NCSG 11/2/20

Branes, Wavefunctions, Sheaves, Clusters

A book report plus work with L. Shen

CV, CCV, BCOV, AV, ADKMV, AKV, AENV, KS, GMN, LZ, B, W, DW, FG, DGG, DGG, CEHRV GV, OV TZ, SZ, Z, (CB)

IIA S.T. on $\mathbb{R}^4 \times X_{CY}$ / M-Thy on $\mathbb{R}^4 \times X_{CY} \times S^1$ N=2 4D

GV BPS M2-brane contributions to F terms of 4D Thy ↓ AGNT

can count BPS states ⇒ Integrality of GW

OV Add D4 $\mathbb{R}^2 \times L$ BPS M2-M5 contribs to F terms of 2D Thy

BPS ⇒ Integrality of OGW Conifold X^n ⇒ knot Invs. (FRAMINGS)

D6 $\mathbb{R}^4 \times L$: superpotential W & other terms

$$F = -\sum_{d,s} N_{d,s} \sum_{n=1}^{\infty} \frac{(q^s x^d)^n}{n(q^{n/2} - q^{-n/2})} \qquad q = e^{\hbar}$$

$$\lim_{\hbar \to 0} \hbar F = \sum N_d \, Li_2(x^d)$$

A-V : WS Disk insts Can coords u, v $v = \partial_u W$ $[u,v] = \hbar$ DUAL COORDS (E.G. BY M.S. HCS)

Lesson : $\mathcal{M}$ c P ex Lag pot'l W

$\mathcal{M}$: moduli of branes W : integrality e^F

These structures have quantizations, too. (CEHRV QM) ADKMV DGG

Witten : BCOV ⇔ $e^F = \mathbb{F}_{Top}(x)$ a state in QM on CY moduli space $\mathcal{H}_{CY}$

? Open-string BCOV s. $\mathbb{F}^{open}_{top}(x)$ a state in QM ~ Brane m.s.

x : CY moduli

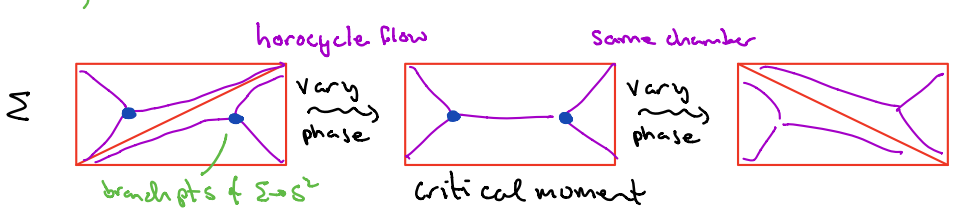
x : Brane moduli

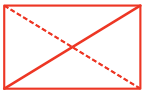
CV, CCV : 4D SW → 3D, EQUATING PHASE w/ INTERNAL DIR

CEHRV (DGG DGG)

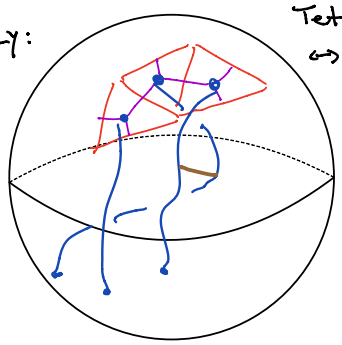
4D BPS STATES CONTRIB TO 3D PARTITION F^n

ALSO: BPS Quiver QM



Locally: $\Sigma \times \mathbb{R}$  Tetrahedron

MORE GENERALLY:



Tetrahedronation ↔ Sing. Tangle:

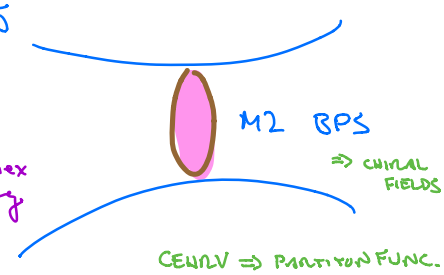
ex Lag

phase ↔ smoothing

nonex Lag

3D Lag L

$M5 : \mathbb{R}^3 \times L$



Recall: 2D Lag brane Σ ⇒ cluster chart $\quad \mathrm{Loc}_1(\Sigma) = P$

3D Lag, $\partial L = \Sigma$ ⇒ Lag ⊂ chart $\quad \mathcal{M} \xrightarrow{\partial}$

ex

Our set-up:

CLUSTER CHART

$f = g+3$
$e = 3g+3$
$v = 2g+2$

Γ cubic graph on S^2 $\quad \Delta_\Gamma$ triangulation

Λ_Γ Legendrian surf branch 2:1 genus g in $J^1(S^2) = S^2 \times \mathbb{R}^3 = T^{\infty,-}(S^2 \times \mathbb{R}^3) \subset T^\infty(\mathbb{R}^3)$

$\mathcal{C}_\Gamma$ $\quad Sh_1(\mathbb{R}^3, \Lambda_\Gamma)_0 \to \mathrm{Loc}_{\Lambda_\Gamma}$

$\mathcal{M}_\Gamma$ $\quad \mathrm{Ob}\,\mathcal{C}_\Gamma$ BRANES $= [(\mathbb{P}^1)^{\Gamma(2)} \setminus \delta] / PGL_2$

L_Γ filling (+ bounding cochain) ⇒ objects $\quad \mathcal{M}_\Gamma \to \mathrm{Loc}_{\Lambda_\Gamma}$

nonexact

exact case w/ $b_1(L_\Gamma) = g$: $\mathrm{Loc}\, L_\Gamma \overset{\text{Lag.}}{\hookrightarrow} \mathrm{Loc}\, \Lambda_\Gamma$ $\quad \subset$ Symp Torus $\underset{A}{\cong} (\mathbb{C}^\times)^{2g}$

$(\mathbb{C}^\times)^g \subset (\mathbb{C}^\times)^{2g}$

CHOICE

Filling: • determines $H_1(\Lambda_\Gamma) \to H_1(L_\Gamma)$

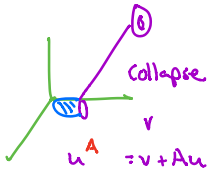
and $K \subset H_1(\Lambda_\Gamma)$ Kernel "phase" isotropic

• "framing" K^+ transverse isotropic
("A" above)

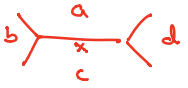
Matches A-V, Large-N CS

A.V. Lift μ_Γ to $\tilde{\mu}_\Gamma \subset T^*(\mathbb{C}^\times)^g$

Express as $\Gamma_{dW_\Gamma^A}$



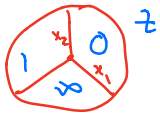
Map $\mu_\Gamma \hookrightarrow \mathcal{P}_\Gamma \overset{A}{=} (\mathbb{C}^\times)^{2g}$:



x = holonomy = cross ratio
(calc, not def)

Ex:

$g=1$



$x = -x_1$ $\quad y = -x_1 x_2$ $\quad x+y=1$ $\quad y = e^{-v}$ $\quad x = e^u$ $\quad v = -\ln(1-e^u)$

$W_\Gamma^A = \mathrm{Li}_2(x)$

Quantization

μ involution

Γ (or Δ_Γ) Labels cluster chart $\mathcal{P}_\Gamma$ in $\mathcal{P} = \{$framed PGL_2 loc systs on $S^2_{1\dots}\}$

