Big Fiber Theorems and Ideal-Valued Measures in Symplectic Topology

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Big Fiber Theorems

Big Fiber Theorems

Various fields of mathematics exhibit big fiber theorems:

(Template) Theorem:

For any map $f\colon X \to Y$ in a suitable class, there exists $y_0 \in Y$, such that the fiber $f^{-1}(y_0)$ is "big".

Example Theorems:

- Topological Centerpoint Theorem (Rado ... Karasev);
- Maximal fiber theorem for maps of the torus (Gromov);
- Non-displaceable fiber theorem in symplectic topology (Entov-Polterovich).

Goal:

Gromov's ideal-valued measures	+	Varolgunes' relative symplectic homology	$\Longrightarrow$	put all three theorems on equal footing.

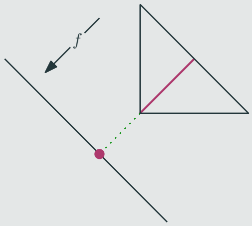
- Y - metric space of covering dimesnion d.
- p - a positive integer.

Topological centerpoint theorem, Karasev (2014)

Let $n = p(d+1)$ and let Δ^n be the n-simplex. Then for any continuous map $f\colon \Delta^n \to Y$, there exists a point $y_0 \in Y$, such that $f^{-1}(y_0)$ intersects **all** pd-dimensional faces of Δ^n.

For affine maps Rado (1946).

Example ($d = p = 1$, $n = 2$)



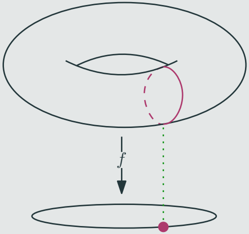
- Y - metric space of covering dimesnion d.
- p - a positive integer.

Torus Theorem, Gromov (2010)

Let $n \geq p(d+1)$. For every continuous map $f\colon \mathbb{T}^n \to Y$, there exists a point $y_0 \in Y$, such that $\operatorname{rank}\left(\check{H}^*(\mathbb{T}^n) \to \check{H}^*\left(f^{-1}(y_0)\right)\right) \geq 2^p$

Example ($d = p = 1$, $n = 2$)

$f : S^1 \times S^1 \to S^1$, proj on the 1st factor.
$f^{-1}(y_0) = S^1$.
$\operatorname{im}\left(H^*(\mathbb{T}^2) \to H^*\left(f^{-1}(y_0)\right)\right) = \langle 1, [dy] \rangle$
$\operatorname{rank}\left(H^*(\mathbb{T}^2) \to H^*\left(f^{-1}(y_0)\right)\right) = 2 = 2^1$.



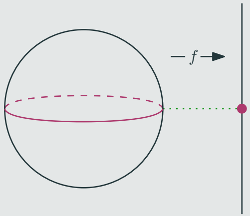
- (M^{2n}, ω) - a closed symplectic manifold.
- $\mathbf{f} = (f_1, \ldots, f_N) \colon M \to \mathbb{R}^N$, such that $\{f_i, f_j\} = 0, \ \forall i, j$.

Non-displaceable Fiber Theorem, Entov-Polterovich (2006)

There exists $p \in R^N$ such that $\mathbf{f}^{-1}(p)$ is non-displaceable.

Example ($n = 1$, $N = 1$)

$\mathbf{f} = (f) \colon S^2 \to \mathbb{R}$
Height function.



Ideal Valued Measures

Ideal Valued Measures

- $(A, *)$ - a graded skew-commutative associative unital algebra.
- $\dim A < \infty$.
- X - a compact Hausdorff topological space.

Think: $A = \check{H}^*(x)$.

Definition (Ideal Valued Measure (Gromov))

An *A-ideal valued measure, (A-IVM)* is an assignment

$$U \subset X \text{ open} \mapsto \mu(U) \subset A \text{ graded ideal, such that:}$$

1. **(Normalization):** $\mu(\emptyset) = 0, \quad \mu(X) = A$.
2. **(Monotonicity):** $U \subset U' \implies \mu(U) \subset \mu(U')$.
3. **(Continuity):** If $U_1 \subset U_2 \subset \ldots$ & $U = \bigcup_i U_i$, then $\mu(U) = \bigcup_i \mu(U_i)$.
4. **(Additivity):** $\mu(U \cup U') = \mu(U) + \mu(U')$ for disjoint U, U'.
5. **(Multiplicativity):** $\mu(U) * \mu(U') \subset \mu(U \cap U')$.
6. **(Intersection):** If U, U' cover X, then $\mu(U \cap U') = \mu(U) \cap \mu(U')$.

