3d Convex Contact Forms
\& The Ruelle Invariant

joint w/ Oliver Edtmair (Berkeley).


\section{Introduction}
Recall one of most standard examples of contact manifolds w/ contact form.

\textbf{Def:} A \underline{star-shaped} \( X \subset \mathbb{R}^n \) is a \underline{domain} with smooth boundary \( Y \) s.t.

\( Y \) is transverse to \( Z \)

where \( Z(p) = \frac{1}{p} \), i.e. \( Z \) is radial Liouville vectorfield.

"Star-shaped" contact forms on \( Y = S^{2n-1} \) admit an intrinsic characterization.

\[ (Y, \lambda|_Y) \cup Y = \partial X \]

is equivalent to a contact form \( \alpha \) on \( (S^{2n-1}, \text{std}) \) where (t)
\[ \omega_{\text{std}} \mapsto \] on \( (\mathbb{S}^{n-1}, \mathbb{S}^{std}) \) where \( \mathbb{S}^{std} := k \mathcal{C}(\lambda |_{\mathbb{S}^2}) \)

\[ \mathbb{S}^{2n-1} - \text{ star-shaped} \]

\[ \text{is equivalent to} \]

\section{2.1 - Convexity:}
There is a special subclass of star-shaped domains: convex ones.

\textbf{Def.} A contact form \( \omega \) on \( (\mathbb{S}^{2n-1}, \mathbb{S}^{std}) \) is \textit{convex} if there exists a convex star-shaped domain \( X \) such that

\[ (\mathbb{S}^{2n-1}, \omega) \overset{\text{strict}}{\rightarrow} (Y, \lambda_Y) \text{ where } Y = \partial X \]

Convex contact forms have many special properties, e.g.

\begin{itemize}
  \item Every convex contact form on \( S^3 \) has a disk-like surface of section for Reeb flow (Hofbauer, Wysocki, Zehnder 98)
  \item There exists a constant \( C > 0 \) such that
    \[ \text{sys}(S^{2n-1}, \omega) \leq C \text{ where } \text{sys}(-) = \frac{\text{min period of closed Reeb orbit}}{(n-1)! \cdot \text{vol}(-)} \]
    For all convex \( \omega \). (Vitushko conjectures that \( C = 1 \))
\end{itemize}

\textbf{Question:} Give intrinsic definition of convexity, with no reference to embedding \( S^{2n-1} \rightarrow \mathbb{C}^n \).

\section{1.2 - Dynamical Convexity:}
Here is a candidate answer...
Def: A contact form $\alpha$ on $S^3$ is **dynamically convex** if every closed Reeb orbit $\gamma$ of $\alpha$ satisfies $C\mathbb{Z}(\gamma) \geq 3$.

This was introduced by HW2 in 1998. We have

$\alpha$ strictly convex $\Rightarrow \alpha$ dynamically convex

Furthermore, properties 1 and 2 above are true for dynamically convex. So a natural question is...

**Question:** Does...

$\alpha$ dynamically convex $\Rightarrow \alpha$ (strictly) convex?

This question has been open since HW298.

§1.3 - Main Result: Today, we prove that

**Thm (C-Edtmaier):** There exist dynamically convex contact forms on $S^3$ that are not convex.

The proof consists of two separate propositions.

The first proposition uses the **Ruelle invariant**

$\text{Ru}(S^3, \tau)$ of $(S^3, \tau)$

Intuitively, $\text{Ru}$ is space-time average of "rotation" of Reeb Flow around trajectory. More later...
Prop 1: There exists a $c > 0$ such that
\[
(1) \quad c < \frac{R_u(S^3, \alpha)}{\operatorname{vol}(S^3, \alpha)^{1/2}} \quad \text{scale invariant}
\]
For all convex contact forms $\alpha$ on $S^3$.

Rmk: (i) RHS of (1) is scale invariant (ii) Prop in paper is slightly stronger.

To prove the main Thm, we simply show that

Prop 2: For all $\epsilon > 0$, there exists a dynamically convex contact form $\gamma$ on $S^3$ s.t.
\[
R_u(S^3, \gamma) < \epsilon \quad \text{and} \quad \operatorname{vol}(S^3, \gamma) = 1
\]

Rest of Talk:
- Detailed discussion of $R_u$.
- Detailed proof of Prop 1.
- Short discussion of Prop 2.

§ 2 - Ruelle Invariant: Here is detailed construction of Ruelle invariant. Let
(\textcircled{1}) be closed contact 3-manifold with \(H^i(Y,\mathbb{R}) = 0\) and \(c_1(\xi) = 0\) e.g. \((S^3, S^{5,1}).\)

\(\alpha\) be a contact form on \((Y, \xi).\)

\(\tau: \xi \cong Y \times \mathbb{R}^2\) be a symplectic trivialization of contact structure.

\(q: \text{Sp}(2) \to \mathbb{R}\) be a quasimorphism of universal cover of symplectic group \(\text{Sp}(2)\) with

\[q(t \mapsto e^{2\pi i \kappa t}) = \kappa\]

From this data, can define \(Ru(Y, \alpha)\) as so.

