Holomorphic Floer theory and the Fueter equation

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Lagrangian Floer theory

Finite-dimensional model

A function $f: M \to \mathbb{R}$ gives rise to Morse homology $H(M, f)$.

Infinite-dimensional case

If (M, ω) symplectic manifold, $L_0, L_1 \subset M$ Lagrangians, then the Lagrangian Floer homology $H_*(L_0, L_1)$ is the Morse homology of the action functional:

$$\mathcal{A}: \mathcal{P}(M; L_0, L_1) \to \mathbb{R}$$

generators = intersection points of L_0 and L_1

differential = count of pseudo-holomorphic strips in M

Lagrangian Floer theory

Infinite-dimensional $\Longrightarrow$ Finite-dimensional

Let $f\colon M \to \mathbb{R}$ be a Morse function on a compact manifold. Let $L_0, L_\epsilon \subset T^*M$ be the zero section and graph of $\epsilon \mathrm{d} f$.

Theorem (Floer)

For every small ϵ there is a compatible almost complex structure I on T^*M such that I-holomorphic strips in T^*M correspond bijectively to gradient trajectories of ϵf in M.

$$\Longrightarrow \quad H(M, f) \cong H(L_0, L_\epsilon)$$

Complexification

cf. Kapranov–Kontsevich–Soibelman

Let (M, I, Ω) be a complex symplectic manifold or equivalently a hyperkahler manifold equipped with three complex structures I, J, K such that $IJ = -JI = K$ and symplectic forms $\omega_I, \omega_J, \omega_K$ so that

$$\Omega = \omega_J + i\omega_K.$$

Let $L_0, L_1 \subset M$ be I-complex and J- and K-Lagrangian.

- What is the finite-dimensional model?
- What is the infinite-dimensional application of that model?
- What is the analog of Floer's theorem?

Category associated with Lefschetz fibrations

Finite-dimensional version

Let $F\colon X \to \mathbb{C}$ be exact Lefschetz fibration, i.e. M has exact symplectic form $\omega = \mathrm{d}\lambda$ and almost complex structure I making F a holomorphic Morse function with prescribed behavior at infinity.

$\implies$ Fukaya–Seidel category: $\mathrm{FS}(X, F)$

Category associated with Lefschetz fibrations

Construction of $\mathrm{FS}(X, F)$ using Morse trajectories

Haydys, Gaiotto–Moore–Witten, Kapranov–Kontsevich–Soibelman

Generators: critical points of F

Morphisms: $\mathrm{Hom}(x, y)$ is generated by gradient trajectories of $f_\theta = \mathrm{Re}(e^{i\theta} F)$ from x to y

A_∞ operations: $\partial \colon \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, y)$ counts Floer planes

$$u \colon \mathbb{R}^2 \to X$$
$$\partial_s u - I(u)(\partial_t u - \nabla f_\theta(u)) = 0.$$

Higher operations count Floer planes with n asymptotic ends.

Category associated with complex Lagrangians

Infinite-dimensional version

(M, I, J, K) hyperkahler, L_0, L_1 complex Lagrangians

We apply the above construction to the complex action functional

$$\mathcal{A}_{\mathbb{C}} = \mathcal{A}_J + i\mathcal{A}_K : \mathcal{P}(M; L_0, L_1) \to \mathbb{C}$$

Generators: intersection points of L_0 and L_1

Morphisms: for $x, y \in L_0 \cap L_1$ generated by J_θ-antiholomorphic strips $[0,1] \times \mathbb{R} \to M$ with boundary on L_0 and L_1, where

$$J_\theta = \cos\theta J + \sin\theta K.$$

Category associated with complex Lagrangians

A_∞ operations count solutions to the Fueter equation
(cf. Hohloch–Noetzel–Salamon, Haydys, Walpuski)

$$U\colon [0,1]_\tau \times \mathbb{R}^2_{s,t} \to M$$
$$I(U)\partial_\tau U + J_\theta(U)\partial_s U + K_\theta(U)\partial_t U = 0.$$

Lagrangian boundary condition

$U(0,s,t) \in L_0$ and $U(1,s,t) \in L_1$.