Think: $\mu(U) = \ker\left(\check{H}^*(X) \to \check{H}^*(X \setminus U)\right)$.

Ideal Valued Measures - Examples

IVM Examples:

1. **Čech cohomology IVM** - $\mu(U) = \ker\left(\check{H}^*(X) \to \check{H}^*(X \setminus U)\right)$.

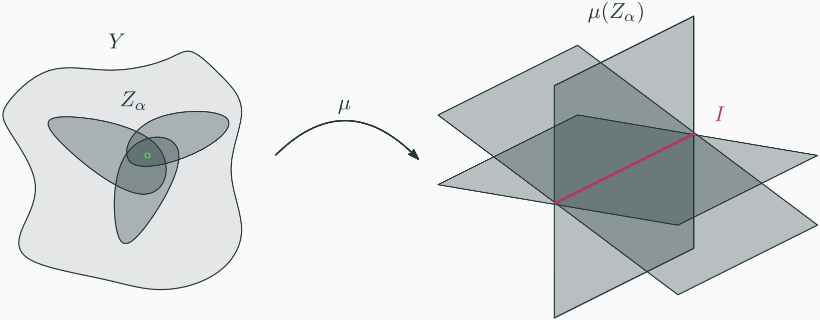
2. **Pushforward IVMs** - Given an IVM μ on X, and $f : X \to Y$ cont.
   Obtain an IVM on Y: $f_*\mu(U) := \mu\left(f^{-1}(U)\right)$ for all $U \subset Y$ open.

Ideal Valued Measures - Abstract Centerpoint Theorem (Karasev)

Theorem (Variation on Karasev, 2014)

Let Y be a compact metric space of covering dim. d.
$(A, *)$ - an algebra. I - an ideal s.t. $I^{*(d+1)} \neq 0$. μ - an IVM on Y. Then:

$$\bigcap \left\{ \underset{cpct}{Z} \subset Y \,\middle|\, I \subset \mu(Z) \right\} \neq \emptyset$$



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Sketch.

Otherwise $(Z^c)_{I \in \mu(Z)}$ is an open cover.
By Palais lemma, and the covering dim. $\exists$ a refinement $\left(V_{ij}\right)_{ij}$, such that $i = 0, \ldots, d$, and $\forall i$, the sets $\left(V_{ij}\right)_j$ are pairwise disjoint.
Put $K_i = \cap_j V_{ij}^c$.
(Multiplicativity) $\implies \prod_{i=0}^{d} \mu(K_i) \subset \mu(\cap_{i=0}^{d} K_i) = \mu(\emptyset) = 0$.

So enough to show: $I \subset \mu(K_i)$, $\implies 0 \neq I^{d+1} \subset 0$.
Follows by (Intersection) and (Monotonicity).
Contradiction! □

Ideal Valued Measures - Topological Centerpoint Theorem - I

Corollary (Variation on Karasev, 2014)

Y a d-dim space, $I^{d+1} \neq 0$ as before.
But now, X – any compact Hausdorff space. μ - IVM on X.
Any continous map $f \colon X \to Y$ has a fiber intersecting every compact $Z \subset X$ with $I \subset \mu(Z)$.

Follows by applying the abstract centerpoint theorem to the pushforward IVM: $f_*\mu$.

Ideal Valued Measures - Topological Centerpoint Theorem - II

- Y - metric space of covering dimesnion d.
- p - a positive integer.

Topological centerpoint theorem, Karasev (2014)

Let $n = p(d+1)$ and let Δ^n be the n-simplex. Then any continuous $f\colon \Delta^n \to Y$, has a fiber $f^{-1}(y_0)$ intersecting **all** pd-dim faces of Δ^n.

Proof (Sketch).

We need an algebra A and an IVM μ on Δ^n, s.t:

- $\exists$ ideal $I \subset A$ s.t. $I^{d+1} \neq 0$.
- For every pd-dimensional face σ, $I \subset \mu(\sigma)$.

Consider the moment map $\Phi\colon \mathbb{C}P^n \to \Delta^n$.
Preimage of face of Δ is a complex projective hyperspace of the same complex dim.

