FOGARTY'S PROOF OF THE FINITE GENERATION OF CERTAIN SUBRINGS

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Abstract. This is an expository note covering Fogarty's geometric approach to proving finite generation of certain subrings, including invariants under *linearly reductive* group actions. We offer a very mild generalization which allows one to conclude that good moduli spaces are finite type.

1. Introduction

In [Fog87], John Fogarty proves the following remarkable result:

Proposition 1.1. [Fog87, Proposition p. 203] Let R be an excellent ring and $\phi : X \to Y$ a surjective R-morphism. If X is irreducible and finite type over R and Y is normal and noetherian, then Y is finite type over R.

If G is linearly reductive group scheme over R acting on an affine, normal R-scheme $X = \operatorname{Spec} A$, then the GIT quotient $X \to \operatorname{Spec} A^G$ is surjective and the ring of invariants A^G is noetherian. The proposition then implies that A^G is finitely generated over R and gives s positive answer to Hilbert's 14th problem for the subring of invariants of a linearly reductive group action. We find it remarkable that surjectivity and noetherianness are enough to guarantee finite typeness.

If R is a field, the standard proof that A^G is finitely generated is via a reduction to the graded case. If G is a *reductive* group scheme over a field, then A^G is not necessarily noetherian. We note that in [Fog87], Fogarty also establishes via similar methods that the ring of invariant for reductive group actions is finitely generated.

Fogarty's proof easily extends to the mild generalization:

Proposition 1.2. Let R be an excellent ring and $\phi : X \to Y$ a surjective R-morphism. If X is finite type over R and each irreducible component of X dominates Y and Y is normal and noetherian, then Y is finite type over R.

In [Alp07], this proposition is applied to prove the following generalization of Hilbert's 14th problem:

Date: December 15, 2007.

Theorem 1.3. [Alp07, Theorem 4.13(xi)] Let $\mathcal{X}$ be an Artin stack finite type over an excellent scheme S. If $\phi : \mathcal{X} \to Y$ is a good moduli space, then Y is finite type over S.

We make an effort to isolate both the tough algebra facts and the properties of excellence that are needed in the proof.

2. Excellent rings

Excellent rings were introduced by Grothendieck to capture the essential properties of rings needed in the algebra behind certain useful facts in algebraic geometry. The class of excellent rings contains almost all noetherian rings of interest. In particular, any field is excellent and $\mathbb{Z}$ is excellent.

We recall the following properties of rings:

Definition 2.1. A ring A is *catenary* if for any primes ideals p and p' of A with $p \subseteq p'$, there exists a saturated finite chain of primes ideals starting from p and ending at p', and all such chains have the same length. A ring A is *universally catenary* if A is noetherian and every finite generated A-algebra is catenary.

Definition 2.2. An integral domain A is *japanese* if the integral closure of A in a finite extension of the fraction field is a finitely generated A-module. A ring A is *universally japanese* if every finitely generated A-algebra that is a domain is japanese.

Definition 2.3. [Gro67, Definition IV.7.8.2] A noetherian ring A is excellent if it satisfies the following condition:

(1) A is universally catenary.
(2) For all primes ideals p of A, the formal fibers of A_p are geometrically regular; that is, for all $q \subseteq p$ the fiber $\widehat{A}_p \otimes_{A_p} k(q)$ is geometrically regular over $k(q)$. (This means that $\widehat{A}_p \otimes_{A_p} k'$ is regular for all finite extensions k' of $k(q)$.)
(3) For every finitely generated A-algebra B, the locus $\operatorname{Reg}(B) = \{p \subseteq B \mid B_p \text{ is regular}\}$ is open in $\operatorname{Spec} B$.

Excellent rings are closed under localization, finitely generated extensions and passing to quotients. Fogarty's argument uses only the following three properties of excellence:

Proposition 2.4. [Gro67, IV.7.8.3]

(i) An excellent ring is universally catenary.
(ii) An excellent ring is universally japanese.
(iii) If A is an integrally closed local excellent ring with maximal ideal m and $\widehat{A}$ is the m-adic completion, then $\widehat{A}$ is integrally closed (and in particular an integral domain). □

3. Some algebra

The following lemma is useful in reducing proving finite typeness to integral schemes.