**Step 1**: Start with Reeb Flow \(\phi: \mathbb{R} \times Y \to Y.\)

**Step 2**: Differential \(d\phi\) combined with \(\tau\) yields the linearized Reeb flow...

\(\overline{\tau}: \mathbb{R} \times Y \to \text{Sp}(2)\)

given by \(\overline{\tau}(t, y) := \tau \phi_y(t, y) \circ d\phi_t \circ \tau^{-1}\)

**Step 3**: Lift \(\overline{\tau}\) uniquely to map

\(\overline{\overline{\tau}}: \mathbb{R} \times Y \to \widetilde{\text{Sp}}(2)\) with \(\overline{\overline{\tau}}(0, y) = \overline{1}\)

**Def:** (Ruelle 1985) The Ruelle invariant is

\[\rho_Y(\alpha) := \lim_{t \to \infty} \int_{\partial Y} q \overline{\overline{\tau}}(t, -) \quad \text{mod} (2)\]
\[
R_u(Y, t) := \lim_{T \to \infty} \sum_Y \int_{(T, -T)}^{T} \text{and} \quad (2)
\]

**Prop.** Limit (2) is well-defined and independent of choice of \(q\) and \(T\) under hypotheses.

A key alternate formula for \(R_u\) is following.

- Flow \(\Xi_t\) on \(Y \times \mathbb{R}^2\) projectivize \(\text{flow \(\Xi_t\) on \(Y \times S^1\) generated by vector-field \(\overline{R}\)}\).

- The derivative \(\overline{R}(0) = \frac{d}{dt}(\Theta(0_t))\) of the angle coordinate measures speed of rotation at \((y, s)\)

\[
Y \quad s \quad \text{speed of this} = \overline{R}(0)
\]

Lemma 1: \(R_u(Y, t) = \lim_{T \to \infty} \frac{1}{T} \int_{Y} \int_{0}^{T} \overline{R}(0) \cdot \overline{\Xi}(s, y) \, ds \, dy\) (3)

\(\text{2.2: Curvature Formula: It turns out that } \overline{R}(0) \text{ admits a beautiful curvature formula if } (Y, \overline{\Xi}) \text{ is convex body.}

(Rmk: Due to Nagata-Salamon and Hryniewicz)
Def: Let $Y \subset \mathbb{C}^2$ be a star-shaped boundary. The quaternionic trivialization is given by composition

$$\xi \xrightarrow{\pi} \text{span}(jv, kv) \to \mathbb{R}^2$$

where

- $j$ and $k$ are quaternions $j$ and $k$ acting on $\mathbb{C}^2$
- $v$ is normal vector field to $Y$
- $\pi$ is projection along Reeb direction
- $\text{span}(jv, kv) \to \mathbb{R}^2$ sends $jv$ to $(1)$ and $kv$ to $(i)$

Lem 2. (Curvature Formula) In the quaternionic trivialization, we have

$$\overline{R}(\xi) \cdot (1, i) = \frac{1}{2\pi \langle z, j \rangle} \cdot \left( S(iv, iv) + S(c \cdot jv, c \cdot jv) \right)$$

Cor 3: If $Y$ is convex boundary, then

$$\text{Ric}(\xi, x) \geq \frac{1}{2\pi} \int_Y S(iv, iv) \, dv$$

Key Point: In the theme park of math

rotation = curvature
bridge -> Lemma 2
§ 3 - Proposition 1: Back to proof of...

Prop 1: (Simple Version): There exist $C > 0$ such that

$$\text{(1)} \quad C \cdot \text{sys}(S^3, \alpha)^{\frac{5}{2}} \cdot \text{vol}(S^3)^{\frac{1}{2}} \leq \text{Ru}(S^3, \alpha)$$

For all convex contact forms $\alpha$ on $S^3$.

Let $X$ be convex, star-shaped $\forall Y \in 2X$. Suffices to bound

$$S \leq \text{civ}(iv, iv) \text{ duly}$$

Step 1: First, show that

$$\text{(1)} \quad S_y \leq \text{civ}(iv, iv) \text{ duly} \leq \frac{\text{area}(Y)^{\frac{1}{2}}}{3 \cdot \text{diam}(Y)^{\frac{1}{2}} \cdot S_y \text{ H duly}}$$

where diam is diameter $\& H$ is mean curvature.

Step 2: Next we apply...
Thm: (John’s Ellipsoid) After applying affine symplectomorphism to $X$, there is an ellipsoid $E = E(a,b)$ such that

$E \subseteq X \subseteq 4E$

Here for $0 < a \leq b$, we have

$$E(a,b) := \{ (z_1, z_2) | \pi \left( \frac{121z_1^2}{a} + \frac{121z_2^2}{b} \right) \leq 1 \}$$

So now assume (t).

Step 3: Next we use standard convexity theory.

Len: Diameter, total mean curvature, area & volume are all monotonic under inclusion of convex bodies.

