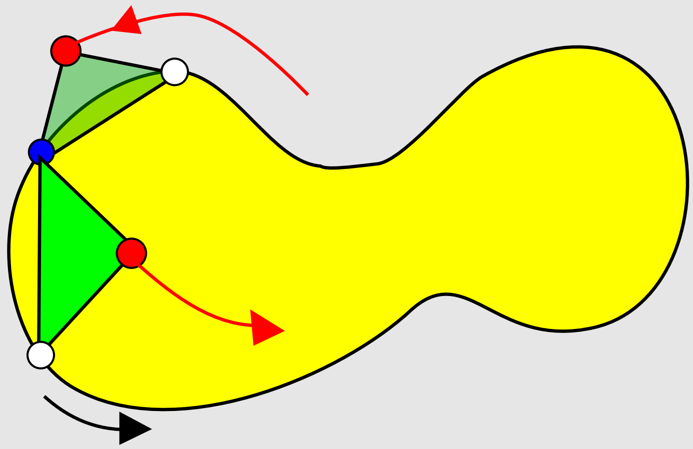
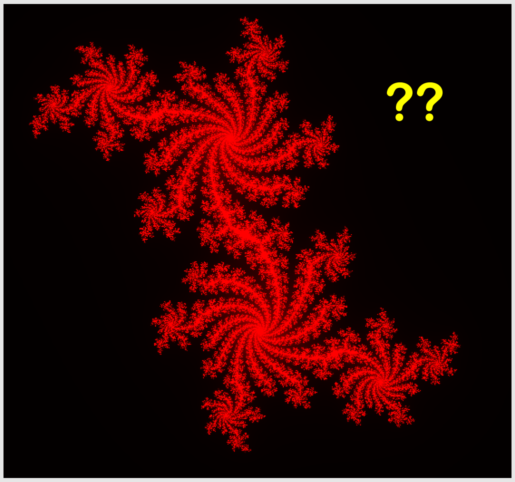
Inscribing Triangles

Easy: Every point on a smooth Jordan curve is the vertex of an inscribed triangle of your favorite shape.

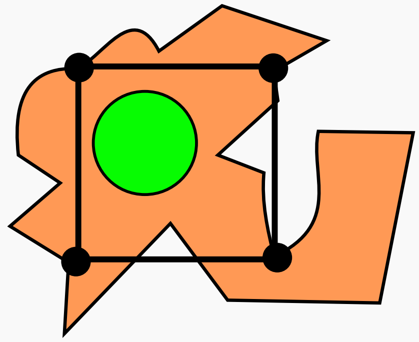




M. Meyerson 1980: All but at most 2 points on a Jordan curve are vertices of an equilateral triangle.

M. Neilsen 1992: Any Jordan curve has a dense set of points which are vertices of any desired triangle.

Inscribing Squares   (Toeplitz, 1911)



L. Schnirelmann 1929: true for smooth loops and polygons

Peter Feller and Marco Golla 2020: true for locally 1-lipschitz curves

Conjecture: If the interior contains
a disk of radius 1 then there is a
square of side length at least 1/100

See 2014 AMS Notices survey by B. Matschke, and
I. Pak's online book "Discrete and Polyhedral Geometry"

Inscribing Rectangles.

Vaughan 1980s:

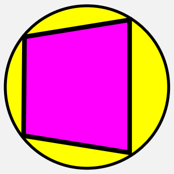
Every JC has an inscribed rectangle.

RES 2018

All but at most 4 points of any JC are vertices of inscribed rects.

Akopyan-Avvakumov 2018

Every smooth convex JC has cyclic quad of any shape.

