + \left(\frac{-7}{5}\right)x_3 & = 0 \\
x_4 & = 0
\end{align*}
\]

\[ \iff x = x_3 \begin{bmatrix} -\frac{7}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix}, \text{ } \text{x}_3 \text{ being free} \]

Hence we can write

\[
N(A) = \{c \begin{bmatrix} -7/5 \\ 4/5 \\ 1 \\ 0 \end{bmatrix} : c \in \mathbb{R}\}
\]
and \( \begin{bmatrix} -7/5 \\ 4/5 \\ 1 \\ 0 \end{bmatrix} \) is a basis for \( N(A) \).

4. Can we find two nonzero, linearly independent vectors in \( N(A) \)? Explain.

**Solution.** No. Since \( \dim N(A) = 1 \), any set of \( n \) vectors where \( n > 1 \) must be linearly dependent (see Proposition 12.1).

5. Suppose \( b \in \mathbb{R}^4 \) is a nonzero vector. Does \( Ax = b \) have a solution?

**Solution.** Not necessarily. It depends on the particular \( b \). (For a specific \( b \), we would need to check the rref of the augmented matrix \([A|b]\) to see if there is any inconsistency.)

6. If \( Ax = b \) has a solution above, describe the solution set (is it a point, line, plane, or... in \( \mathbb{R}^4 \), and does it pass through the origin?)

**Solution.** If there is a particular solution, then the solution set is a translation of \( N(A) \) (see Proposition 8.2) away from the origin. So in this case, it is a line not through the origin.

7. With \( b \) still as above, is \( \{ \mathbf{x} : Ax = b \} \) a subspace of \( \mathbb{R}^4 \)?

**Solution.** No. (There are three conditions that a subspace must satisfy, the first of which says the zero vector \( \mathbf{0} \) must be in it.)

**Quick Example.** Let \( A' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Give a \( b \) such that \( Ax = b \) has a solution in \( x \). Give a \( b \) such that \( A'x = b \) does not have a solution in \( x \). Find \( C(A'), N(A') \).

**Solution.** \( C(A') = \{ \begin{bmatrix} c \\ 0 \end{bmatrix} : c \in \mathbb{R} \} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \). So if \( b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) then \( A'x = b \) would have a solution (since \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in C(A') \)). If \( b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) then \( A'x = b \) would not have a solution (since \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin C(A') \)).

The matrix \( A' \) is already in the rref. So \( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in N(A') \) iff \( x_1 = 0 \). So \( N(A) = \{ \begin{bmatrix} 0 \\ c \end{bmatrix} : c \in \mathbb{R} \} = \text{span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \).
Example 2  Suppose $B$ is a 3 by 5 matrix over $\mathbb{R}$.

1. Can rank $(B)$ be 4?

Solution  No. $(\text{rank } (B) = \dim C(A)$ equals the number of pivots in $\text{rref}(B)$. However, there is at most one pivot in each row of $\text{rref}(B)$, while $B$ has 3 rows. So rank $(B) \leq 3$.)

2. Can nullity$(B)$ be 4?

Solution  Yes. A particular example of such a $B$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3. What is $\dim C(B)$ if nullity$(B) = 4$?

Solution  Recall

$$\dim C(B) + \dim N(B) = \#(\text{columns of } B)$$

(Proposition 12.4).

Example 3  Let $v_1, v_2, v_3, v_4$ be vectors in $\mathbb{R}^3$.

1. Are the $v_i$ linearly independent?

Solution  No. $\mathbb{R}^3$ is 3-dimensional, so any collection of 4 vectors in it must be linearly dependent.

2. Give a range for $\dim N(D)$ where $D = [v_1|v_2|v_3|v_4]$ (i.e. the columns of $D$ are the $v_i$).

Solution  $\text{rref}(D)$ has at most 3 pivots (at most one for each row), meaning $\text{rank } (D) = \dim C(D)$ is between 0 and 3 inclusive. Since $\dim C(D) + \dim N(D) = 4$, $\dim N(D)$ is between 1 and 4.

Extra Examples  Ex 10.20, 10.23, 11.15, 11.5, 12.12