Asymptotic as $s \to \pm\infty$

$U(\tau,s,t) \to u_\pm(\tau,t)$ as $s \to \pm\infty$ where $u_\pm\colon [0,1]\times\mathbb{R} \to M$ is a J_θ-antiholomorphic strip

Asymptotic as $t \to \pm\infty$

$U(\tau,s,t) \to p_\pm$ as $t \to \pm\infty$ where $p_\pm \in L_0 \cap L_1$

Category associated with complex Lagrangians

Summary of the proposal

Given I-complex Lagrangians L_0, L_1 in a hyperkähler manifold (M, I, J, K) define an A_∞ category $\mathrm{FS}(L_0, L_1)$.

- generators are intersection points of L_0 and L_1,
- morphism groups are generated by J_θ-antiholomorphic strips
- A_∞ operations count Fueter maps

It is not necessary that M is hyperkähler. The equations make sense for any triple of almost complex structures (I, J, K).

Category associated with complex Lagrangians

Basic questions

- Fredholm theory
- Transversality
- Compactness
- Examples & computations

I will briefly discuss compactness and then give an example.

Convexity and maximum principle

The Levi form of a function ρ with respect to I is

$$\sigma_\rho^I = -\mathrm{d}(\mathrm{d}\rho \circ I)$$

Recall that ρ is I-convex if $\sigma_\rho^I > 0$ on I-complex planes.
This gives us a maximum principle for I-holomorphic maps.

Definition

A function $\rho\colon M \to \mathbb{R}$ is *IJK-convex* if

$$U^*\sigma_\rho^I \wedge \mathrm{d}\tau + U^*\sigma_\rho^J \wedge \mathrm{d}s + U^*\sigma_\rho^K \wedge \mathrm{d}t < 0.$$

for every Fueter map $U\colon \mathbb{R}^3_{\tau,s,t} \to M$.

Convexity and maximum principle

Definition

(M, I, J, K) has conical end if there is an exhaustion function $\rho\colon M \to \mathbb{R}$ which, outside compact set, is IJK-convex and has no critical points.

An I-complex submanifold $L \subset M$ is conical if $\mathrm{d}\rho$ vanishes on $J(TL)$ and $K(TL)$ but not on TL, outside a compact set.

Convexity and maximum principle

Proposition (D.–Rezchikov)

If (M, I, J, K) has conical ends and $L_0, L_1 \subset M$ are conical, then there is a compact set in M which contains the image of every Fueter map $U\colon [0,1] \times \mathbb{R}^2 \to M$ with boundary on L_0, L_1 and asymptotic at infinity described before.

Energy and compactness

The energy of a Fueter map is

$$E(U) = \frac{1}{2}\int_{[0,1]\times\mathbb{R}^2} |\partial_s U|^2 + |\partial_t U - J(U)\partial_\tau U|^2$$

Suppose (M, I, J, K) is such that $\omega_I = \mathrm{d}\lambda_I$, and so on.

Proposition (D.–Rezchikov)

The energy of a Fueter map asymptotic to $u_\pm$ as $s \to \pm\infty$ is

$$E(U) = L(u_+) - L(u_-)$$

where for a map $u\colon [0,1]\times\mathbb{R} \to M$ we define

$$L(u) = \int_{[0,1]\times\mathbb{R}} u^*\lambda_I \wedge \mathrm{d}\tau$$

Energy and compactness

Given a C^0 bound and energy bound, there is an analog of Gromov's compactness theorem for Fueter maps, due to Walpuski.

Bubbling can now happen along a codimension one subset of the domain. In addition, there can be non-removable singularities.

If M is flat, singularities don't occur and the moduli space of Fueter maps is compact (Hohloch–Noetzel–Salamon).

Understanding singularities of Fueter maps is the main challenge of this proposal, which requires new analytic ideas. This is a very active area of research, related to recent developments in gauge theory (Taubes, Haydys, Walpuski, Zhang, Takahashi, Donaldson).

Taming forms and transversality

The requirement that $\omega_I, \omega_J, \omega_K$ are closed is restrictive.

Instead, we can consider (I, J, K) which are tamed by a triple closed forms $(\sigma_I, \sigma_J, \sigma_K)$ in the sense that for any Fueter map U

$$E(U) \leq c \int U^*\sigma_I \wedge \mathrm{d}\tau + U^*\sigma_K \wedge \mathrm{d}t.$$

(And similarly for the rotated equation.)

If $\sigma_I, \sigma_J, \sigma_K$ are exact, we get the same energy bound. In general, we get energy bound if we fix a relative homology class of U.

Being tamed is a pointwise open condition, so there is an infinite-dimensional space of tamed (I, J, K).

Cotangent bundles

What is the analog of Floer's theorem in this setting?

