Northern California Symplectic Geometry Seminar Talk

Sunday, February 7, 2021 12:45 PM

Thanks to Dan + organizers for invite!

3d Convex Contact Forms & The Ruelle Invariant

joint w/ Oliver Edtmair (Berkeley).

preprint at arXiv: 2012.12869.

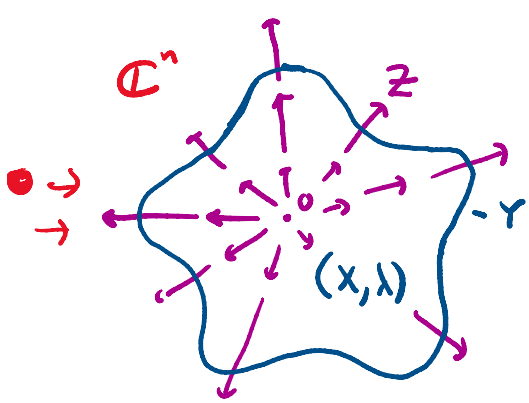
§ 1 - Introduction:

Recall one of most standard examples of contact manifolds w/ contact form.

Def: A star-shaped domain $X \subset \mathbb{R}^{2n}$ is a domain with smooth boundary Y s.t.

$$Y \text{ is transverse to } Z$$

where $Z(p) = \frac{1}{2}p$, i.e. Z is radial Liouville vectorfield.



- $\lambda := i_Z \omega$ is the radial Liouville form
- $\alpha = \lambda|_Y$ is contact form on $Y = \partial X$

"Star-shaped" contact forms on $Y \simeq S^{2n-1}$ admit an intrinsic characterization.

$(Y, \lambda|_Y)$ w/ $Y = \partial X$ & X star-shaped $\xleftrightarrow{\text{is equivalent to}}$ a contact form α on (S^{2n-1}, ξ_{std}) where $\alpha = \ker(\lambda|_Y)$ (†)

$(\mathbb{C}^n, \omega_{std})$ ... X star-shaped $\xleftrightarrow{\text{is equivalent to}}$ on (S^{2n-1}, ξ_{std}) where $\xi_{std} := \ker(\lambda|_{S^3})$ (†)

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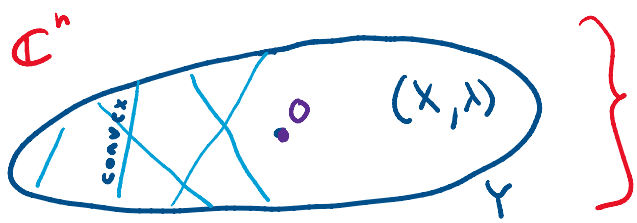
§ 2.1 – Convexity: There is a special sub-class of star-shaped domains: convex ones.

Def: A contact form α on (S^{2n-1}, ξ_{std}) is convex if there exists a convex star-shaped domain X such that

$$(S^{2n-1}, \alpha) \underset{\text{contactomorphism}}{\overset{\text{strict}}{\cong}} (Y, \lambda|_Y) \quad \text{where } Y = \partial X$$



Convex contact forms have many special properties e.g.

- (a) Every (strictly) convex contact form on S^3 has a disk-like surface of section for Reeb flow (Hofer-Wysocki-Zehnder 98)
- (b) There exists a constant $C > 0$ such that

$$\mathrm{sys}(S^{2n-1}, \alpha) \leq C \quad \text{where} \quad \mathrm{sys}(-) = \frac{(\text{min period of closed Reeb orbit})^n}{(n-1)! \cdot \mathrm{vol}(-)}$$

for all convex α. (Viterbo conjectures that $C = 1$)

$$\boxed{\text{Question: Give intrinsic definition of convexity, with no reference to embedding } S^{2n-1} \to \mathbb{C}^n.}$$

§ 1.2 – Dynamical Convexity: Here is a candidate answer....

Def: A contact form α on S^3 is **dynamically convex** if every closed Reeb orbit γ of α satisfies $CZ(\gamma) \geq 3$.

This was introduced by HWZ in 1998. We have

$$\alpha \text{ strictly convex} \Rightarrow \alpha \text{ dynamically convex}$$

Furthermore, properties ① & ② above are true for dynamically convex. So a natural question is...

Question: Does...

$$\alpha \text{ dynamically convex} \overset{??}{\Longrightarrow} \alpha \text{ (strictly) convex?}$$

This question has been open since HWZ 98.

