Morava K-theory & Arnol'd Conjecture

(jnt with A. Blumberg)

(NOT PROOFREAD USE AT YOUR OWN RISK!)

1. Statement
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1. Statement

Thm (A-Blumberg): Let $\varphi \circlearrowright M$ be a Hamiltonian diffeo on a cpct symplectic manifold M. If all fixed pts are non-degenerate, then

$$\# \mathrm{Fix}(\varphi) \geq \mathrm{rk}\, H^{*}(M; \mathbb{F}_p) \quad \forall p.$$

The result that

$$\# \mathrm{Fix}(\varphi) \geq \mathrm{rk}\, H^*(M;\mathbb{Q})$$

is due to Fukaya-Ono & Liu-Tian but can also be proved using our methods.

<u>Rem:</u> The result is strictly weaker than what can currently be proved if M is semi-positive.

It is a consequence of

Thm: Let n be a natural number. $\exists$ a "Floer Morava K-theory" group $HF^*(\varphi; K(n))$ with the following prop:

Lazarev $\Leftarrow\!\!\Rightarrow$ (1) $H^*(M, K(n)) \otimes \Lambda$ is a summand ← Novikov field

(2) The rk of $HF^*(\varphi; K(n))$ is smaller that the $\# \mathrm{Fix}\, \varphi$

(2) Morava K-theory

A cohomology thry of "non-geometric nature"

Basic Properties

(a) Theory depends on prime p & $n \in \mathbb{N}$ ~~$\cup\{\infty\}$~~

(b) Coefficients are

$\boxed{\underbrace{\mathbb{F}_p[v_n, v_n^{-1}]}}$ $\quad |v_n| = \underbrace{2(p^n-1)}$

As $n \to \infty$ $\quad |v_n| \to +\infty$ $\quad K(n)$ approach $\mathbb{F}_p$

This implies that, for every finite CW cplex, we have

$$\boxed{\mathrm{rk}_{K(n)} H^*(M; K(n)) \leq \mathrm{rk}_{\mathbb{F}_p} H^*(M; \mathbb{F}_p)}$$

with equality if $n \gg 0$.

(This can be used to deduce the $\#\mathrm{Fix}\,\varphi \geq \mathrm{rk}\, H^*(M, \mathbb{F}_p) \ldots$)

Key Property:
There is a natural map from complex bordism to $K(n)$.
The first Chern class of the degree p circle bundle over $\mathbb{CP}^\infty$ is

Degree p $\ni$ $\boxed{v_n^{-1} \cdot u^{p^n}}$

$L_\infty(p,1) \to \mathbb{CP}^\infty$
$\quad \| \quad$
BC_p cyclic gp

Gysin: $H^*(BC_p, K(n))$ is finite dimensional & $\exists$ a fundamental class in degree 0.

Borel

Ambidexterity (Greenlees - Sadofsky)
For every finite group Γ, we have a natural degree 0 isomorphism
$$H^*(B\Gamma; K(n)) \xrightarrow{\sim} H_*(B\Gamma; K(n))$$

This can be combined with Poincaré duality

Thm (Cheng, A-Blumberg) If $\mathfrak{X}$ is a cpct almost cplex orbifold of dim n, $\exists$ a degree n iso

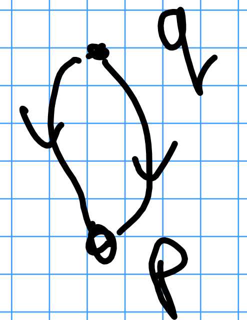
$\mathfrak{X} = M/G$
$M \times EG/G$

$$H^*(B\mathfrak{X}, K(n)) \xrightarrow{\sim} H_*(B\mathfrak{X}; K(n))$$

③ VFC in Floer htpy

What does the above discussion have to do with Floer hlgy?

$\boxed{\text{Toy case:}}$ Say we have a pair of Hamiltonian orbits p & q & a single moduli space



$\overline{\mathcal{M}}(p,q)$ which is regular

Analysing the linearisation of $\bar{\partial}$ operator implies that $\overline{\mathcal{M}}(p,q)$ is almost cplex manifold.

$\Longrightarrow$ Fundamental class in $H_*(\overline{\mathcal{M}}(p,q), MU)$

cplex bordism

The image in $H_*(pt, MU) \cong MU_*$ allows us to build a "Floer cplex" as

$$MU_*\langle p\rangle \xrightarrow{\cdot[\overline{\mathcal{M}}(p,q)]} MU_*\langle q\rangle$$

This only works for two orbits.

In general we have to work at the level of spectra: map of spectra

$$\underline{\mathrm{Cone}}\left(MU\langle q\rangle \xrightarrow{[\mathcal{M}(p,q)]} MU\cdot p \right)$$

For example, if there are 3 orbits, we set

$$\begin{tikzcd} MU\cdot r \arrow[r, "{[\mathcal{M}(q,r)]}"] \arrow[rr, bend left, "{[\mathcal{M}(p,r)]}"] & MU\cdot q \arrow[r, "{[\mathcal{M}(p,q)]}"] & MU\cdot p \end{tikzcd}$$

"Twisted cplex"

We have Cohen-Jones-Segal

$$* \quad \boxed{\partial \mathcal{M}(p,r) \simeq \mathcal{M}(p,q) \times \mathcal{M}(q,r)}$$

This corresponds to a null htpy at the level of maps of spectra.

Problem: Moduli spaces may not be regular.

This is why we can't use complex bordism.

So we need to take as input "cplex oriented Kuranishi spaces".

Thm (A–Blumberg) If Z is a cpct Hausdorff Top space then a cplex oriented Kuranishi structure with boundary of dim n determines a map $H^*_{rel}(Z; k(n)) \longrightarrow K(n)_*$ which is functorial for cobordisms (of cplex oriented Kuranishi)

Which version of Kur do we use?

Our own new version...
(Philosophically close to Pardon)

Z has a "global Kuranishi chart" if
- X top manifold
- $X \xrightarrow{s} V \leftarrow$ vector space

$\circlearrowright G$ finite.

+ homeo $s^{-1}(0)/_{G} \cong Z$.

Kuranishi structure is this data locally + patching maps.

Build a cat Charts of Kuranishi charts.

Diagram

A ⟶ Charts

+ properties about A . .

How are Kur charts constructed in Ham?

Add marked pts to your orbifold + perturbation ⟹ globally op

$\mathrm{Sym}_n \curvearrowright X$

$\downarrow \pi$

Abstract curves with marked pts. $\to \mathcal{M}_n \leftarrow$ Smooth manifold cplex

Fibers of $X \to \underline{\mathcal{M}}_n$ are also smooth.

$\Longrightarrow$ Refine

$TX = T^{\pi}X \oplus T\mathcal{M}_n$. cplex

Stable a.c stably iso to index cplex SL op

$G \curvearrowright M$ cpct top manifold &

$\mathbb{K}$ any spectrum

$$H_\alpha^G(M;\mathbb{K}) \xrightarrow{\sim} H^\alpha_G(M, S^{TM} \wedge \mathbb{K})$$