Simply connected surfaces of general type with $p_g = 0$ and $K^2 > 0$

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One of the fundamental problems in the classification of complex surfaces is to find a new family of simply connected surfaces with $p_g = 0$ and $K^2 > 0$. It has been studied intensively by algebraic surface theorists for more than 100 years since F. Enriques constructed a new surface with $p_g = q = 0$ and $\pi_1 = \mathbf{Z}_2$ in 1894. Although a large number of non-simply connected complex surfaces of general type with $p_g = 0$ and $K^2 > 0$ have been known ([BHPV], Chapter VII), until recently the only previously known simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 > 0$ was Barlow surface [B]. Barlow surface has $K^2 = 1$. The natural question arises if there are other simply connected surfaces of general type with $p_g = 0$ and $K^2 > 0$ except Barlow surface.

In 2004 I constructed a new simply connected symplectic 4-manifold with $b_2^+ = 1$ (equivalently $p_g = 0$) and $K^2 = 2$ [P] by using a rational blow-down surgery. After this construction, it has been a very intriguing question whether such a symplectic 4-manifold admit a complex structure.

In 2006 Y. Lee and myself [LP] constructed a new family of simply connected, minimal, complex surfaces of general type with $p_g = 0$ and $1 \leq K^2 \leq 2$ by modifying Park's symplectic 4-manifolds in [P]. Our main techniques are a rational blow-down surgery [FS] and a $\mathbf{Q}$-Gorenstein smoothing theory [KSB, M].

In this year, using the same technique as above, H. Park, D. Shin and myself successfully found a right configuration to construct a simply connected, minimal, complex surface of general type with $p_g = 0$ and $K^2 = 3$ [PPS].

In this talk, I would like to review some basic facts about a rational blow-down surgery and a $\mathbf{Q}$-Gorenstein smoothing theory. And then I'll sketch how to construct a new family of simply connected symplectic 4-manifolds using a rational blow-down surgery and how to show that such 4-manifolds admit a complex structure using a $\mathbf{Q}$-Gorenstein smoothing theory. Finally we show that such surfaces are in fact minimal surfaces of general type with $p_g = 0$ and $1 \leq K^2 \leq 3$.

If a time is allowed, I'll also sketch how to construct a simply connected, minimal, symplectic 4-manifold with $b_2^+ = 1$ and $K^2 = 4$ [PPS] using a rational blow-down surgery.

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