§1.3 - Main Result: Today, we prove that

Thm (C-Edtmair): There exist dynamically convex contact forms on S^3 that are not convex.

The proof consists of two separate propositions.

The first proposition uses the **Ruelle invariant**

$$Ru(S^3, \xi) \in \mathbb{R} \text{ of } (S^3, \xi)$$

Intuitively, Ru is space-time average of "rotation" of Reeb flow around trajectory. More later...

Prop 1: There exists a $c>0$ such that

$$(1) \qquad c < \frac{Ru(S^3,\alpha)}{vol(S^3,\alpha)^{1/2}}$$

scale invariant

for all convex contact forms α on S^3.

Rmks: (i) RHS of (1) is scale invariant (ii) Prop in paper is slightly stronger.

To prove the main Thm, we simply show that

Prop 2: For all $\epsilon>0$, there exists a dynamically convex contact form α on S^3 s.t.

$$Ru(S^3,\alpha) < \epsilon \quad \text{and} \quad vol(S^3,\alpha) = 1$$

Rest of Talk:

- Detailed discussion of Ru.
- Detailed proof of Prop 1
- Short discussion of Prop 2.

§2 - Ruelle Invariant: Here is detailed construction of Ruelle invariant. Let

- (Y, ξ) be closed contact 3-manifold with $H^1(Y;\mathbb{Z}) = 0$ and $c_1(\xi) = 0$ e.g. (S^3, ξ_{std}).
- α be a contact form on (Y, ξ).
- $\tau : \xi \cong Y \times \mathbb{R}^2$ be a symplectic trivialization of contact structure.
- $q : \widetilde{Sp}(2) \to \mathbb{R}$ be a quasimorphism of universal cover of symplectic group $Sp(2)$ with

$$q(t \mapsto e^{2\pi i k t}) = k \}$$

From this data, can define $Ru(Y, \alpha)$ as so.

Step 1: Start with Reeb flow $\phi : \mathbb{R} \times Y \to Y$.

Step 2: Differential $d\phi$ combined with τ yields the linearized Reeb flow...

$$\Phi_\tau : \mathbb{R} \times Y \to Sp(2)$$

given by $\Phi_\tau(t, y) := \tau_{\phi(t,y)} \circ d\phi_t \circ \tau_y^{-1}$

Step 3: Lift Φ_τ uniquely to map

$$\widetilde{\Phi}_\tau : \mathbb{R} \times Y \to \widetilde{Sp}(2) \quad \text{with} \quad \widetilde{\Phi}_\tau(0, y) = \widetilde{Id}$$

Def: (Ruelle 1985) The Ruelle invariant is

$$Ru(Y, \alpha) := \lim \int \frac{q \circ \widetilde{\Phi}_\tau(T, -)}{T} \cdot \alpha \wedge d\alpha \qquad (2)$$

Def: Define

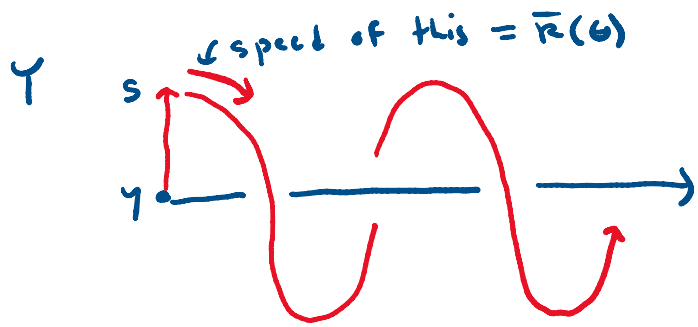
$$\boxed{Ru(Y,\alpha) := \lim_{T\to\infty} \int_Y \frac{q\circ \Phi_\tau(\tau,-)}{T} \cdot \alpha\wedge d\alpha \qquad (2)}$$

<u>Prop</u>: Limit (2) is well-defined & independent of choice of q and τ under hypotheses.

A key alternate formula for Ru is following.

- Flow Φ_τ on $Y\times\mathbb{R}^2$ $\xrightarrow{\text{projectivize}}$ flow $\bar{\Phi}$ on $Y\times S^1$ generated by vector-field $\bar{R}$.