Take $\mu = \Phi_* \nu$, where is the cohomological IVM on $\mathbb{C}P^n$,
$I = \langle PD[\mathbb{C}P^{pd}] \rangle$.

□

Ideal Valued Measures - Gromov's Torus Theorem

The proof has a similar structure.

Two ingredients

- A suitable abstract centerpoint theorem:
  "There exists a point y_0 with $\operatorname{codim} \mu(Y \setminus y_0) \geq$ [something]".
- Pushforward IVMs.

Ideal Valued Quasi Measures

Goals:

- Adapt IVMs to the symplectic setting.
- Be able to apply centerpoint theorems.
- Explore symplectic rigidity through the ideal-valued lens.

Commuting Subsets and Involutive Maps

- (M, ω) - a closed symplectic manifold.

Definition

A map $f = (f_1, f_2, \ldots, f_k) : M \to \mathbb{R}^k,$ where $\{f_i, f_j\} = 0$ for all i, j is called *involutive*.

Definition

More generally a smooth map $f\colon M \to B$ is called *involutive* if

$$\{f^*F, f^*G\} = 0$$

for all $F, G \in C^\infty(B)$.

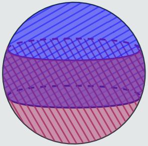
Remark: One can always embed B into $\mathbb{R}^N$ and use the first definition.

Commuting Subsets and Involutive Maps

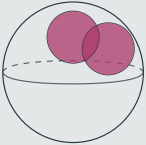
Definition

Say that compact $K, K' \subset M$ *commute* if there exist Poisson commuting $f, g \in C^\infty(M)$ with $K = f^{-1}(0)$, $K' = g^{-1}(0)$.
We say that open sets commute if their complements commute.

Example



Non intersecting boundaries.
Commuting.



Boundaries intersect.
Doesn't commute.
(True in $\dim 2$).

Ideal Valued Quasi Measures

- (M, ω) - a closed symplectic manifold.

Definition (Ideal Valued Quasi Measure)

An *A-ideal valued quasi measure, (A-IVQM)* is the same as an A-IVM, **except for multiplicativity**, which is replaced by the weaker:

1. **(Quasi-Multiplicativity):** $\tau(U) * \tau(U') \subset \tau(U \cap U')$,
   if U and U' commute.

Ideal Valued Quasi Measures

To adapt to the symplectic setting, we require two extra axioms:

2. **(Invariance):** $\tau(U) = \tau(\phi(U))$ for $\phi \in \mathsf{Symp}_0(M)$.
3. **(Vanishing):** If a compact K is (Hamiltonianly) displaceable, then there exists $U \supset K$ with $\tau(U) = 0$. Moreover $\tau(M \setminus K) = A$.

Ideal Valued Quasi Measures - Main Theorem

Theorem (Dickstein–G–Polterovich–Zapolsky)

Let (M, ω) be a closed symplectic manifold. Then there exists an A-IVQM on M, for some algebra A.

Upshot: Preimages of sets under involutive maps commute, hence:
IVQMs push to IVMs under involutive maps!

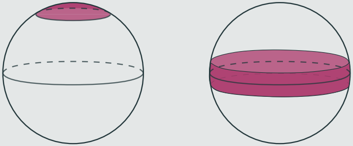
— Gain symplectic analogues to Karasev's and Gromov's theorems.

Ideal Valued Quasi Measures - Example

Example (In dim 2: take $M = S^2$ of area$= 1$)

- Enough to define IVQM on 2-dim closed connected submanifolds with boundary Q.
-

$$\tau(Q) = \begin{cases} 0 & Q \text{ is contained in a smooth closed disc of area} < 1/2 \\ A & \text{else} \end{cases}$$



$\mu(Q) = 0.$ $\mu(Q) = A.$

Ideal Valued Quasi Measures - General Construction - I

Relative symplectic cohomology (Varolgunes):

- A homology $SH(K)$ for every compact $K \subset M$.
- Also, a ring. (Varolgunes-Tonkonog).
- Restriction maps $SH(K) \to SH(K')$.
- Mayer Vietoris sequence for commuting pairs.

$$SH(A \cup B) \longrightarrow SH(A) \oplus SH(B) \longrightarrow SH(A \cap B) \xrightarrow{+1} SH(A \cup B)$$

- Vanishes for displaceable sets.

