This talk reports on various joint works: with Shende, Kucharski, Longhi, Georgieva, Ng

Plan:

- Skein valued open GW-invariants
- Skein recursion for the toric brane and for knot conormals in the resolved conifold
- Some applications: quantum curves, basic holomorphic disks, and quivers.
Geometric setting:

- \((X, \omega)\) 3-dim symplectic Calabi-Yau, \(c_1(X) = 0\).
  Main examples: \(\mathbb{C}^3\), \(T^*S^3\), and \(\mathcal{O}(-1)^{\oplus 2} \to \mathbb{C}P^1\).

- \(L \subset X\) Maslov zero Lagrangian.
  Main examples: toric brane, knot conormals, 0-section.

Holomorphic curves:

- \(J\) acs on \(X\) compatible with \(\omega\). \((S, j)\) Riemann surface. A \textit{holomorphic curve} is a map \(u: (S, \partial S) \to (X, L)\) that solves the Cauchy-Riemann equation: \(\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0\).

- The Cauchy-Riemann equation is Fredholm and the expected dimension of the moduli space of solutions is

\[
(dim_{\mathbb{C}} X - 3)\chi(S) + 2c^\text{rel}_1(u^*TX) = 0 + 0.
\]
The dimension count indicates that after perturbation, the moduli space of solutions to the Cauchy-Riemann equation is an oriented zero-manifold. For closed curves nodal solutions appear in codim 2 and curve counts are invariant under deformation. For open curves boundary nodes have codimension one and curve counts are not invariant. There are invariant curve counts in this setting, in the skein.

For general curves we use the HOMFLY skein. For curves invariant under an involution that fixes the Lagrangian we use the Kauffmann skein.
Skeins on branes

\[ \mathbb{Q} \left[ a^\pm, z^\pm \right] \text{- modules} \quad \left( \text{or } \mathbb{Q} \left[ a^\pm, q^{\pm \frac{1}{2}} \right], \quad z = q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right) \]

**HOMFLY**
- oriented framed links

\[ \begin{align*}
\begin{array}{c}
\xrightarrow{\uparrow} - \xrightarrow{\uparrow} = z \bigcirc \\
\downarrow a - a^{-1} = z \bigcirc
\end{array}
\end{align*} \]

**Kauffman**
- unoriented framed links

\[ \begin{align*}
\begin{array}{c}
\begin{array}{c}
\xrightarrow{\uparrow} - \xrightarrow{\uparrow} = z \bigcirc \\
\downarrow a - a^{-1} = z \bigcirc
\end{array} \\
\bigcirc \bigcirc = a_1 \\
\bigcirc \bigcirc = a_1
\end{array}
\end{align*} \]

For example, \( \text{Sk}(S^3) = \mathbb{C}[q^{\pm 1}, a^{\pm 1}] \), \( \text{Sk}(S^1 \times \mathbb{R}^2) \) is a free commutative algebra on countably many generators \( A_m \) (\( m - 1 \) crossings, \( m \) times around).
**Bare curves:** A stable map $u : S \rightarrow X$ is *bare* if all its irreducible components have positive symplectic area.

Skein valued curve counts are based on counting bare holomorphic curves by their boundary in the framed skein.

**Auxiliary framing data:** Generic vector field $\xi$ on $L$ and 4-chain $C$ with $\partial C = 2 \cdot L$ and $\pm J \cdot \xi$ along the boundary.

$$\text{lk} (L, u) = u_{J\nu} \cdot C$$
The skein valued curve count is then a sum over all disconnected bare holomorphic curves where the contribution of 
\( u: (S, \partial S) \to (X, L) \) is

\[
w(u) \, z^{-\chi(S)} \, a^{\text{lk}(L,u)} \, \langle u(\partial S) \rangle \in \text{Sk}(L)
\]

- \( w(u) \) – rational weight of \( u \) as a weighted point in the moduli space
- \( \chi(S) \) – Euler characteristic of \( S \)
- \( \text{lk}(L, u) \) – linking between \( u \) and \( L \)
- \( \langle u(\partial S) \rangle \) – the boundary of \( u \) in the skein of \( L \).
The skein valued curve count is invariant under deformations. This is proved by constructing a perturbation scheme for the Cauchy-Riemann equation. A cartoon version is as follows.