FG: quantization $\mathcal{P}_\Gamma$ q-torus $\mathcal{D}_\Gamma$

A: $\quad yx = qxy$

$\{\mu_\Gamma\}_{\mathrm{Leg}} \rightsquigarrow$ holonomic (LEFT) ideals $\mathcal{I}_\Gamma = \mathcal{D}_\Gamma L_\Gamma$ $\quad \mathcal{D}\text{-mod } \mathcal{D}_\Gamma/\mathcal{I}_\Gamma$

principal $L_\Gamma \Psi_\Gamma = 0$

$\Psi_\Gamma^A(x)$ depends on A

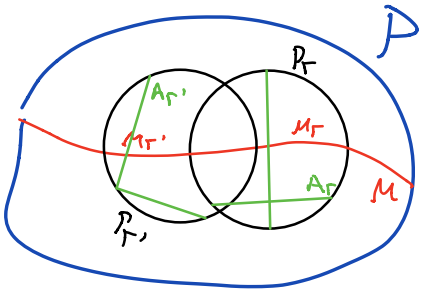
Ex: $x+y=1 \Rightarrow \Psi(qx) = (1-x)\Psi(x)$

$yx = qxy$

$\Psi = \Phi = \prod_{n\geq 0}(1-q^n x)^{-1}$ q-dilog

$\lim_{\hbar\to 0} \hbar \log \Psi = 1 \cdot \mathrm{Li}_2(x) = W$ $\quad \Psi \sim e^{W/\hbar}$

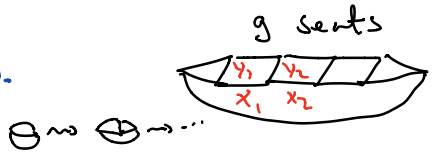
Warning: nonconvp. q-dilog.



PLAN: 0 • understand some Γ, A in single chart

1 • understand Δ framings, phases* within chart

2 • transfer from chart to chart via mutation

0. 

$\mathcal{M} = (\text{p.o.p.})^g : \Psi_{\text{canoe}} = \prod_{i=1}^{g} \Phi(x_i)$ $(A=0)$

1. A symm $g \times g$ mat. $\quad v_i^A = v_i$, $u_i^A = u_i + A_{ij} v_j$ $\quad L\Psi = 0$, $L^A \Psi^A = 0$ $\quad$ * Δ phase = F.T.

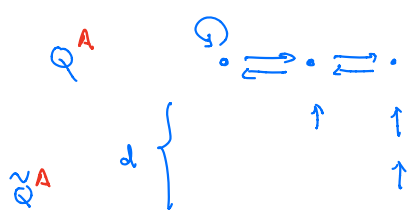
KS $\quad \Psi = \sum c_d x^d \qquad \Psi^A = \sum (-q^{1/2})^{x^T A x} c_d x^d$

integrality!

KS form for symm quivers: Framing Duality conj.: $\Psi_{DT}^{Q^A}(x,q) = \Psi_{OGW}^{A}(x,q)$

2. Mutation along $x_j \longleftrightarrow$ conj by $\Phi(x_j) \circ \exp^{\text{linear}}$ $\quad \Psi \to \Phi\Psi$

F.D. $\quad Q^A$



$$\Psi_{DT}^{Q^A} = \sum_{d \in \mathbb{Z}_{\geq 0}^{|Q^0|}, s} (-1)^s \dim \mathcal{H}_{d,s}\, q^{s/2} x^d$$

$$= \prod \Phi(q^{s/2} x^d)^{N_{d,s}}$$

$$\underset{\text{NLRV}}{=} \sum H_c^{2i}(\tilde{Q}_d^A)^W t^{i - d\tilde{v}}$$

E.g. $g=1$, $A=1$

$\dim \mathcal{H}_{d,s} = \dim H^{2s}(B\mathbb{C}_i^*)^{S_d}$

$\Psi = \sum_{d,n} \#\{\text{partitions of } n \text{ into } d \text{ parts}\} x^d q^n = \Phi$

$g=1$, $d=1$, $A=h$ $\quad Q_1^A$ = twisted char vty of genus h !?