PFH spectral invariants and the large-scale geometry of Hofer's metric

joint work with Vincent Humilière and Sobhan Seyfaddini

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Two old questions

Basic notions and notation

(M, ω) symplectic manifold.

Hamiltonian diffeomorphisms:

- Take $H : [0,1] \times M \to \mathbb{R}$. (Hamiltonian)
- Vector field X_H: $\omega(X_H, \cdot) = dH$.
- Hamiltonian flow: φ_H^t. Hamiltonian diffeo:= time-1 map φ_H^1.
- $\mathrm{Ham}(M, \omega) := \{\varphi_H^1\} \triangleleft \mathrm{Symp}(M, \omega)$.

Hofer norm :

- To a Hamiltonian H, define

$$||H||_{1,\infty} := \int_0^1 (\max_M H_t - \min_M H_t)dt.$$

- Now for $\varphi \in \mathrm{Ham}$, define

$$||\varphi|| := \inf\{||H||_{1,\infty} \mid \varphi = \varphi_H^1\}.$$

Hofer's metric

Hofer metric: For $\varphi, \psi \in \mathrm{Ham}$,

$$d_H(\varphi, \psi) := ||\varphi^{-1}\psi||.$$

. Defines a bi-invariant metric on $\mathrm{Ham}(M, \omega)$:

- bi-invariant: $d_H(\varphi, \psi) = d_H(\theta\varphi, \theta\psi) = d_H(\varphi\theta, \psi\theta)$.
- $d_H(\varphi, \psi) = d_H(\psi, \varphi)$.
- $d_H(\varphi, \psi) \leq d_H(\varphi, \theta) + d_H(\theta, \varphi)$.
- non-degeneracy: $d_H(\varphi, \psi) = 0 \iff \varphi = \psi$. (Hofer, Polterovich, Lalonde-McDuff)

Rather remarkable due to lack of compactness

Basic notions from large-scale geometry

$\Phi : (X_1, d_1) \to (X_2, d_2)$ a map between metric spaces.

Quasi-isometric embedding: if $\exists\, A \geq 1, B > 0$ s.t.

$$\frac{1}{A} d_1(x, y) - B \leq d_2(\Phi(x), \Phi(y)) \leq A d_1(x, y) + B.$$

Eg: 1. $\mathbb{Z} \hookrightarrow \mathbb{R}$, 2. $\mathbb{R} \longrightarrow \mathbb{Z}$, $x \mapsto \lfloor x \rfloor$.

Quasi-isometry: Φ QI embedding and $\exists\, C$ s.t. $\forall y \in X_2$

$$d_2(y, \Phi(X_1)) \leq C.$$

Eg: 1. $\mathbb{Z} \overset{QI}{\sim} \mathbb{R}$, 2. $\mathbb{R} \overset{QI}{\not\sim} \mathbb{R}^2$, 3. X bdd $\implies X \overset{QI}{\sim} pt$. “space viewed from far away"

The Kapovich-Polterovich Question

Theorem (Polterovich 1998)

$\mathrm{Ham}(\mathbb{S}^2)$ *admits a QI embedding of* $\mathbb{R}$.

Question (Kapovich-Polterovich 2006, McDuff-Salamon: Problem 21)

$\mathrm{Ham}(\mathbb{S}^2) \overset{QI}{\sim} \mathbb{R}$?

Our first theorem

Theorem (CG., Humilière, Seyfaddini; Polterovich-Shelukhin)

$\mathrm{Ham}(\mathbb{S}^2)$ *admits QI embedding of* $\mathbb{R}^n$ *for every n.*

Corollary: $\mathrm{Ham}(\mathbb{S}^2) \overset{\text{QI}}{\not\sim} \mathbb{R}$. But we can say more.

Quasi-flat rank: $\mathrm{rank}(\mathrm{X}, \mathrm{d}) = \max\{n : \mathbb{R}^n \overset{QI}{\hookrightarrow} X\}$.