For area & total mean curvature, follows from theory of mixed volumes (both & are mixed volumes).
Thus, by John’s thy,

- \( \text{vol}(X) \sim \text{vol}(E) \)
- \( S_t \text{dul}_4 \sim S_{2t} \text{dul}_{26} \)
- \( \text{area}(Y) \sim \text{area}(E) \)
- \( \text{diam}(Y) \sim \text{diam}(E) \)

and \( \text{sys}(Y) \sim \text{sys}(OE) \) (follows from min orbit period)

\( = \) 1st EHE capacity in convex case.

**Step 4:** Finally, directly calculate that

\[*\]

- \( \text{area}(OE) \geq a^{3/2} \)
- \( \text{diam}(E) = b^{1/2} \)
- \( \text{vol}(E) \sim a \cdot b \)
- \( \text{sys}(E) \sim \frac{a}{b} \)

\[\Rightarrow \]

\[\frac{\text{area}(OE)^2}{\text{diam}(E)^3 \cdot S_{2t} \text{dul}_{26}} \geq \frac{\frac{3}{a \cdot b}}{b \cdot b} \leftarrow \text{SAME!!} \]

\[\Rightarrow \]

\[\sqrt{\text{vol}(E)} \cdot \sqrt[5/2]{\text{sys}(E)} \sim \left(\frac{a}{b}\right)^{5/2} \cdot (ab)^{1/2} \]

By Step 3, we’re done!

**Remark:** Full Prop 2 handles special case of \( \text{sys}(Y) \) very small by replacing \( \text{diam}(Y) \) with smaller quantity in Step 1.
§4 – Proposition 2: Uses, variant of

A construction by Abbondandolo-Branham-Hryniewicz-
Salamno (ABHS).

**Step 1:** By open book correspondence we have

<table>
<thead>
<tr>
<th>Hamiltonian Map $\phi: \mathcal{D} \to \mathcal{D}$</th>
<th>Contact from $y$ on $S^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>periodic points $p$</td>
<td>closed orbits $\sigma$</td>
</tr>
<tr>
<td>action $A(p) = \sum_{k=1}^{n} \sigma_k^{(1)}(p)$</td>
<td>period $A(\sigma)$ of $\sigma$,</td>
</tr>
<tr>
<td>$C\tau(p) + \text{period of } p$</td>
<td>$C\tau(\sigma)$</td>
</tr>
<tr>
<td>Calabi invariant $\text{Cal}(\psi)$</td>
<td>volume $\text{Vol}(S^3, \mu)$</td>
</tr>
<tr>
<td>$Ru(D_3\psi) + \pi$</td>
<td>$Ru(S^3, \mu)$</td>
</tr>
</tbody>
</table>

**Step 2:** Make map $\phi: \mathcal{D} \to \mathcal{D}$ that

- rotates counter-clockwise by $2\pi(1 + \frac{1}{n})$ angle
- then rotates a bunch of little disks in $\mathcal{D}$ by a little less than $4\pi$ angle clockwise
a little less than $4\pi$ angle clockwise

Can show that

\[ \sum_{D} \text{area}(D) \]
\[ \cdot \ \Cal(D, \phi) = \pi^2 - 2 \sum_{D} \text{area}(D)^2 \]
\[ \cdot \Ru(D, \phi) + \pi n  = 2\pi - 2 \sum_{D} \text{area}(D)^2 \]

By taking disks $D$ to be small if all $D$, we get

\[ \cdot \Ru(D, \phi) + \pi n \text{ small} \]
\[ \cdot \Cal(D, \phi) = 2 \]

Can also show $\text{sys}(Y) = 2$. $\phi$, $Y$ is dynamically convex

Lots of future work... for discussion.

Thanks!
Proof of Step 1

Consider vector-field \( iv \) on \( Y \). Can calculate

\[
| \nabla_{iv}(iv) |^2 \leq 3 \cdot S(iv, iv) \cdot H
\]

"acceleration of iv" 

\[
\Rightarrow \left( \int_Y | \nabla_{iv}(iv) | \text{d}u \text{d}y \right)^2 \leq 3 \int_Y S(iv, iv) \cdot S_{H}
\]

Replan \( | \nabla_{iv}(iv) | \) with average over trajectory...

\[
A_T(y) = \frac{1}{T} \int_0^T | \nabla_{iv}(iv) | \cdot Y(t) \text{d}t = \int_0^T \bar{v}_i \text{d}t
\]

where \( Y : [0, T] \rightarrow Y \) satisfies \( Y(0) = y \) \& \( \bar{v} = iv \). Any unit parametrized trajectory on \( Y \)

\[
\frac{1}{T} \int_0^T | \bar{v}_i | \text{d}t \geq \frac{c}{\text{diam}(Y)} \Rightarrow \text{must accelerate enough}
\]

\[
\Rightarrow \left( \frac{c}{\text{diam}(Y) \cdot \text{area}(T)} \right)^2 \leq \left( \int_Y A_T \right)^2 = \left( \int_Y | \nabla_{iv}(iv) | \right)^2 \leq S_{H} \cdot S(iv, iv)
\]