- The derivative $\bar{R}(\theta) = \frac{d}{dt}(\theta\circ\bar{\Phi}_t)|_{t=0}$ of the angle coordinate measures speed of rotation at (y,s)



<u>Lemma 1</u>: $Ru(Y,\alpha) = \lim_{T\to\infty} \frac{1}{T}\int_Y\int_0^T \bar{R}(\theta)\circ\bar{\Phi}(s,y)\, ds\wedge\alpha\wedge d\alpha$

(*)

<u>§2.2 : Curvature Formula</u>: It turns out that $\bar{R}(\theta)$ admits a beautiful <u>curvature</u> formula if (Y,α) is convex bdry.

(Rmk: Due to <u>Ragazzo-Salamão</u> & <u>Hryniewicz</u>)

[illegible] $\mathbb{R}^2$ has a star-shaped boundary. The

(Rmk: Due to [illegible])

Def: Let $Y \subset \mathbb{C}^2$ be a star-shaped boundary. The quaternion trivialization is given by composition

$$\xi \xrightarrow{\pi} \mathrm{span}(\overset{\mathbb{C}}{j\nu}, \overset{\mathbb{R}}{k\nu}) \to \mathbb{R}^2$$

where

- j and k are quaternions j and k acting on $\mathbb{C}^2$
- ν is normal vector-field to Y,
- π is projection along Reeb direction,
- $\mathrm{span}(j\nu, k\nu) \to \mathbb{R}^2$ sends $j\nu$ to $\binom{1}{0}$ & $k\nu$ to $\binom{0}{1}$

Lem 2. (Curvature Formula) In the quaternionic trivialization, we have

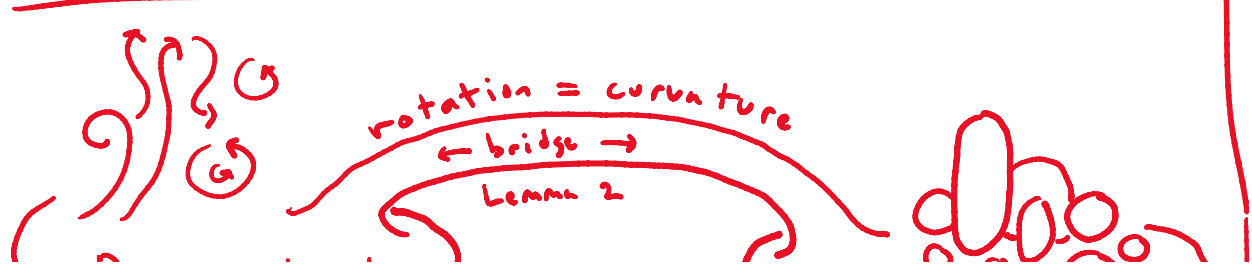
second fundamental form of Y

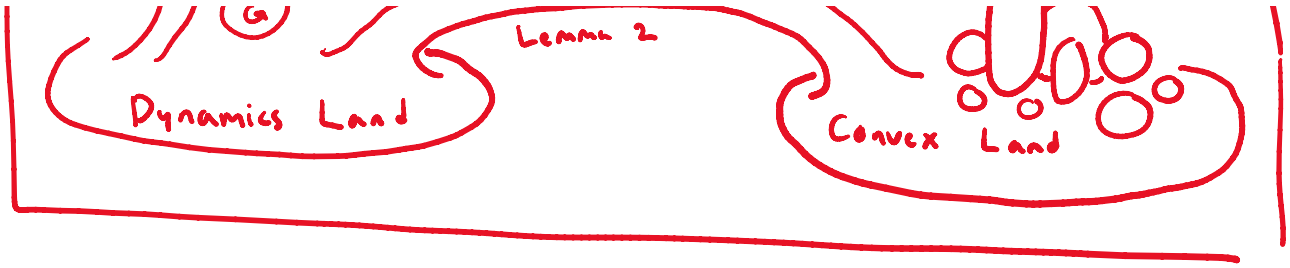
$$\boxed{\bar{R}(G)(y,s) = \frac{1}{2\pi\langle Z,\nu\rangle} \cdot \left(\overset{\mathbb{C}}{S(i\nu,i\nu)} + S(e^{2\pi i s}\cdot j\nu, e^{2\pi i s}\cdot j\nu)\right)}$$

Cor 3: If Y is convex boundary, then

$$Ru(Y,\alpha) \geq \frac{1}{2\pi}\int_Y S(i\nu,i\nu)\, d\mathrm{vol}_Y$$

Key Point: In the theme park of math





§3 – Proposition 1: Back to proof of....