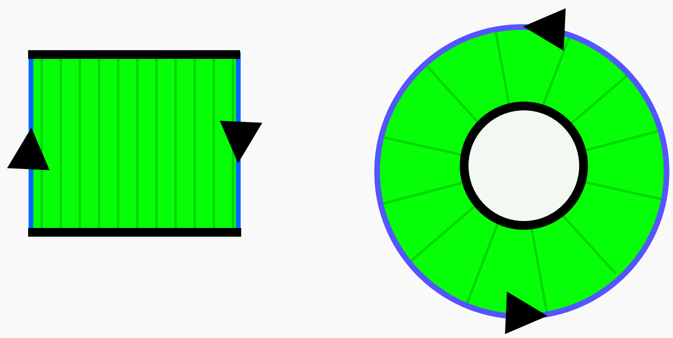


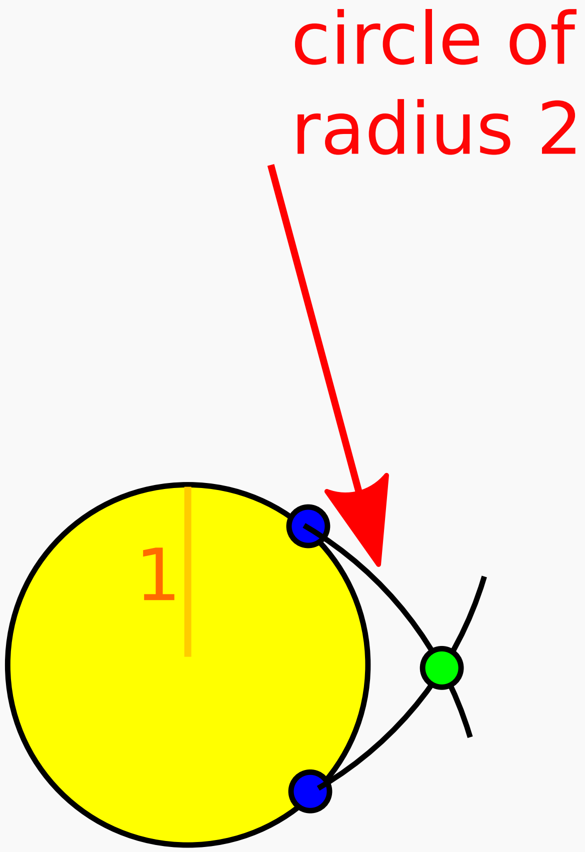
Greene-Lobb 2020:

Every smooth JC has an inscribed rectangle of any aspect ratio !

Every smooth JC has a cyclic quad of any shape!!

3 views of a Mobius band:

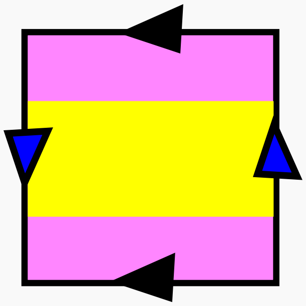




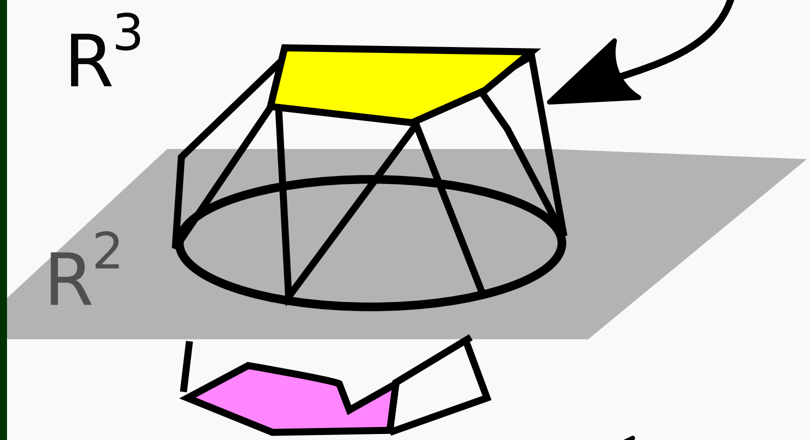
Sym2:

The space of
pairs of
unoriented
points on a circle.

2 views of the Klein bottle:

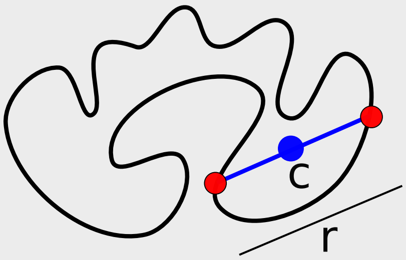


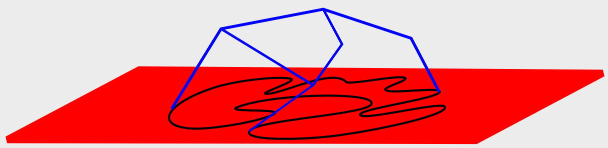
pretend this is beautiful
yellow a Moebius band.



Now pretend there
is a beautiful pink
reflected copy
underneath.