Lemma 3.1. [Fog83, p. 169] Suppose B is a noetherian ring and A is a B-algebra.

(1) A is finitely generated over B if and only if A_{red} is finitely generated over B.
(2) Let $\operatorname{Spec} A = \bigcup_i \operatorname{Spec} A_i$ be the irreducible decomposition. Then A is finitely generated over B if and only if each A_i is finitely generated over B.

Proof. For (i), it is clear that if A is finitely generated over B, then A_{red} is. For the converse, let $I = \sqrt{0}$ be the radical ideal in A. Since A is noetherian, I is nilpotent so by induction it suffices to assume $I^2 = 0$. Choose $a_i \in A$ for $i = 1, \dots, n$ such that the image $\bar{a_i}$ in A_{red} generate A_{red} as a B-algebra. Let $(f_1, \dots, f_m)$ be generators of the ideal I. We claim that $a_1, \dots, a_n, f_1, \dots, f_m$ generate A. Indeed, given $a \in A$, we can write $\bar{a} = g(\bar{a_1}, \dots, \bar{a_n})$ for some polynomial g. Then $a' = a - g(a_1, \dots, a_n) \in I$ and we can write $a' = a'_1 f_1 + \cdots + a'_m f_m$. Write $\bar{a'_i} = g_i(\bar{a_1}, \dots, \bar{a_n})$ for polynomials g_i. Then $a' - (g_1 f_1 + \cdots + g_m f_m) = (a'_1 - g_1) f_1 + \cdots + (a'_m - g_m) f_m = 0$ so $a = g + g_1 f_1 + \cdots g_m f_m$. We note that the noetherian hypothesis on B is not necessary for (i).

For (ii), one direction is clear. Let $p_1, \dots, p_n$ be the minimal primes of A and $A_i = A/p_i$. Then $p_1 \cap \cdots \cap p_n = (0)$. By induction on n, we may assume that $A/(p_1 \cap \cdots \cap p_{n-1})$ is finite type. Let $a_1, \dots a_m \in A$ be such that their images generate $A/(p_1 \cap \cdots \cap p_{n-1})$ and A/p_n. Because A/p_n is noetherian, we can find generators of $p_1 \cap \cdots \cap p_{n-1} \hookrightarrow A/p_n$. We claim $a_1, \dots, a_m, f_1, \dots, f_k$ generate A. Given $a \in A$, there is a polynomial g in $a_1, \dots a_m$ such that $a - g \in p_1 \cap \cdots \cap p_{n-1}$. Write $a - g = b_1 f_1 + \cdots b_k f_k$ with $b_i \in A$. We can find polynomials h_i in $a_1, \dots, a_m$ so that $b_i - h_i \in p_n$. But $(b_i - h_i) f_i = 0$ so that $a = g + h_1 f_1 + \cdots h_k f_k$. □

The following lemma is essential in Fogarty's argument and explains the relevance of the normality assumption.

Lemma 3.2. [Fog83, Lemma 3] Let A be a normal noetherian domain with fraction field K and B a ring between A and K with $\operatorname{Spec} B \to \operatorname{Spec} A$ surjective. Then $B = A$.

Proof. The result is clear if A is a field. Otherwise, if p is a height 1 prime of A, then A_p is a discrete valuation ring of K. By localizing $A \subseteq B \subseteq K$, we have the inclusion of rings

$$A_p \subseteq (A - p)^{-1} B \subseteq K$$

Since $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective, $(A-p)^{-1}B$ is not a field. The DVR A_p is a maximal subring of K and therefore $A_p = (A-p)^{-1}B$. Therefore

$$B \subseteq \bigcap_{\operatorname{ht} p=1} (A-p)^{-1}B = \bigcap_{\operatorname{ht} p=1} A_p = A$$

where the equality is Hartog's lemma. Thus, $B = A$. □

Corollary 3.3. If $f : X \to Y$ is an affine morphism of integral schemes with Y noetherian and normal. Suppose $f(X) \subseteq Y$ is open and the induced map $FF(Y) \xrightarrow{\sim} FF(X)$ is an isomorphism. Then f is an open immersion. □

We isolate the algebra results needed in the proof.

