The Mean Euler Characteristic and Contact Structures

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- Suppose that there exists an integer l_+ such that $\dim \mathrm{HC}_l(M, a) < \infty$ for all $l \geq l_+$. The **positive mean Euler characteristic** is set as

$$\chi^+(M, \xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{l=l_+}^{N} (-1)^l \dim \mathrm{HC}_l(M, \xi) \tag{1}$$

provided that the limit exists.

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- Similarly, if there exists an integer l_- is and integrer such that $\dim \mathrm{HC}_l(M,a) < \infty$ for all $l \leq l_-$. The **negative mean Euler characteristic** is set as

$$\chi^-(M,\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{l=l_-}^{N} (-1)^l \dim \mathrm{HC}_{-l}(M,\xi) \tag{2}$$

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- This works for a variety of flavors of contact homology.
- Can also be restricted to homotopy classes of the closed Reeb orbits.

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Ginzburg-Kerman:

If the Reeb flow has finitely many simple periodic Reeb orbits, then

$$\sum{}^{\pm} \frac{\sigma(x_i)}{\Delta(x_i)} + \frac{1}{2}\sum{}^{\pm} \frac{\sigma(y_i)}{\Delta(y_i)} = \chi^{\pm}(M, \xi), \tag{3}$$

where the two different types of good Reeb orbits are distinguished by x_i and y_i, and $\sum^{+}$ (respectively, $\sum^{-}$) stands for the sum over all orbits with positive (respectively, negative) mean index.

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Goal

Extend this expression to contact forms with infinitely many simple closed Reeb orbits in a useful way.

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  (a) $\lim_{k\to\infty} \frac{\mu_{\text{CZ}}(\Psi^k)}{k} = \Delta(\Psi)$

  (b) $|\,\mu_{\text{CZ}}(\Psi) - \Delta(\Psi)| < n$.

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- J a complex structure on ξ compatible with $d\,\alpha$, $(J\colon \xi \to \xi \mid J^2 = -Id,\ d\,\alpha(J\cdot, J\cdot) = d\,\alpha(\cdot,\cdot),\ d\,\alpha(\cdot, J\cdot) > 0\ )$
- R_α the Reeb vector field associated to α,
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- We can always take a small perturbation of α to get a non-degenerate contact form.

Contact chain groups

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- In the presents of homologically non-trivial Reeb orbits, ambiguity arises due to the dependency of $|\gamma|$ on Φ_γ.
- $\mathrm{C}_*(M, \alpha) = \bigoplus_a \mathrm{C}_*^a(M, \alpha)$, where a are free homotopy classes of Reeb orbits.

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  - We'll return to homologically non-trivial Reeb orbits.

Asymptotically finite contact manifolds

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Example

The standard contact sphere, $\mathrm{C}_*(S^{2n-1}, \xi_0)$ is essentially generated by one Reeb orbit when we "stretch" it out in all directions but one.

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Let $(M, \xi) = \{(M, \alpha_r)\}$ *be a closed asymptotically finite contact manifold. Assume that for each* α_r*, the contact homology is defined* **. Then the limits in the mean Euler characteristics are defined and*

$$\chi^{\pm}(M, \xi) = \sum^{\pm} \frac{\sigma_{x_i}}{\Delta_{x_i}} + \frac{1}{2}\sum^{\pm} \frac{\sigma_{y_i}}{\Delta_{y_i}}, \tag{4}$$

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where $\sum^{+}$ (respectively $\sum^{-}$) stands for the sum over the sequences of orbits with positive asymptotic mean index (respectively, negative), and we use x_i and y_i to distinguish the two types of good Reeb orbits.

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Let $(M, \xi) = \{(M, \alpha_r)\}$ *be a closed AF contact manifold and assume each* (M, α_r) *is weakly index-positive with respect to a fixed section* $\mathfrak{s}$ *of* $S^1[(\Lambda^{\text{n-1}}_{\mathbb{C}}\, \xi)^{\otimes 2}]$.

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- Application:

Corollary

The mean Euler characteristic of $(M', \{\alpha'_r\})$ *converges and*

$$\chi(M', \xi') = \chi(M, \xi) + (-1)^k \frac{1}{2},$$

where k *is the index of the subcrirical contact surgery* $(k < n)$*.*

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- Moreover, if $\gamma(0) = \gamma(T)$ and hence $V_T = V_0$, then $\mu(\gamma;\mathfrak{s}) = \mu(\gamma;\mathfrak{s}')$.
- When $\pi_1(M) = 0$, all sections of $S^1[(\Lambda_{\mathbb{C}}^{\text{n-1}}\,\xi)^{\otimes 2}]$ are homotopic. Then $\mu(\gamma;\mathfrak{s})$ is independent of the section $\mathfrak{s}$ for any closed orbit γ.

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- When $\pi_1(M) = 0$, all sections of $S^1[(\Lambda_{\mathbb{C}}^{\text{n-1}}\,\xi)^{\otimes 2}]$ are homotopic. Then $\mu(\gamma;\mathfrak{s})$ is independent of the section $\mathfrak{s}$ for any closed orbit γ.
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Lemma (μ-Catenation lemma)

For the paths of symplectic maps Ψ_1 and Ψ_2 over $\gamma_1 = \gamma|_{[0,T_1]}$ and $\gamma_2 = \gamma|_{[T_1,T]}$, we have

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where b is a constant that depends on n.

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Note: Since $\pi_1(\mathcal{H}) = 0$, the handle is weakly index-positive with respect to any section of $S^1[(\Lambda^{n-1}_{\mathbb{C}}\, \xi_{\mathcal{H}})^{\otimes 2}]$.