Ideal Valued Quasi Measures - General Construction - II

Our IVQM

For U open: $\quad \tau(U) = \ker\left(SH^*(M) \to SH^*(M \setminus U)\right).$

Remarks:

- Can either discuss IV(Q)Ms on compacts or on open sets.
- To achieve continuity one has to alter this definition a bit.
- Quasi-multiplicativty is nontrivial and requires new ideas.

Big Fiber Theorems, Revisited

IVQMs - Non Displaceable Fiber - Analog to Gromov's Thm

A quantitative version of Entov-Polterovich non displaceable fiber:

Theorem (Dickstein–G–Polterovich–Zapolsky)

Every involutive map $f\colon M \to B$ has a fiber $f^{-1}(b_0)$ with $\operatorname{codim} \tau(M \setminus f^{-1}(b_0))$ at least 1.

— Displaceabliliy implies $\operatorname{codim} \tau \left(M \setminus f^{-1}(b_0)\right) = 0$.

— Gromov gives lower bounds for $\operatorname{codim} \tau(M \setminus f^{-1}(b_0))$.

IVQMs - Symplectic Centerpoint - Analog to Karasev's Thm

Given:

- I - graded ideal, $I^{d+1} \neq 0$.
- B of covering dim d.

Theorem (Dickstein–G–Polterovich–Zapolsky)

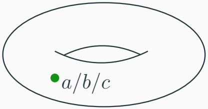
Every involutive map $f : M \to B$ has a fiber intersecting all members of the collection:

$$\left\{ \underset{cpct}{Z} \subset M \,\middle|\, I \subset \tau(Z) \right\}$$

— A source for a new kind of examples of symplectic rigidity.

IVQMs - Symplectic Centerpoint - Concrete Example

Take the torus $\mathbb{T}^6$, with coordinates $p_i, q_i \in \mathbb{T}^2$, $\omega = \sum dp \wedge dq$.
For every $a, b, c \in \mathbb{T}^2$ consider the following coisotropic subtori in $\mathbb{T}^6$:



$$T_1(a) = \{(\mathbf{p}, \mathbf{q}) \,|\, (q_1, q_2) = a\},$$
$$T_2(b) = \{(\mathbf{p}, \mathbf{q}) \,|\, (p_1, p_3) = b\},$$
$$T_3(c) = \{(\mathbf{p}, \mathbf{q}) \,|\, (p_2, q_3) = c\}.$$

Set $T(a, b, c) = T_1(a) \cup T_2(b) \cup T_3(c)$.

Theorem (Dickstein–G–Polterovich–Zapolsky)

Every involutive map $\mathbb{T}^6 \times S^2 \to Y^2$ has a fiber intersecting all sets of the form:

$$T(a, b, c) \times \textit{equator}.$$

An *equator* in S^2 is any loop dividing S^2 to two discs of equal area.

The involutivity is essential:

Project $\pi\colon \mathbb{T}^6 \times S^2 \to S^2$.
For $y_0 \in S^2$ and $L \subset S^2$ an equator not containing y_0,
the fiber $f^{-1}(y_0)$ disjoint from any $T(a,b,c) \times L$.

Rigidity

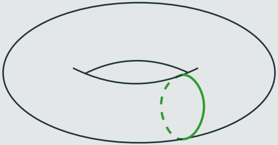
SH-Heaviness - Definition

- (M, ω) - a closed symplectic manifold.
- τ - The $SH(M)$-IVQM on M.

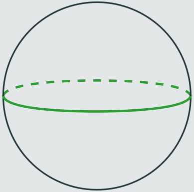
Definition

A compact $K \in M$ is *SH-heavy* if $\tau(K) \neq 0$.

Example



$\tau(K) =$ same as in cohomological IVM



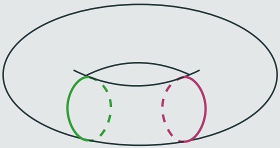
$\tau(K) =$ full measure

SH-Heaviness - Properties

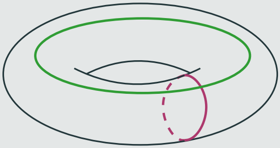
Properties

- SH-heavy sets are **Ham** non-displaceable.
- For K, K', if $\tau(K) * \tau(K') \neq 0$ then:
  — K, and K' are SH-heavy.
  — K is Symp_0 non-displaceable from K'.