- $X \overset{\text{QI}}{\sim} Y \implies \mathrm{rank}(X) = \mathrm{rank}(Y)$.
- $\mathrm{rank}(\mathrm{Ham}(\mathbb{S}^2)) = \infty$ by our theorem
- $\mathrm{rank}(\mathbb{R}^n) = n$, $\mathrm{rank}(G) < \infty$ for G connected finite-dim Lie group. (Bell-Dranishnikov)

So, $\mathrm{Ham}(\mathbb{S}^2)$ is quite “large". Remark: We also show it is not coarsely proper.

A question of Fathi

$Homeo_0(S^n, \omega)$: group of volume-preserving homeomorphisms of the n-sphere, in component of the identity.

Theorem (Fathi, 70s)

$Homeo_0(S^n, \omega)$ is simple when $n \geq 3$.

(Definition of simple: no non-trivial proper normal subgroups.) Simple $\implies$ no quotient groups.)

Question (Fathi, 70s)

Is $Homeo_0(S^2, \omega)$ simple?

Only closed manifold M for which simplicity of $Homeo_0(M, \omega)$ not known.

Our second theorem

Recall: commutator subgroup $[G, G] \triangleleft G$. A group is **perfect** if $G = [G, G]$. “Perfect group has no additive invariants."

Theorem (CG., Humilière, Seyfaddini)

$Homeo_0(S^2, \omega)$ is not perfect.

In particular, $Homeo_0(S^2, \omega)$ is not simple.

Although at first glance unrelated to the first theorem, proof also uses ideas from Hofer geometry.

Historical Remarks: results on QI type of Ham

Σ surface of positive genus:

- Lalonde-McDuff: $\mathbb{R}^n \overset{QI}{\hookrightarrow} \mathrm{Ham}(\Sigma)$, for every n. (1995)
- Polterovich: $(C([0,1]), \|\cdot\|_\infty) \overset{QI}{\hookrightarrow} \mathrm{Ham}(\Sigma)$. (1998).
- Other results: Polterovich-Shelukhin (2014), Alvarez-Gavela-Kaminker-Kislev-Kliakhandler-Polterovich-Rigolli-Rosen-Shabtai-Stevenson-Zhang (2016).

More general manifolds: Entov-Polterovich, Kawamoto, Khanevsky, Lalonde-Polterovich, Lalonde-McDuff, McDuff, Ostrover, Polterovich-Shelukhin, Py, Schwarz, Usher, Stojisavljevic-Zhang, ...

Historical Remarks: results on the simplicity question

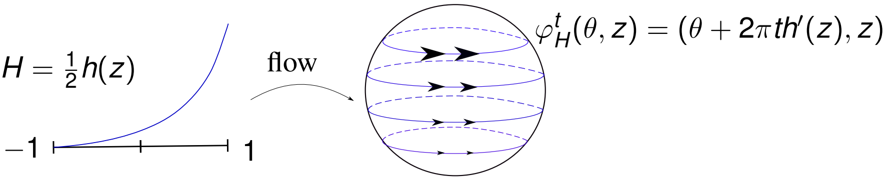
- Ulam ("Scottish book", 1930s): Is $Homeo_0(S^n)$ simple?
- 30s-60s: $Homeo_0(M)$ simple (Ulam, von Neumann, Anderson, Fisher, Chernavski, Edwards-Kirby)
- 60s-70s: $Diff_0^\infty(M)$ simple (Smale, Epstein, Herman, Mather, Thurston)
- More structure: Volume preserving diffeos (Thurston), symplectic case (Banyaga), volume preserving homeomorphisms with $n \geq 3$ (Fathi) — here there is a natural homomorphism (eg flux), kernel is simple.
- 2020: $Homeo_c(D^2)$ not simple (CG., Humilière, Seyfaddini). Very different from diffeomorphism group case (Le Roux)

Remark: Fathi's proof uses a "fragmentation" property; our work + Le Roux shows it fails in dimension 2.