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- (M, ξ) is a weakly index-positive with respect to a section $\mathfrak{s}_M$ of $S^1[(\Lambda_{\mathbb{C}}^{n-1}\xi)^{\otimes 2}]$,
- (M', ξ') is a contact manifold obtained from M by a subcritical k-surgery with $c_1(\xi') = 0$, and
- $\mathfrak{s}_M$ extends over $\mathcal{H}$.

Then for every integer r, there exists a non-degenerate contact form α'_r such that the following hold:

- (M', α'_r) is weakly index-positive with respect to some extension $\mathfrak{s}'$ of $\mathfrak{s}_M$;
- if c_j and c'_j denote the number of degree j generators of (M, α) and (M', α'_r), then for $j \leq r$,

$$c'_j = c_j + b_j,$$

where

$$b_j = \begin{cases} 1, & \text{if } j = 2n - k - 4 + 2i, \text{ for } i \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

- Hence asymptotic finiteness + weak index-positivity are preserved under subcritical contact surgery if $c_1(\xi') = 0$ and if $\mathfrak{s}_M$ extends over $\mathcal{H}$.

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- Hence asymptotic finiteness + weak index-positivity are preserved under subcritical contact surgery if $c_1(\xi') = 0$ and if $\mathfrak{s}_M$ extends over $\mathcal{H}$.
- A given section $\mathfrak{s}$ is almost always compatible with a given surgery.

Homologically non-trivial Reeb orbits

Suppose γ is a homologically non-trivial Reeb orbit, then $\mu_{CZ}(\gamma)$ depends on the trivialization of ξ along γ and one must keep track of such choices.

It is convenient to consistently assign such trivializations along closed paths in M of any homotopy class using a fixed section $\mathfrak{s}$ of $S^1[(\Lambda_{\mathbb{C}}^{\text{n-1}}\,\xi)^{\otimes 2}]$.

- Pick a unitary frame $F = \{e_1, \cdots, e_{n-1}\}$ of $\xi|_\gamma$ with $\mathfrak{s}_F := (\Lambda_{j=1}^{n-1} e_j)^{\otimes 2}$ such that $\mathfrak{s}_F \sim \mathfrak{s}_\gamma$
- $\mu_{CZ}(\gamma; [\mathfrak{s}])$
    - $\mu_{cz}(\gamma; \mathfrak{s})$ Depends only on the homotopy class of F.
    - All such frames are homotopic.
    - Therefore, take $\mu_{cz}(\gamma; [\mathfrak{s}])$.

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- apply the following MEC formula:

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- Set $\sigma(S) := (-1)^{|S|}$, where $|S| := \mu_{\mathrm{RS}}(S) - \frac{1}{2}\dim S + n - 3$, and $\mu_{\mathrm{RS}}(S)$ is the generalized Conley-Zehnder index (Robbin-Salamon).
- The **mean index** of S_T is given by $\Delta(S_T) := \lim_{k\to\infty} \frac{\mu_{\mathrm{RS}}(S_{kT})}{k} = \Delta(\Psi)$.

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- Given an orbifold CW decomposition of a Reeb orbifold S, set

$$e(S) = \sum_{\bar{\sigma}} (-1)^{\dim \bar{\sigma}} |Stab(\bar{\sigma})|,$$

where the sum runs over the q-cells $\bar{\sigma}$ of S, and $|Stab(\bar{\sigma})|$ is the order of the *stabilizer* subgroup of $\bar{\sigma}$.

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- The Morse-Bott version of the MEC formula generalizes the relation established by Rademacher in [Ra] for geodesic flows.

Example: Ustilosky spheres

Brieskorn manifold

$\Sigma(a_0, \cdots, a_n) := V(a_0, \cdots, a_n)) \cap S^{2n+1}$, where

$$V(a_0, \cdots, a_n) := \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} : z_0^{a_0} + \cdots + z_n^{a_n} = 0\},$$

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  1. $S_\pi (z_0 = 0)$
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- By the MEC formula, we get

$$\chi^+(M,\xi_p) = \frac{1}{2}\frac{((n-1)p+1)}{((n-2)p+2)} \quad \text{and} \quad \chi^-(M,\xi_p) = 0.$$

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- Similar for weakly index-negative case ($\lambda < 0$ with the roles of χ^+ and χ^- interchanged).

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- $\Delta(B)/2$ is the rotation number of $\mathfrak{s}'$ with respect to $\mathfrak{s}$.

References

A. Adem, J. Leida, Y. Ruan, *Orbifolds and Stringy Topology*, Cambridge Tracts in Mathematics **171**, Cambridge University Press, New York, 2007.

F. Bourgeois, A Morse-Bott approach to contact homology. PhD Thesis, Stanford, 2002 University

I. Ekeland, H. Hofer, Convex Hamiltonian energy surfaces and their periodic trajectories, *Comm. Math. Phys.*, **113** (1987), 419–469.

V.Ginzburg, E. Kerman, Homological Resonances for Hamiltonian Diffeomorphisms and Reeb Flows, *IMRN*, vol. 2009;2010 no.1,53– 68.

H.B. Rademacher, On the average indices of closed geodesics, *J. Differential Geom.*, **29** (1989), 65–83.

J. Robbin and D. Salamon, The Maslov index for paths, *Topology* **32** (1993), 827–844

I. Ustilovsky, Infinitely many contact structures on S^{4m+1}, *Internat. Math. Res. Notices* 1999, no. 14, 781–791.

O. van Koert, Open books for contact five-manifolds and applications of contact homology, Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln, 2005.

C. Viterbo, Equivariant Morse theory for starshaped Hamiltonian systems, *Trans. Amer. Math. Soc.*, **311** (1989), 621–655.

M.-L. Yau, Cylindrical contact homology of Stein-fillable conatct manifolds, *Geometry & Topology*, **8** (2004), 1243–1280.