Example



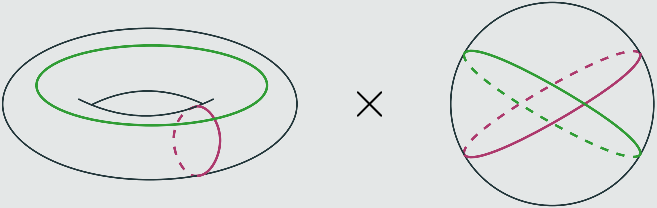
$$\tau(K) * \tau(K') = 0$$



$$\tau(K) * \tau(K') \neq 0$$

SH-Heaviness - Nontrivial Symplectic Example

Example



Two product tori in $\mathbb{T}^2 \times S^2$, in green and cherry,
Non displaceable from each other.

Note: They are smoothly displaceable.

SH-Heaviness - Proof

$\tau(K) * \tau(K') \neq 0 \implies K$ is non-displaceable from K':

Proof.

Assume $\phi \in \mathrm{Symp}_0$ displaces K from K': $\quad \phi(K) \cap K' = \emptyset$.
Then $\phi(K)$ and K commute.
(Quasi-Multiplicativity) $\implies$

$$\tau\big(\phi(K)\big) * \tau(K') \subset \tau\big(\phi(K) \cap K'\big) = \tau(\emptyset) = 0.$$

(Invariance) $\implies$ $\tau(\phi(K)) = \tau(K)$, hence
$0 \neq \tau(K) * \tau(K') \subset 0.$ Contradiciton! $\square$

Categorification of Heaviness?

- (M, ω) - a closed symplectic manifold.
- e - an idempotent in $QH(M)$.

Definition (Heavy sets (Entov and Polterovich))

$F \in C^\infty(M) \mapsto \zeta(F) := \lim_{k \to \infty} \frac{c(kF;e)}{k}$

Partial symplectic quasi state

$K \subset M$ is *heavy* if $\forall F \in C^\infty(M)$, one has $\zeta(F) \geq \inf_K F$.

Properties

- Heavy sets are non-displaceable.
- Heavy sets need not necessarily intersect! (e.g. two parallel meridians on $\mathbb{T}^2$).
- Unclear how to detect intersections, in contrast to SH-heavy sets.

Conjecture: Heavy $\implies$ SH-heavy.

Proven for a simple case:

index bounded incompressible domains in aspherical manifolds.

The other direction is more speculative.

Construction of IVQMs

Varolgunes' Relative Symplectic Cohomology

The Novikov field

The Novkiov ring

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i T^{\alpha_i} \,\middle|\, c_i \in \mathbb{Q},\ \alpha_i \in \mathbb{R},\ \alpha_i \nearrow \infty \right\}, \quad \Lambda_{\geq 0} = \left\{ \sum_{i=0}^{\infty} c_i T^{\alpha_i} \in \Lambda \,\middle|\, \alpha_i \geq 0 \right\}$$

- H - a Hamiltonian,
- $\mathcal{P}(H)$ - 1-periodic orbits. (graded by mod-2 CZ-index)

Floer Complex: $CF(H) := \bigoplus_{\gamma \in \mathcal{P}(H)} \Lambda_{\geq 0} \cdot \gamma.$

— Note: no cappings.

Floer Differential:

— Positive gradient flow of action functional (cohomology).

— Weighted by the topological energy of Floer solutions:

$$d\gamma_- = \sum_{\gamma_+ \in \mathcal{P}(H)} \sum_{B \in \pi_2(M, \gamma_-, \gamma_+)} \#\mathcal{M}(\gamma_-, \gamma_+) T^{\underset{\uparrow\ \text{"action difference"}}{\omega(B) + \int_{\gamma_+} H - \int_{\gamma_-} H}} \gamma_+$$

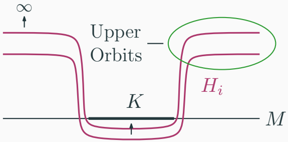
Continuation Maps:

— Also weighted by top. energy ("action difference").

— Defined over $\Lambda_{\geq 0}$ for $H_1 \leq H_2$ (going from low to high).

Varolgunes' Relative Symplectic Cohomology

- $K \subset M$ compact.
- H_i Hamiltonians adapted to K:



Symplectic Cohomological Complex: $SC(K) := \widehat{\varinjlim_{i \to \infty} CF(H_i)}$

— Completion: $\widehat{A} := \varprojlim_{r \to \infty} \left(A \otimes_{\Lambda_{\geq 0}} {}^{\Lambda_{\geq 0}}\!/\!_{\Lambda_{\geq r}} \right)$.