Configuration space for bare maps with ghosts.
Define, inductively in Euler characteristic, a (multi-) section \( \lambda: \mathbb{Z} \to \mathbb{W} \), which is zero on ghost components and which have the following properties:

1) Bare solutions transversely cut out, embeddings, tangent along boundary spans together with \( \xi \) a 2-plane everywhere.

2) Constant curves bubble off only in codimension \( \geq 2 \) \( \Rightarrow \) for 1-parameter families, all solutions near boundary are bare with ghosts.
3) Degeneracies in 1-parameter families of solutions have standard form.
4) At tangencies with $\xi$ a kink is traded for a 4-chain intersection.
Skein counts are inductive in Euler characteristic. Usual perturbative treatment is not.
Geometric setup: \( K \subset S^3 \) — knot. \( L_K \subset T^* S^3 \) — Lagrangian conormal \( \approx S^1 \times \mathbb{R}^2 \). Shift \( L_K \) off of 0-section \( S^3 \) (non-exact). Transition to resolved conifold \( X = \{ \mathcal{O}(-1)^{\oplus 2} \to \mathbb{C}P^1 \} \).
Conjecture

The GW partition function equals the generating function for the colored HOMFLY:

$$\Psi_K(x, a, g_s) = \sum_{k \geq 0} H_{K,n}(a, e^{\frac{g_s}{2}}) e^{nx},$$
For a small shift of the conormal there is a unique holomorphic cylinder. SFT stretching removes all boundaries from the 0-section (outside curves asymptotic to Reeb orbits of index 2 gives negative dimension). Calculating the skein valued invariant gives the colored HOMFLY (obvious for once around, for many times we use info about the unknot). Curves in the stretched structure are the same as in the conifold for small area $\mathbb{C}P^1$. 
Moduli spaces for planar unknot.

*Real curves* can be counted as in ordinary GW theory. For any knot \( K \) the count in the basic homology class in \( H_2(T^* S^3; L_K \cup S^3) \) is one cylinder, i.e., 1. The skein count corresponds in the stretched picture to a count in \( H_2(T^* S^3 \setminus S^3; L_K) \).
The toric brane in $\mathbb{C}^3$ provides a universal model for ‘crossing a basic disk’ and illustrates how to calculate skein invariants ‘from infinity’.

**Strategy for curve counts from infinity:** $(X, L)$ has ideal contact boundary $(\partial X, \partial L)$. The boundary of 1-dimensional moduli spaces consists of $\mathbb{R}$-invariant curves at infinity and rigid curves in the bulk. The boundary is zero in the skein. The outside then determines the inside.
The toric brane in $\mathbb{C}^3$

$\mathbb{C}^3$ with coordinates $z = (z_1, z_2, z_3)$.

$\mathbb{C}^3 \to \mathbb{R}^3$, $z \mapsto (r_\alpha(z), r_\beta(z), r_\gamma(z))$,

$r_\alpha(z) = |z_1|^2 - |z_2|^2$, $r_\beta(z) = |z_2|^2 - |z_3|^2$, $r_\gamma(z) = \text{Im}(z_1 z_2 z_3)$.

Fiber at $(r_\alpha, r_\beta, r_\gamma)$:

$|z_1|^2 = r_\alpha + |z_3|^2$, $|z_2|^2 = r_\beta + |z_3|^2$,

$\text{Im}(|z_1||z_2||z_3|e^{i(\theta_1+\theta_2+\theta_3)}) = r_\gamma$

$\Rightarrow$ generic fiber $T^2 \times \mathbb{R}$. 
The toric brane in $\mathbb{C}^3$

Lagrangians $L_1, L_2, L_3 \approx S^1 \times \mathbb{R}^2$.