Vaughan's proof:



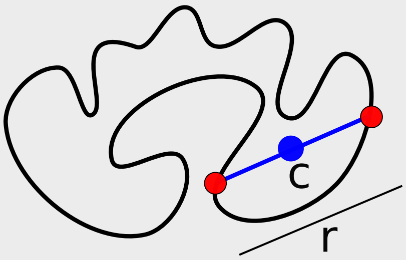


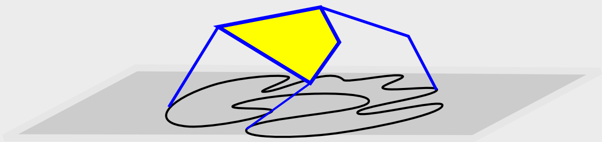
Let M = unordered pairs of pts...

The map f: M $\rightarrow$ R^3
carries a Moebius band to R^2
so that the boundary lies in R^2
and the rest lies above R^2.

Can't be an embedding!

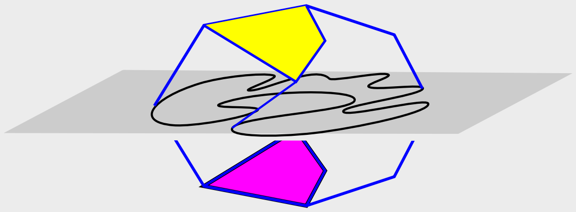
Vaughan's proof, Continued:





Can't be an embedding!

why not?
Double the
picture to
get an
embedded
Klein bottle, contradiction.



Theorem 1: A generic polygon either has a continuous sweepout of rectangles having all aspect ratios or a rotating family that contains the same square with all 4 labelings.

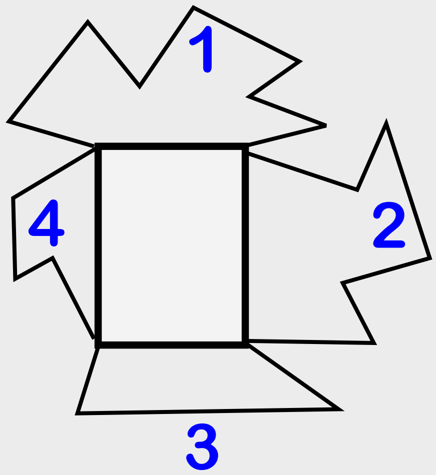
Proof:

1. Each sweepout contains an odd # of squares.
2. Each rotating family contains an odd #.
3. All other components have an even #.

So, (# sweeps) + (#rotaters) = odd.

Conjecture: 1. The rotaters do not appear generically. 2. Each arc component connects a saddle to a min or a max. (True for quads.)

Theorem 2: All but at most 4 points of any JC are vertices of an inscribed rectangle.



Proof: 1. Approximate by polygons.

2. Define A(R) = ([1]+[3])/([2]+[4]).

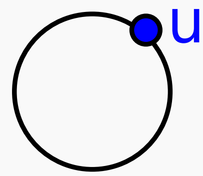
3. A^{-1}([1/N,N] is precompact.

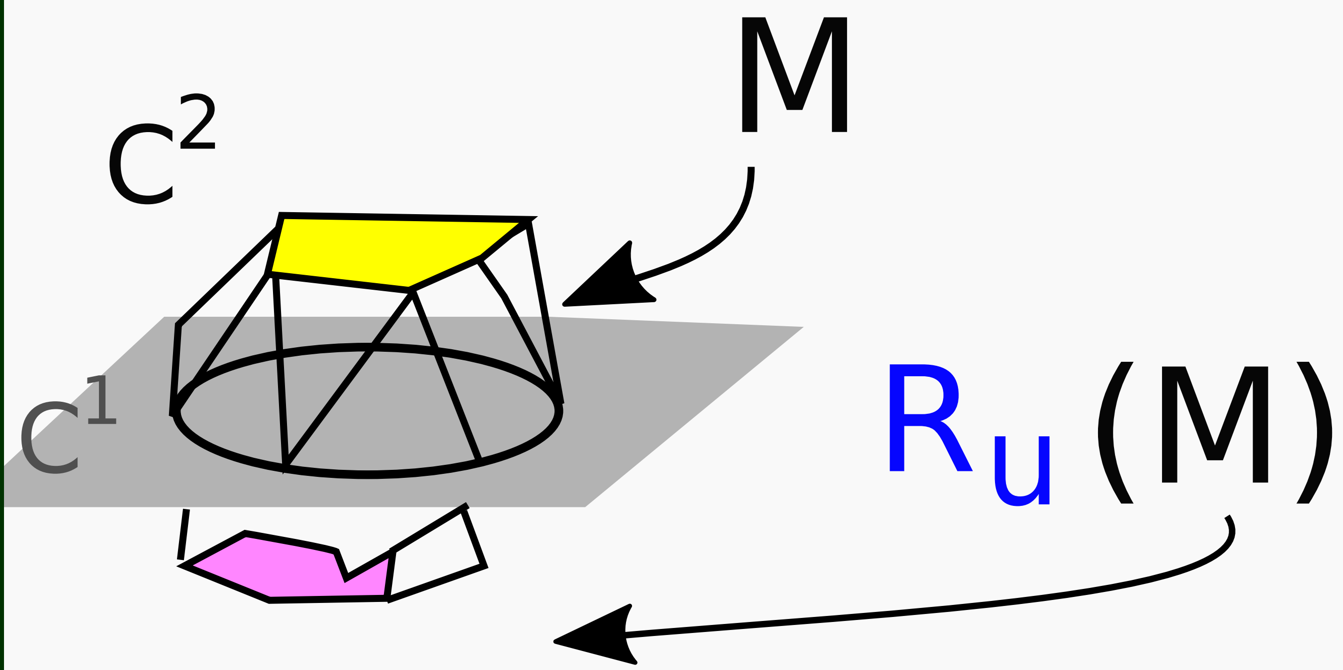
4. Either there is a persistent rotating family, or there is a sequence of arcs connecting A^{-1}(1/N) to A^{-1}(N) for N=1,2,3,...