— Eliminates contributions of upper orbits, since, e.g.:

$$\begin{array}{ccccccccc}
\Lambda_{\geq 0} & \xrightarrow{T\cdot} & \Lambda_{\geq 0} & \xrightarrow{T\cdot} & \Lambda_{\geq 0} & \xrightarrow{T\cdot} & \dots & \longrightarrow & \varinjlim \\
\big\| \wr & & \big\| \wr & & \big\| \wr & & & & \big\| \wr \\
\Lambda_{\geq 0} & \hookrightarrow & \Lambda_{\geq -1} & \hookrightarrow & \Lambda_{\geq -2} & \hookrightarrow & \dots & \longrightarrow & \Lambda
\end{array}$$

Symplectic Cohomology:

$SH(K; \Lambda_{\geq 0}) := H^*(SC(K))$, – a $\Lambda_{\geq 0}$-Module.

$\boldsymbol{SH(K) := SH(K; \Lambda_{\geq 0}) \otimes_{\Lambda_{\geq 0}} \Lambda}$. – (eliminates torsion. "finite bars")

Relative Symplectic Cohomology - Properties

- **Restriction maps:** For $K' \subset K \subset M$ compacts, there are restriction maps $SH(K) \to SH(K')$.

- **Mayer-Vietoris (Varolgunes):** For $A, B \subset M$ compact commuting subsets, there exists an exact triangle:

$$SH(A \cup B) \longrightarrow SH(A) \oplus SH(B) \longrightarrow SH(A \cap B) \xrightarrow{+1} SH(A \cup B)$$

- **Product (Tonkonog-Varolgunes):** $SH(K) \otimes SH(K) \xrightarrow{*} SH(K)$, compatible with restriction, making $SH(K)$ a unital ring. Moreover, $SH(M) = QH(M)$.

Ideal Valued Quasi Measures - Construction

Recall our IVQM:

- For a compact K: $\tau(K) = \bigcap_{\substack{U \supset K \\ \text{open}}} \ker\left(SH^*(M) \to SH^*(M \setminus U)\right).$

Quasi-multiplicalivity is nontrivial:

We define SH of a pair $K' \subset K$:
$SH(K, K') = H^*\left(\text{cocone}\left(SH(K) \to SH(K')\right)\right)$

$$\text{cocone}(r\colon V \to W) := \left(V \oplus W[-1], \begin{pmatrix} d_V & 0 \\ r & d_W \end{pmatrix}\right).$$

cocone = "homotopy kernel".

We have an exact triangle: $SH(K, K') \to SH(K) \to SH(K') \xrightarrow{+1}$

Main ingredient: lift the product to pairs, for A, B commuting:

$$\begin{array}{ccc} SH(M,A) \otimes SH(M,B) & & SH(M, A \cup B) \\ \downarrow & & \downarrow \\ SH(M) \otimes SH(M) & \xrightarrow{\quad * \quad} & SH(M) \end{array}$$

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$$\begin{array}{ccc} SH(M, A) \otimes SH(M, B) & \xrightarrow{*} & SH(M, A \cup B) \\ \downarrow & & \downarrow \\ SH(M) \otimes SH(M) & \xrightarrow{*} & SH(M) \end{array}$$

SH of Pairs - Why The Sets Have to Commute in the Product?

Varolgunes' Mayer-Vietoris: $SH(A \cup B) \to SH(A) \oplus SH(B) \to SH(A \cap B) \xrightarrow{+1}$
Requires: A, B commuting.

Algebraic Topology: A, B satisfy M-V, $\iff$ (A, B) is an excisive-pair:
The natural chain map $C_*(A) + C_*(B) \to C_*(A \cup B)$ is an isomorphism in homology.

"commuting" is the symplectic analogue of "excisive-pair"

Similarly, Classically **relative cup product** exists for an excisive-pair:

$$H^*(M, A) \otimes H^*(M, B) \to H^*(M, A \cup B)$$

Expect: A, B should commute for:

$$SH(M, A) \otimes SH(M, B) \to SH(M, A \cup B)$$

Thank You!

Questions?