Prop 1: (Simple Version): There exists $C>0$ such that

(1) $$C \cdot \mathrm{sys}(S^3,\alpha)^{5/2} \cdot \mathrm{vol}(S^3,\alpha)^{1/2} \leq Ru(S^3,\alpha)$$

for all convex contact forms α on S^3.

Let X be convex, star-shaped w/ $Y=\partial X$. Suffices to bound

$$\boxed{\int_Y s(iv,iv)\, d\mathrm{vol}_Y}$$

Step 1: First, show that

(1) $$\boxed{\int_Y s(iv,iv)\, d\mathrm{vol}_Y} \geq \frac{\mathrm{area}(Y)^2}{3 \cdot \mathrm{diam}(Y)^2 \cdot \int_Y H\, d\mathrm{vol}_Y}$$

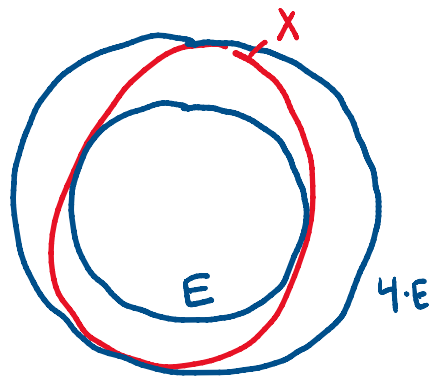
where diam is diameter & H is mean curvature.

Step 2: Next we apply...

Thm: (John's Ellipsoid) After applying affine symplectomorphism to X:

Thm: (John's Ellipsoid) After applying affine symplectomorphism to X, there is an ellipsoid $E = E(a,b)$ such that

$$(\dagger) \qquad E \subset X \subset 4 \cdot E$$



Here for $0 < a \leq b$, we have

$$E(a,b) := \{ \underset{\mathbb{C}^2}{(z_1, z_2)} \mid \pi \left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \right) \leq 1 \}$$

So now assume $(\dagger)$.

Step 3: Next we use standard convexity theory..

Lem: Diameter, total mean curvature, area & volume are **all** monotonic under inclusion of convex bodies

For area & total mean curvature, follows from theory of mixed volumes (both $\nwarrow$ $\nearrow$ are mixed volumes).

Thus, by John's thm,

- $\mathrm{vol}(X) \sim \mathrm{vol}(E)$
- $\int_Y H \, d\mathrm{vol}_Y \sim \int_{\partial E} H \, d\mathrm{vol}_{\partial E}$
- $\mathrm{area}(Y) \sim \mathrm{area}(E)$
- $\mathrm{diam}(Y) \sim \mathrm{diam}(E)$

<u>and</u> $\cdot\ \mathrm{sys}(Y) \sim \mathrm{sys}(\partial E)$ (follows from min orbit period = 1st EHZ capacity in convex case.)

<u>Step 4</u>: Finally, directly calculate that

$$\boxed{\begin{array}{ll} (*) \ \cdot\ \mathrm{area}(\partial E) \gtrsim a^{3/2} & \cdot\ \mathrm{diam}(E) \sim b^{1/2} \\ \cdot\ \mathrm{vol}(E) \sim ab & \cdot\ \int_{\partial E} H \, d\mathrm{vol}_{\partial E} \lesssim b \\ \multicolumn{2}{c}{\cdot\ \mathrm{sys}(E) \sim \frac{a}{b}} \end{array}}$$

- $\Rightarrow \dfrac{\overbrace{\mathrm{area}(\partial E)^2}}{\mathrm{diam}(E)^2 \cdot \int_{\partial E} H \, d\mathrm{vol}_{\partial E}} \gtrsim \boxed{\dfrac{a^3}{b \cdot b}}$ ← SAME!!
- $\Rightarrow \boxed{\mathrm{vol}(E)^{1/2}} \cdot \boxed{\mathrm{sys}(E)^{5/2}} \sim \boxed{\left(\frac{a}{b}\right)^{5/2} \cdot (ab)^{1/2}}$

By Step 3, we're done! $\square$

<u>Rmk</u>: Full Prop 1 handles special case of $\mathrm{sys}(Y)$ very small by replacing $\mathrm{diam}(Y)$ with smaller quantity in Step 1.

§4 – Proposition 2: Uses a variant of a construction by Abbondandolo–Branham–Hryniewicz–Salamao (ABHS).