5. Take a suitable limit in either case.

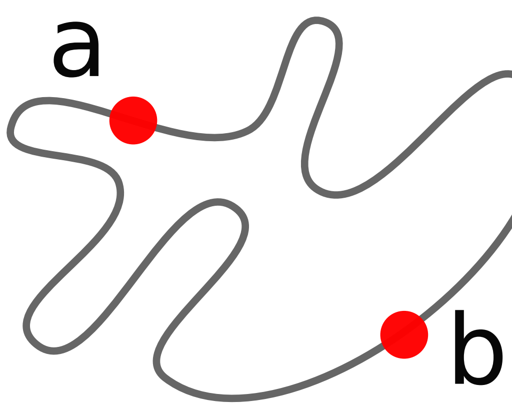
1 more view of the Klein bottle:

$R_u(z,w) = (z,uw)$





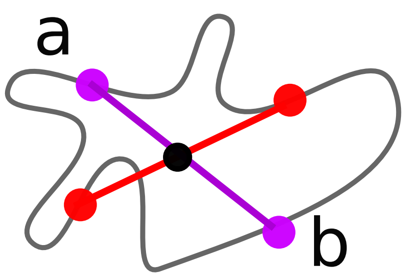
A new Map into $C^2 (=R^4)$

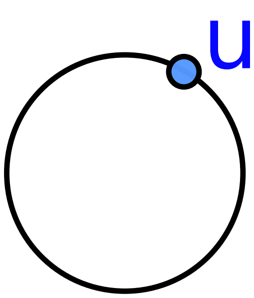


Idea:
(Hugelmeyer,
Greene - Lobb)

$$f(a,b) = \left(\frac{a+b}{2}, \frac{(a-b)^2}{2\sqrt{2}|a-b|}\right).$$

IDEA: look at 2 of these images

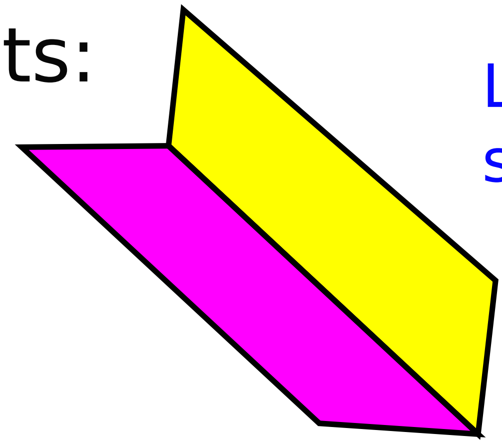




$$f(a,b) = \left(\frac{a+b}{2}, \frac{(a-b)^2}{2\sqrt{2}|a-b|}\right).$$

$$f(a,b) = \left(\frac{a+b}{2}, \frac{u(a-b)^2}{2\sqrt{2}|a-b|}\right).$$

2 ingredients:



1. smooth out the crease.

Lagrangian smoothing

2. show that the resulting Klein bottle cannot be embedded.

Shevchishin's Lagrangian non-embedding theorem (Nemirovski...).