Proposition 3.4. [Gro67, Proposition IV.5.6.5],[Mat80, Theorem 15.5-6] Let Y be an irreducible locally noetherian scheme, X an irreducible scheme, and $f : X \to Y$ a dominant morphism locally of finite type. Let ξ (resp. η) be the generic point of X (resp. Y) and $d = \dim f^{-1}(\eta) = \operatorname{tr.deg}_{k(\eta)} k(\xi)$ be the dimension of the generic fiber. Let $x \in X$ and $y = f(x)$. Then

$$d + \dim \mathcal{O}_y \geq \operatorname{tr.deg}_{k(y)} k(x) + \dim \mathcal{O}_x$$

If Y is universally catenary, there is equality. If $x \in f^{-1}(y)$ is closed, then

$$\dim \mathcal{O}_x = \dim \mathcal{O}_y + d$$

□

In fact the equality in the first expression above characterizes universally catenary rings (see [Mat80, Theorem 15.6]).

Proposition 3.5. [Mat80, Theorem 8.4] Let A be a ring, I an ideal, and M an A-module. Suppose A is I-adically complete and M is separated for I-adic topology. If M/IM is generated over A/I by $\bar{\omega}_1, \dots, \bar{\omega}_n$, and $\omega_i \in M$ is an arbitrary inverse image of $\bar{\omega}_i$, then M is generated over A by $\omega_1, \dots, \omega_n$.

We recall that a module M is I-adically separated if $\bigcap_n I^n M = 0$. The above proposition can be viewed as a version of Nakayama's lemma for separated (but not necessarily finitely generated!) modules over complete rings.

4. Proof of Proposition 1.2

Summary of proof: We adjoin elements of A to form a finitely generated sub-R-algebra $A' \subseteq A$ such that the induced morphism $\eta : Y \to Z = \operatorname{Spec} A'$ is injective and birational. Since R is universally japanese, we may assume A' is normal. By Corollary 3.3 if it is shown that the image $\eta(y)$ is open, then η is an open immersion and therefore finite type. Since ϕ is surjective, the image is constructible so it suffices to show that it is closed under generization which can be reduced to showing that for closed points

$y \in Y$ with $z = \eta(y)$, $\mathcal{O}_z \to \mathcal{O}_y$ is an isomorphism. Using that Z is universally catenary, $\dim \mathcal{O}_y \geq \dim \mathcal{O}_z$. We then show via a dimension argument that the induced map on completions $\widehat{\mathcal{O}}_z \to \widehat{\mathcal{O}}_y$ is injective. It follows that $\mathcal{O}_z \to \mathcal{O}_y$ is injective and a normality argument is used to conclude that it is an isomorphism.

We stress that Fogarty uses the irreducibility assumption on X only in the conclusion that $\dim \mathcal{O}_y \geq \dim \mathcal{O}_z$ and this easily extends to the case when the irreducible components of X dominate Y.

Proof. We may suppose $Y = \operatorname{Spec} A$ with A noetherian and integrally closed in its fraction field K. By replacing $\operatorname{Spec} R$ with the scheme theoretic image of $Y \to \operatorname{Spec} R$, we may assume $R \subseteq A$. By using Lemma 3.1, we may assume R is an integral domain. Let $L = \operatorname{Frac}(R)$. Let $\operatorname{Spec} B \to X$ be a finite type morphism with B an integral domain such that the composition $\operatorname{Spec} B \to X \to Y$ is dominant. The composition $\operatorname{Spec} B \to \operatorname{Spec} A \to \operatorname{Spec} R$ yields the inclusions of function fields $L \hookrightarrow K \hookrightarrow \operatorname{Frac}(B)$. Since $L \hookrightarrow \operatorname{Frac}(B)$ is a finitely generated field extension, $L \hookrightarrow K$ is as well. Therefore we may adjoin finitely many elements to form a finitely generated sub-R-algebra $A_0 \subseteq A$ with the same fraction field.