Idea of the proofs

Twist maps

Monotone twist Hamiltonians: $H : \mathbb{S}^2 \to \mathbb{R}$ of the form $H(\theta, z) = \frac{1}{2}h(z)$, where $h \geq 0, h' \geq 0, h'' \geq 0$.



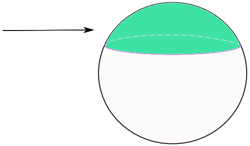
To prove our first theorem, suffices to produce QI embedding of $\mathbb{R}^n_{\geq 0} = \{(t_1, \ldots, t_n) : t_i \geq 0\}$.

Our embedding:

Discs: $D_i = \{(z, \theta) : 1 - \frac{1}{i+1} \leq z \leq 1\}$. Note: $D_i \supset D_{i+1}$, $\text{Area}(D_i) = \frac{1}{2(i+1)}$.

H_i: monotone twists such that $\mathrm{supp}(\mathrm{H_i}) = D_i$.

$\mathrm{supp}(H_i) = D_i$



$z = 1 - \frac{1}{i+1}$

Define

$$\Phi : \mathbb{R}^n_{\geq 0, ||\cdot||_\infty} \to \mathrm{Ham}(\mathbb{S}^2, d_{hofer}), \ (t_1, \ldots, t_n) \longrightarrow \varphi^{t_1}_{H_1} \circ \ldots \circ \varphi^{t_n}_{H_n}.$$

Claim A: Φ is a QI embedding.

Non-simplicity of $Homeo_0(S^2, \omega)$.

To prove our second theorem, we write down a particular normal subgroup.

Say that $\varphi \in Homeo_0(S^2, \omega)$ has **finite energy** if there exists a sequence of Hamiltonian diffeomorphisms that are bounded in Hofer's distance and converge in C^0 to φ.

Definition: $FHomeo_0(S^2, \omega) = \{\text{finite Hofer energy homeomorphisms}\}$.

Theorem B: $FHomeo_0(S^2, \omega) \triangleleft Homeo_0(S^2, \omega)$.

- Non-perfectness follows from this by an old argument of Epstein-Higman.

Hard part: why proper?

Remark: Our results on QI type should extend to FHomeo.

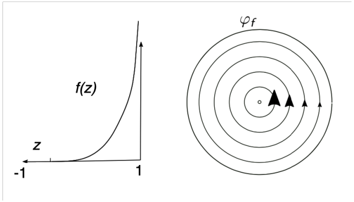
Why proper? Infinite twists

Let p_+ be the north pole. An **infinite twist Hamiltonian** is an $F : S^2 \setminus \{p_+\} \longrightarrow \mathbb{R}$ such that

$$F(z, \theta) = \frac{1}{2} f(z),$$

where $f : [-1, 1) \longrightarrow \mathbb{R}$ satisfies $f', f'' \geq 0$ and the **growth condition**

$$\lim_{d \longrightarrow \infty} \frac{1}{d} f(1 - \frac{2}{d+1}) = \infty.$$



Claim B: Any infinite twist is not in FHomeo.

New spectral invariants

To prove Claims A and B, we use Hutchings' Periodic Floer Homology PFH to construct

$$\mu_d, \eta_d : \mathrm{Ham}(\mathbb{S}^2) \longrightarrow \mathbb{R},$$

every even $d \in \mathbb{N}$.

We show:

- The μ_d and η_d are **Hofer Lipschitz**, eg

  $$|\mu_d(\varphi) - \mu_d(\psi)| \leq C_d \, d_H(\varphi, \psi), C_d = 2d$$

  so bound Hofer's distance from below.
- They can be computed for Monotone twists.
- The η_d are C^0 continuous and extend to Homeo. The μ_d are linear for compositions of monotone twists.

Comparison with previous work

We also used PFH spectral invariants to show $Homeo_c(D^2, \omega)$ is not simple in previous work.

New challenge here: need invariants that depend only on the time-1 map, not the choice of Hamiltonian. In the disc case, can restrict to Hamiltonians that vanish near boundary. No clear analogue here.