$L_1: \quad r_\alpha = 0, \quad r_\beta = r_1^*, \quad r_\gamma \geq 0$ and $\text{Re}(z_1 z_2 z_3) = 0$,

$L_2: \quad r_\beta = 0, \quad r_\alpha = r_2^*, \quad r_\gamma \geq 0$ and $\text{Re}(z_1 z_2 z_3) = 0$,

$L_3: \quad r_\alpha - r_\beta = 0, \quad r_\alpha = r_3^*, \quad r_\gamma \geq 0$ and $\text{Re}(z_1 z_2 z_3) = 0$,

We restrict attention to $L_1$ and parameterize it

$$\left( |z_3| e^{i\theta}, (|z_3| + r_1^*) e^{i\phi}, |z_3| e^{-i(\theta + \phi - \frac{\pi}{2})} \right)$$
As \(|z_3| \to \infty\), \(L_1\) is asymptotic to the \(\mathbb{R}\)-invariant Lagrangian

\[
\left( \rho e^{i\theta}, \rho e^{i\phi}, \rho e^{-i(\theta+\phi-\frac{\pi}{2})} \right).
\]

Consider the image under the Hopf map \(S^5 \to \mathbb{C}P^2\):

\[
[e^{i(2\theta+\phi)} : e^{i(2\phi+\theta)} : i].
\]

A Clifford torus and the Legendrian \(\partial L_1\) is a 3-fold cover (Bohr-Sommerfeld). The Reeb chords of \(\partial L_1\) are Bott degenerate and come in \(T^2\)-families, length \(k\frac{2\pi}{3}\), index \(\geq 1\), with equality only for \(k = 1\).
To find holomorphic curves one can either use curves on the Clifford torus or draw the front of $\partial L_1$ in $\mathbb{R}^5 \subset S^5$:
We learn then that the skein valued curve count $\Psi$ on the toric brane satisfies the operator equation:

$$\left( \bigcirc - P_{1,0} - P_{0,1} \right) \Psi = 0.$$ 

The operators $\bigcirc - P_{1,0}$ and $P_{0,1}$ have a common eigen-basis in the positive skein $W_\lambda$ where $\lambda$ runs over partitions of positive integers. The operator equation has a unique solution in $Sk^+$:

$$\Psi = \sum_\lambda W_\lambda \prod_{\square \in \lambda} \frac{q^{-c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}},$$

where $h$ is the hook length and $c$ the content, here we use $z = q^{1/2} - q^{-1/2}$. 

The toric brane in $\mathbb{C}^3$
The toric brane in $\mathbb{C}^3$

Interpretation of the equation

$$\left(\bigcirc - P_{1,0} - P_{0,1}\right) \psi = 0.$$
Consider the conormal $L_K \subset T^*S^3$ of a knot in the resolved conifold. If $K = U$ there is similarly an immediate recursion relation of the form

$$(\bigcirc - P_{1,0} - P_{0,1} + a^2 P_{1,1}) \Psi_U = 0.$$ 

For more general knots the recursion will appear from a skein valued Legendrian SFT. Schematically, the boundary of a 1-dimensional moduli space looks as follows:

![Diagram of moduli space]

The recursion relation will then appear after the degree 0 chords have been eliminated from the equation.
Generalized curves

The standard approach to open GW invariants with one copy of the Lagrangian correspond to $U(1)$ gauge theory and in the case of bare curves to $a = q = e^{gs}$ after projection to ‘homology + linking’, we call them *generalized curves*.

