Today’s topics:

- partial derivatives vs continuity
- more on tangent plane
- higher-order partial derivatives
- (Newton’s method)
- visualising multi-variable functions

1 Partial derivatives vs Continuity

Continuing the discussion from last section (Tuesday), just to make sure that we are all on the same page regarding the relations between partial derivatives and continuity for a multi-variable function, let us write down what we know (and hopefully resolve lingering confusions once and for all). Given a function \( f : X \subset \mathbb{R}^n \to \mathbb{R}^m \), we know:

- Theorem 3.10. If all partial derivatives \( \partial f_i / \partial x_j \) exist and are continuous in a neighbourhood of \( a \) in \( X \), then \( f \) is differentiable at \( a \)

- Theorem 3.9. If \( f \) is differentiable at \( a \), then it is continuous at \( a \).

As a consequence, if all partial derivatives of \( f \) exist and are continuous in a neighbourhood of \( a \), then \( f \) would be continuous at \( a \).

For our function from last time,

\[
f(x, y) = \begin{cases} 
  x^2 y & \text{if } (x, y) \neq (0, 0) \\
  0 & \text{if } (x, y) = (0, 0) 
\end{cases}
\]

both partial derivatives \( f_x, f_y \) exist everywhere. Unfortunately, they are not continuous at the origin (you can check this), and so the function \( f(x, y) \) itself is not continuous at the origin.

Note that this is different from the single-variable scenario: for a single variable function \( f : X \subset \mathbb{R} \to \mathbb{R} \), as soon as the derivative \( f'(x) \) exists at a point \( a \in X \), \( f \) would be continuous at \( x = a \).

The function that I drew on the board and erased last time,

\[
g(x) = \begin{cases} 
  x^2 & \text{if } x \neq 0 \\
  1 & \text{if } x = 0 
\end{cases}
\]
is not continuous at the origin, and so its derivative cannot exist at the origin. We can see this directly:

$$\lim_{h \to 0^+} \frac{g(0+h)-g(0)}{h} = \lim_{h \to 0^+} \frac{h^2 - 1}{h} = -\infty$$

It was also asked last time: what do functions that are non-differentiable, but have well-defined partial derivatives, look like? Figure 2.50 on p.112 is the picture of one such function. The function $f$ above is another such function, but you would’ve had a hard time trying to visualise it using level curves!

The slides give you an idea of what the function looks like. Note the "non-smooth" behavior of the function near the origin. We can also tell from the slides that the partial derivatives are both equal to 0 at the origin.

Matlab commands I used:

```matlab
syms x y
f = (x^3*y)/(x^6+y^2)
ezsurf(f,[-4,4,-4,4])
```

## 2 More on Partial Derivatives

### Example 1.

Use the tangent plane to approximate:

$f(0.9, 0.1)$, where $f(x, y) = 3 + \cos \pi xy$,

Solution. The partial derivatives of $f$ exist and are continuous everywhere, and so $f$ is differentiable on all of $\mathbb{R}^2$. So near any point on $\mathbb{R}^2$, we can approximate the function by the tangent plane at that point. Since $(0.9, 0.1)$ is near $(1, 0)$, we can approximate $f(0.99, 0.01)$ using the tangent plane at $(1, 0)$.

$$\frac{\partial f}{\partial x} = \pi y \cos \pi xy$$
$$\frac{\partial f}{\partial y} = \pi x \cos \pi xy$$

and

$$\frac{\partial f}{\partial x}(1, 0) = 0$$
$$\frac{\partial f}{\partial y}(1, 0) = \pi \cos 0 = \pi$$

The tangent plane at the point is thus given by

$$z = f(1, 0) = 0 \cdot (x - 1) + \pi \cdot (y - 0) \iff z - 4 = \pi y$$

(where $z$ approximates $f(x, y)$ near $(1, 0)$).

So $f(0.99, 0.01) \approx \pi \cdot (0.01) + 4 = 4.031415....$ (Compare this with the actual value $f(0.99, 0.01) = 3.999....$)
Example 2 (Ex 2.4.11). Determine all second-order partial derivatives of
\[ f(x, y) = e^{y/x} - ye^{-x} \]

What is the domain? Note that the two mixed partials agree on the domain of \( f \) (on which \( f \) is continuous).

3 Newon’s Method
Equation (6) on p.131.

4 Visualising a Multi-Variable Function

Example 3. Consider the function \( f(x, y) := \frac{1}{\sqrt{x^2+y^2}} \), whose domain is \( \mathbb{R}^2 \setminus \{(0,0)\} \). We will try to visualise the behaviour of this function using level curves.
So consider \( f(x, y) = c \), where 1, 4, 9. Would it make sense if we put \( c = 0 \) or \( c < 0 \)?

- When \( c = 1 \), we have \( x^2 + y^2 = 1 \), so we get a circle of radius 1 centred at the origin.
- When \( c = 4 \), we have \( x^2 + y^2 = (\frac{1}{2})^2 \), so we get a circle of radius 1/2 centred at the origin.
- When \( c = 9 \), we have \( x^2 + y^2 = (\frac{1}{3})^2 \), so we get a circle of radius 1/3 centred at the origin.

So the graph of the function is like a spike reaching out to infinity at the origin, and extending towards 0 away from the origin.

5 Extra examples

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