One of our solutions to get around this: take a suitable linear combination of spectral invariants $\longrightarrow \eta_d$. (For the μ_d, the idea is to homogenize and restrict to mean normalize Hamiltonians.)

More about the proofs

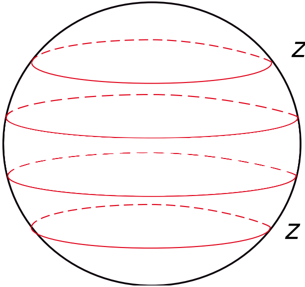
Computing the μ_d

The μ_d are used to prove our QI theorem. We first establish:

Monotone twist formula:

$$\mu_d(\varphi_H^1) = \sum_{i=1}^d H\left(-1 + \frac{2i}{d+1}\right) - d\,H(0).$$

$d = 4$



Linearity for monotone twists: $\mu_d(\varphi_{H_1}^{t_1} \circ \varphi_{H_2}^{t_2}) = t_1 \mu_d(\varphi_{H_1}^1) + t_2 \mu_d(\varphi_{H_2}^1)$.

Sketch of Proof ($n = 2$ case)

$$\Phi : \mathbb{R}^2_{\geq 0} \to \mathrm{Ham}(\mathbb{S}^2), \ (t_1, t_2) \longrightarrow \varphi^{t_1}_{H_1} \circ \varphi^{t_2}_{H_2}.$$

(Recall, H_i: monotone twist, $\mathrm{supp}(H_i) = \{(\theta, z) : 1 - \frac{2}{d_i} \leq z \leq 1\}$, $d_i = 2(i+1)$.)
Let $\mathbf{t} = (t_1, t_2)$. Recall $\Phi(\mathbf{t}) = \varphi^{t_1}_{H_1} \circ \varphi^{t_2}_{H_2}$. Goal: show $\exists C_1, C_2$ st

$$C_1 \|\mathbf{t} - \mathbf{s}\|_\infty \leq d_H\left(\Phi(\mathbf{t}), \Phi(\mathbf{s})\right) \leq C_2 \|\mathbf{t} - \mathbf{s}\|_\infty.$$

We'll just do the **lower bound:** By Hofer Lipschitz property (which says $|\mu_d(\varphi) - \mu_d(\psi)| \leq 2d \ \ d_H(\varphi, \psi)$)

$$\max_i \left| \frac{\mu_{d_i}(\Phi(\mathbf{t}))}{2d_i} - \frac{\mu_{d_i}(\Phi(\mathbf{s}))}{2d_i} \right| \leq d_H\left(\Phi(\mathbf{t}), \Phi(\mathbf{s})\right).$$

From previous slide:

$$\max_i \left| \frac{\mu_{d_i}(\Phi(\mathbf{t}))}{2d_i} - \frac{\mu_{d_i}(\Phi(\mathbf{s}))}{2d_i} \right| \leq d_H(\Phi(\mathbf{t}), \Phi(\mathbf{s})) .$$

Claim: LHS = $\|A(\mathbf{t}-\mathbf{s})\|_\infty$ where $A = \frac{1}{2d_i}\begin{bmatrix} \mu_{d_1}(\varphi^1_{H_1}) & \mu_{d_1}(\varphi^1_{H_2}) \\ \mu_{d_2}(\varphi^1_{H_2}) & \mu_{d_2}(\varphi^1_{H_2}) \end{bmatrix}$ Proof: By Linearity of μ_d (details left as an exercise.)

Claim: A is invertible. Proof: see next slide.

Since A is invertible can write

$$\frac{\|\mathbf{t}-\mathbf{s}\|_\infty}{\|A^{-1}\|_{op}} \leq \|A(\mathbf{t}-\mathbf{s})\|_\infty ,$$

where $\|A^{-1}\|_{op} =$ denotes the operator norm of $A^{-1} : (\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_\infty)$.

So, take $C_1 = \frac{1}{\|A^{-1}\|_{op}}$, hence the lower bound. Remark: We can arrange that our QI embedding is in the kernel of Calabi, answering a 2012 question of Polterovich.