Let $U \to X$ be a surjective finite type morphism with $U = \operatorname{Spec} B$ an affine scheme. There is a cartesian diagram

$$\begin{array}{ccc} U \times_Y U & \longrightarrow & U \times_R U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_R Y \end{array}$$

Since U is finite type over R, there are a finite number of elements $f_1, \cdots, f_n \in A$ such that $(1 \otimes f_i - f_i \otimes 1)$ defines the closed subscheme $U \times_Y U \hookrightarrow U \times_R U$. Let A_1 be the finitely generated sub-R-algebra obtained by adjoining the elements f_i to A_0. We claim that the induced map $\eta : \operatorname{Spec} A \to \operatorname{Spec} A_1$ is geometrically injective. Suppose $y_1, y_2 : \operatorname{Spec} k \to \operatorname{Spec} A$ are geometric points with $\eta(y_1) = \eta(y_2)$. In particular, for all i, $f_i(y_1) = f_i(y_2)$ (where $f_i(y_1)$ is the image of f_i under $y_1^{\#} : A \to k$). Let $u_1, u_2 : \operatorname{Spec} k \to U$ be lifts of y_1, y_2. To show that $y_1 = y_2$, it suffices to show that the geometric point $(u_1, u_2) : \operatorname{Spec} k \to U \times_R U$ factors through the closed subscheme $U \times_Y U$ but this is clear since $(1 \otimes f_i - f_i \otimes 1)(u_1, u_2) = 0$ for all i (indeed, the diagram

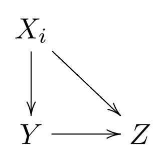
$$\begin{array}{ccc} k & \xleftarrow{(u_1,u_2)^{\#}} & B \otimes_R B \\ & \nwarrow^{(y_1,y_2)^{\#}} & \uparrow \\ & & A \otimes_R A \end{array}$$

implies that $(1 \otimes f_i - f_i \otimes 1)(u_1, u_2) = (1 \otimes f_i - f_i \otimes 1)(y_1, y_2) = f_i(y_1) - f_i(y_2) = 0$).

Since R is universally japanese (Proposition 2.4(ii)), if A_2 is the integral closure of A_1 in K, A_2 a finitely generated R-algebra. Let $Z = \operatorname{Spec} A_2$ and $\eta : Y \to Z$. Since $\eta \circ \phi : X \to Z$ is a finite type morphism of noetherian schemes and since ϕ is surjective (this is the only place the surjectivity assumption is used), $\eta(Y)$ is constructible. Since η is dominant, $\eta(Y)$ has non-empty interior U. In summary, η is a geometrically injective, birational morphism with dense constructible image. If $\eta(Y)$ is open, Corollary 3.3 would imply that η is an open immersion and therefore finite type. To show that $\eta(Y)$ is open, it suffices to show that for all closed points $y \in Y$, any generization of $\eta(y)$ is contained in $\eta(Y)$. Indeed, it certainly suffices to check that $\eta(Y)$ is closed under generization. If $z = \eta(y)$ and z' is a generization of z. Then $\eta^{-1}(\overline{\{z'\}})$ is closed and any closed point $y' \in \overline{\{y\}}$ is in $\eta^{-1}(\overline{\{z'\}})$ so that z' is a generization of $\eta(y')$.

Let $y \in Y$ be a closed point with $\eta(y) = z$. If $\mathcal{O}_z \to \mathcal{O}_y$ is an isomorphism, then any generization of z is contained in $\eta(Y)$. Therefore, we have reduced to show that $\mathcal{O}_z \to \mathcal{O}_y$ is an isomorphism for all closed points y with $z = \eta(y)$.