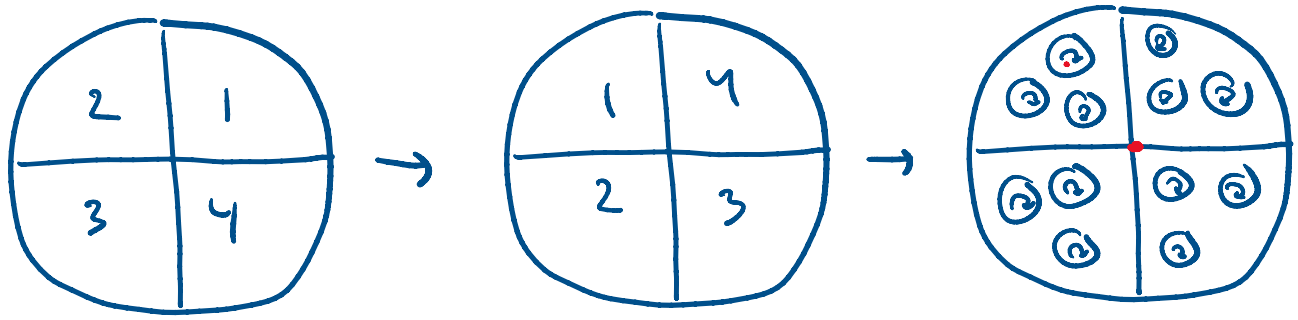
Step 1: By open book correspondence we have

Hamiltonian Map $\phi : \mathbb{D} \to \mathbb{D}$	contact form α on S^3
periodic points p	closed orbits γ
action $A(p) = \sum_{k=1}^{\ell} \sigma(\phi^k(p))$	period $A(\gamma)$ of γ.
$CZ(p)$ + period of p	$CZ(\gamma)$
Calabi invariant $Cal(\phi)$	volume $Vol(S^3, \alpha)$
$Ru(\mathbb{D}, \phi) + \pi$	$Ru(S^3, \alpha)$

Step 2: Make map $\phi : \mathbb{D} \to \mathbb{D}$ that

- rotates counter-clockwise by $2\pi(1 + \frac{1}{\pi})$ angle
- then rotates a bunch of little disks in $\mathbb{D}$ by a little less than 4π angle clockwise

a little less than 4π angle clockwise



Can show that

$$\cdot \quad Cal(\mathbb{D}, \phi) \sim \pi^2 - 2 \overbrace{\sum_{D} \text{area}(D^c)^2}^{\sum \text{ sum over all small disks}}$$

$$\cdot \quad Ru(\mathbb{D}, \phi) + \pi \sim 2\pi - 2 \sum_{D} \text{area}(D) \Big\} \text{ small}$$

By taking disks D to be small & fill $\mathbb{D}$, we get

$$\cdot \quad Ru(\mathbb{D}, \phi) + \pi \sim \text{small}$$

$$\cdot \quad Cal(\mathbb{D}, \phi) \sim 1$$

Can also show $sys(Y) \sim 1$, & Y is dynamically convex

Lots of future work... for discussion.

Thanks!

Proof of Step 1

- Consider vector-field iv on Y. Can calculate

$$\underbrace{|\nabla_{iv}(iv)|}_{\text{"acceleration of } iv\text{"}}{}^2 \leq 3 \cdot S(Iv, Iv) \cdot H$$

$$\Rightarrow \quad \left(\int_Y |\nabla_{iv}(iv)| \, dvol_Y\right)^2 \leq 3 \int_Y S(Iv, Iv) \cdot \int_Y H$$

Replace $|\nabla_{iv}(iv)|$ with average over trajectory...

$$A_T(y) = \frac{1}{T}\int_0^T |\nabla_{iv}(iv)|\circ\gamma(t)\, dt = \int_0^T |\ddot{\gamma}| \, dt$$

where $\gamma : [0,T] \to Y$ satisfies $\gamma(0) = y$ & $\dot{\gamma} = iv$. <u>Any</u> unit parametrized trajectory on Y

$$\frac{1}{T}\int_0^T |\ddot{\gamma}| \, dt \geq \frac{c}{\operatorname{diam}(Y)} \quad \Big\} \text{ must accelerate enough to stay within } Y$$

$$\Rightarrow \left(\frac{c}{\operatorname{diam}(Y)} \cdot \operatorname{area}(Y)\right)^2 \leq \left(\int_Y A_T\right)^2 = \left(\int |\nabla_{iv}(iv)|\right)^2 \leq \int_Y H \cdot \int S(iv, iv)$$

$\square$