$$w(u) \ z^{-\chi(S)} \ a^{\text{lk}(L,u)} \ \langle u(\partial S) \rangle \in \text{Sk}(L) \to$$

$$w(u) \ (q - q^{-1})^{-\chi(S)} \ q^{\text{lk}(L,u)} \ [u(\partial S)] \in \mathbb{Q}[q^\pm][\widehat{H_2(X,L)}]$$
E.g., for the toric brane and the unknot conormal the recursion relations then read:

\[(1 - e^\hat{x} - e^\hat{p})\psi(x) = 0, \quad \psi(x) = \sum_k c_k(q)e^{kx},\]

\[c_k(q) = (1 - q)^{-1}(1 - q^2)^{-1} \ldots (1 - q^k)^{-1},\]

\[(1 - e^\hat{p} - e^\hat{x} + a^2 e^\hat{x} e^\hat{p})\Psi(x) = 0, \quad \Psi(x) = \sum_k C_k(a, q)e^{kx},\]

where \(x\) generates \(H_1(L)\) and \(\log a\) generates \(H_2(X)\), \(e^\hat{x}\) is multiplication by \(e^x\) and \(e^\hat{p} = e^{gs\partial_x}\).
Generalized curves

For the trefoil $T$, $\hat{A}_T \Psi_T(x) = 0$,

$$
\hat{A}_T(e^x, e^p, a, q) = qa^6 e^{3p}(a^2 - q^{-3} e^{2p})(a^2 - q^{-1} e^p)
+ q^{-5/2}(a^2 - q^{-2} e^{2p}) \left( (q^2 e^{2p} + q^3 e^{2p} - q^3 e^p + q^4) a^4 \\
- (qe^{3p} + q^3 e^{2p} + qe^{2p}) a^2 + e^{4p} \right) e^x
+ (a^2 - q^{-1} e^{2p})(e^p - q)e^{2x}.
$$
It was observed that the generating function for the colored HOMFLY can be written as a quiver partition function for a symmetric quiver. The geometry behind such expressions can be understood if we assume that there is a finite set of basic holomorphic disks (the quiver nodes) attached to $L_K$ such that all holomorphic curves lie in a neighborhood of $L_K \cup \{\text{basic disks}\}$.
As for generalized curves, we must keep track of the linking number between disks to count generalized curves. The result is an expression of the following form:

$$
\Psi_K(e^x, a, q) = \psi \left( e^{x_1} e^{\sum_{j=1}^{n} C_{1j} g_s \partial x_j} \right) \ldots \psi \left( e^{x_m} e^{\sum_{j=1}^{n} C_{mj} g_s \partial x_j} \right) \\
= \sum_{(d_1, \ldots, d_m) \in \mathbb{Z}_+^m} (-q)^{\sum_{i\neq j} C_{ij} d_i d_j} \prod_{j=1}^{m} \frac{e^{d_j x_j}}{(q^2, q^2)_{d_j}},
$$

where $$e^{x_i} = q^{n_i} a^{k_i} e^{l_i x}.$$  

**Geometric characters of nodes:** $C_{ij}$ is linking between disks $i$ and $j$, $C_{ii}$ self-linking or framing data for attaching the disk, $n_i$ is 4-chain intersections (invariant self-linking minus framing), $(k_i, l_i)$ homology class in $H_2(X, L_K)$. 
For the unknot the desired form can be obtained from toric geometry.

For conormals of other knots the quiver picture might come from viewing their conormals as a ‘cover’ or the unknot conormal.
Basic holomorphic disks and quivers

\textit{Unknot}

\[ e^x \quad a^2 e^x \]

\textit{Trefoil}

\[ a^2 q^{-2} e^x \]

\[ a^4 q^{-3} e^x \]

\[ a^6 q^{-4} e^x \]

\[ a^4 q^{-1} e^x \]

\[ a^4 q^{-3} e^x \]
Non-uniqueness of quivers

Different quivers can give rise to the same partition function. There are two main sources.
Conjecture (E, Kucharski, Longhi)

The partition function of any knot conormal has the form of a generating function of a finite quiver. The quiver nodes come in unknot pairs and the pairs come in ‘sl(1)-pairs’ and such a quiver representation is unique.