Let $F\colon X \to \mathbb{C}$ be an exact Lefschetz fibration. Let $L_0, L_\epsilon \subset T^*X$ be the zero section and graph of $\epsilon \mathrm{d}f$, where $f = \mathrm{Re}(F)$.

T^*X has an almost complex structure I induced from X and a canonical I-bilinear complex two-form. L_0, L_ϵ are I-complex.

Conjecture

For ϵ sufficiently small,

$$\mathrm{FS}(X, F) \cong \mathrm{FS}(L_0, L_\epsilon)$$

Cotangent bundles

Generators

Critical points of F correspond to intersections of L_0, L_ϵ.

Morphisms

There is an almost complex structure J on T^*X such that gradient trajectories of $f = \mathrm{Re}(F)$ correspond to J-antiholomorphic strips in X with boundary on L_0, L_ϵ. This is Floer's theorem.

A_∞ operations

Fueter maps in T^*X = Floer planes in X?

Cotangent bundles

Theorem (D.–Rezchikov)

For every small $\epsilon > 0$ there is a quaternionic triple (I, J, K) on T^*X such that Fueter maps $U\colon [0,1] \times \mathbb{R}^2 \to T^*X$ with boundary on L_0, L_ϵ correspond bijectively to Floer planes

$$u\colon \mathbb{R}^2 \to X,$$
$$\partial_s u - J(u)(\partial_t u - \epsilon \nabla f(u)) = 0.$$

Proof: Canonical quaternionic triple

Given a metric on X compatible with I, we have

$$T(T^*X) = TX \oplus TX$$

which gives as a natural quaternionic triple I, J, K on T^*X:

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Proof: Floer's correspondence

Let $\varphi_\tau \colon T^*X \to T^*X$ be the Hamiltonian flow of ϵf.
Given $u \colon \mathbb{R}^2 \to X$, define

$$U(\tau, s, t) = \varphi_\tau(u(s, t))$$

This maps Floer planes in X to Fueter maps in T^*X.

Lemma

If the image of U is contained in a sufficiently small neighborhood of the zero section, then U is of the above form.

Proof: Integration by parts

Proof of Lemma

Given U, define $\tilde{U} = \varphi_\tau^{-1} \circ U$ and split $\tilde{U}$ into base and fiber direction: $\tilde{U} = (u, \xi^*)$ where

$$u\colon [0,1] \times \mathbb{R}^2 \to X \quad \text{and} \quad \xi \in \Gamma(u^* TX)$$

The Fueter equation and integration by parts yield

$$0 \geq \|\nabla_\tau \xi\|_{L^2}^2 + \|\nabla^{0,1}\xi\|_{L^2}^2 - C(\epsilon + \|\xi\|_{C^0})(\|\nabla_\tau \xi\|_{L^2}\|\nabla^{0,1}\xi\|_{L^2})$$

which implies that $\xi = 0$ and $\nabla^{0,1}\xi = \partial_\tau u = 0$. □

Proof: Convex completion

Unfortunately, it's not obvious that the image of U must lie in a small neighborhood of the zero section. However, we have

Lemma

If $\varphi\colon X \to \mathbb{R}$ is an exhausting I-convex function, then

$$\rho\colon T^*X \to \mathbb{R}$$
$$\rho(x,\xi) = |\xi|^2 + \epsilon\varphi(x)$$

is IJK-convex in the region $\{\rho \leq c\}$ for some c, and L_0, L_ϵ have conical ends with respect to ρ.

Proof: Convex completion

A technical step in the proof is to complete the region $\{\rho \leq c\}$ to a conical manifold (M, I, J, K) and use convexity to constraint Fueter maps to stay in the original region $\{\rho \leq c\}$.

Playing with the I-convex function on X and ϵ we can arrange it so that $\{\rho \leq c\}$ lies within the tubular neighborhood of the zero section where the Lemma works. $\square$

Open problems

- Analysis of the Fueter equation, singularities
- Non-transverse intersections
  What is $\mathrm{FS}(X, F)$ for non-Morse F?
- What is the invariant associated with (M, I, J, K)?
  Categorification of the Fukaya category
- Functoriality; relation between $\mathrm{FS}(X, F)$ and $\mathrm{FS}(X, G)$
- Relation to perverse sheaves invariants
- Complex Chern–Simons theory (cf. Abouzaid–Manolescu)
- Path integrals and resurgence (cf. Kontsevich–Soibelman)

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