We first show that $\dim \mathcal{O}_y \geq \dim \mathcal{O}_z$. There exists an irreducible component X_i of X and closed point $x_i \in X_i$ with $\phi(x_i) = y$. In the diagram,



the morphisms $X_i \to Y$ and $X_i \to Z$ are dominant. The scheme Z is universally catenary since R is excellent. Proposition 3.4 now gives the inequalities

$$\dim \mathcal{O}_y + d \geq \dim \mathcal{O}_x = \dim \mathcal{O}_z + d$$

where d is the dimension of the generic fiber. It follows that $\dim \mathcal{O}_y \geq \dim \mathcal{O}_z$.

We will now show that $\widehat{\mathcal{O}}_z \to \widehat{\mathcal{O}}_y$ is injective. Since η is injective, $m_z \mathcal{O}_y$ is m_y-primary so $\widehat{\mathcal{O}}_y / m_z \widehat{\mathcal{O}}_y$ is a finite $k(z)$-vector space. Since $\widehat{\mathcal{O}}_y$ is clearly m_z-adically separated, by applying Proposition 3.5, $\widehat{\mathcal{O}}_z \to \widehat{\mathcal{O}}_y$ is finite. Therefore $\dim \widehat{\mathcal{O}}_y = \dim \widehat{\mathcal{O}}_z$. Since R is excellent and $\mathcal{O}_z$ is integrally closed, by Proposition 2.4, $\widehat{\mathcal{O}}_z$ is an integral domain. It follows that $\widehat{\mathcal{O}}_z \to \widehat{\mathcal{O}}_y$ is injective. Indeed, if $g \in \ker(\widehat{\mathcal{O}}_z \to \widehat{\mathcal{O}}_y)$ is non-zero, then $\dim \widehat{\mathcal{O}}_y \leq \dim \widehat{\mathcal{O}}_z / g < \dim \widehat{\mathcal{O}}_z$, a contradiction.

We can now conclude that $\mathcal{O}_z \to \mathcal{O}_y$ is an isomorphism. The m_z-adic completion of $\mathcal{O}_z \to \mathcal{O}_y$ is the injective morphism $\widehat{\mathcal{O}}_z \hookrightarrow \widehat{\mathcal{O}}_y$. Since $\mathcal{O}_z \to \widehat{\mathcal{O}}_z$ is faithfully flat and $\widehat{\mathcal{O}}_y = \mathcal{O}_y \otimes_{\mathcal{O}_z} \widehat{\mathcal{O}}_z$, $\mathcal{O}_z \to \mathcal{O}_y$ is injective. It remains to show that $\mathcal{O}_z \to \mathcal{O}_y$ is surjective. Faithfully flatness also implies that for any ideal $I \subseteq \mathcal{O}_z$, $(I \widehat{\mathcal{O}}_z) \cap \mathcal{O}_z = I$. Let $a/b \in \mathcal{O}_y$ with $a, b \in \mathcal{O}_z$. Since $\widehat{\mathcal{O}}_y$

is integral over $\widehat{\mathcal{O}}_z$, there exist $\alpha_0, \dots \alpha_{n-1} \in \widehat{\mathcal{O}}_z$ with

$$(\frac{a}{b})^n + \alpha_{n-1}(\frac{a}{b})^{n-1} + \cdots + \alpha_1(\frac{a}{b}) + \alpha_0 = 0$$

which gives

$$a^n + \alpha_{n-1}a^{n-1}b + \cdots + \alpha_1 ab^{n-1} + b^n = 0$$

This implies

$$a^n \in \left(\sum_{i=1}^{n} (a^{n-i}b^i)\widehat{\mathcal{O}}_z \right) \cap \mathcal{O}_z = \sum_{i=1}^{n} (a^{n-i}b^i)\mathcal{O}_z$$

Therefore a/b is integral over $\mathcal{O}_z$ and since $\mathcal{O}_z$ is integrally closed, $a/b \in \mathcal{O}_z$. This establishes that $\mathcal{O}_z = \mathcal{O}_y$ and finishes the proof. □

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