Ideal Valued Quasi Measures - General Construction - II

We define our IVQM τ by:

- For a compact K: $\tau(K) = \bigcap_{\substack{U \supset K \\ \text{open}}} \ker\left(SH^*(M) \to SH^*(M \setminus U)\right)$,
- For an open U: $\tau(U) = \bigcup_{\substack{K \subset U \\ \text{compact}}} \tau(K)$.

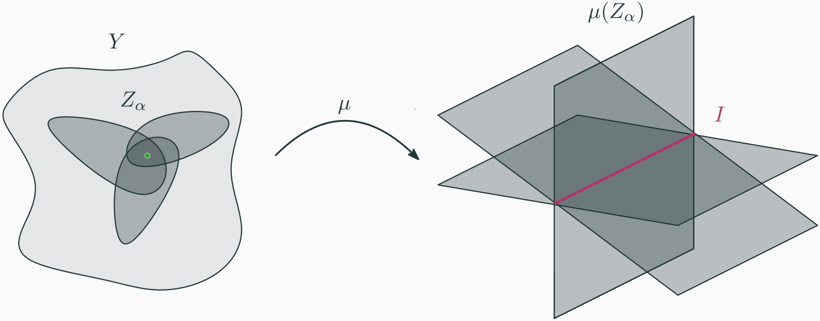
Remark: quasi-multiplicativty is nontrivial and requires new ideas.

Ideal Valued Measures - Abstract Centerpoint Theorem (Karasev)

Theorem ([Variation on Karasev, 2014)

Let Y be a compact metric space of covering dim. d.
$(A, *)$ - an algebra. I - an ideal s.t. $I^{*(d+1)} \neq 0$. μ - an IVM on Y. Then:

$$\bigcap \left\{ \underset{cpct}{Z} \subset Y \,\middle|\, I \subset \mu(Z) \right\} \neq \emptyset$$



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Proof.

Otherwise $(Z^c)_{I \in \mu(Z)}$ is an open cover.
By Palais lemma, and the covering dim. $\exists$ a refinement $\left(V_{ij}\right)_{ij}$, such that $i = 0, \ldots, d$, and $\forall i$, the sets $\left(V_{ij}\right)_j$ are pairwise disjoint.
(Monotonicity) $\implies \mu(V_{ij}^c) \supset \mu(Z_{ij}) \supset I$.
Note that $V_{ij}^c \cup V_{ij'}^c = Y$, for $j \neq j'$, and put $K_i = \cap_j V_{ij}^c$.
(Intersection) $\implies \mu(K_i) = \mu(\cap_j V_{ij}^c) = \cap_j \mu(V_{ij}^c) \supset I$.
(Product) $\implies 0 \neq I^{d+1} \subset \prod_{i=0}^{d} \mu(K_i) \subset \mu(\cap_{i=0}^{d} K_i) = \mu(\emptyset) = 0$.
Contradiction! $\square$

Ideal Valued Measures - Topological Centerpoint Theorem - I

Corollary (Variation on Karasev, 2014)

Let X be a compact Hausdorff space.
$(A, *)$ *- an algebra.* I *- an ideal s.t.* $I^{*(d+1)} \neq 0$. μ *- an IVM on* X.
Let Y *be a compact metric space of covering dim.* d.
Then any continous map $f: X \to Y$ *has a fiber intersecting every compact* $Z \subset X$ *with* $I \subset \mu(Z)$.

Proof.

Consider the pushforward IVM, $f_*\mu$, on Y, defined by
$f_*\mu(U) := \mu(f^{-1}(U))$ for $U \subset Y$ open.
If $Z \subset X$ is such that $I \subset \mu(Z)$ then $I \subset f_*\mu(f(Z))$, since:

$$I \subset \mu(Z) \subset \mu(f^{-1}(f(Z))) = f_*\mu(f(Z)).$$

By the abstract centerpoint theorem for $f_*\mu$, there exists

$$y_0 \in \bigcap_{I \subset f_*\mu(W)} W \neq \emptyset.$$

In particular, $\forall$ such Z, $y_0 \in f(Z)$, namely $f^{-1}(y_0) \cap Z \neq \emptyset$, as claimed. □

Ideal Valued Measures - Gromov's Torus Theorem

The proof has a similar structure.

Two ingredients

- A suitable abstract centerpoint theorem:
  "There exists a point y_0 with $\operatorname{codim} \mu(Y \setminus y_0) \geq$ [something]".
- Pushforward IVMs.