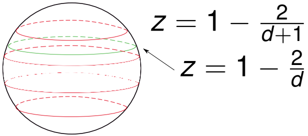
Why is A invertible?

Claim: A is invertible. Recall from previous slide: $A = \frac{1}{2d_i}\begin{bmatrix} \mu_{d_1}(\varphi^1_{H_1}) & \mu_{d_1}(\varphi^1_{H_2}) \\ \mu_{d_2}(\varphi^1_{H_2}) & \mu_{d_2}(\varphi^1_{H_2}) \end{bmatrix}$

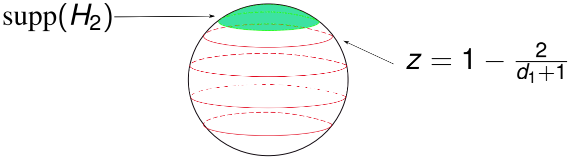
Proof: follows from the next two observations.

Observation 1: $\mu_{d_i}(\varphi^1_{H_i}) > 0$. Proof:

$$\mu_{d_i}\left(\varphi^1_{H_i}\right) = H_i\left(1 - \frac{2}{d_i+1}\right) > 0$$



Observation 2: $\mu_{d_1}(\varphi^1_{H_2}) = 0$. Proof:



Next theorem: properness of FHomeo

Claim: an infinite twist does not have finite energy.

Proof: The η_d are C^0 continuous and extend to $Homeo_0$. By Hofer continuity, we get the **linear growth** property: for any $\psi \in FHomeo_0$,

$$\limsup_{d \longrightarrow \infty} \frac{\eta_d(\psi)}{d} < \infty.$$

On the other hand, for infinite twists we show

$$\lim_{d \longrightarrow \infty} \frac{\eta_d(\psi)}{d} = \infty.$$

To do this, we use a combinatorial model for the η_d of Monotone twists. Rough idea: the η_d should recover the “Calabi invariant" asymptotically, can verify this for monotone twists by direct computation.

PFH spectral invariants

(impressionistic sketch of the construction)

The PFH of φ: the setup

Let $\varphi \in \mathrm{Ham}(\mathbb{S}^2, \omega)$. Recall the **mapping torus**

$$Y_\varphi = \mathbb{S}^2_x \times [0,1]_t / \sim, \quad (x,1) \sim (\varphi(x), 0).$$

Canonical two-form ω_φ induced by ω.

Canonical vector field $R := \partial_t$. Captures the dynamics of φ.

$$\{\text{Periodic Points of } \varphi\} \stackrel{1:1}{\longleftrightarrow} \{\text{Closed Orbits of } R\}$$

R is the "Reeb" vector field of the Stable Hamiltonian Structure (dt, ω_φ).

PFH = ECH in this setting. (Hutchings)

There exists PFH spectral invariants c_d "=" ECH spectral invariants in this setting. (Hutchings)

More details about PFH

$PFH(\varphi)$ is homology of a chain complex $PFC(\varphi)$. (φ non-degenerate)

$PFC(\varphi)$: generated by (certain) "Reeb orbit sets" $\{(\alpha_i, m_i)\}$

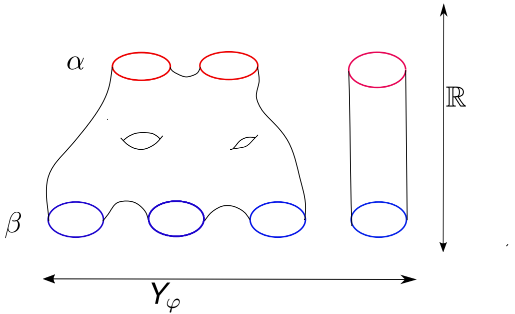
- α_i distinct, embedded closed orbits of R
- m_i positive integer. ($m_i = 1$ if α_i is hyperbolic)

∂: counts certain J-holomorphic curves in $\mathbb{R} \times Y_\varphi$.

$PFH(\varphi)$ is the homology of this chain complex.

Lee-Taubes: $PFH(\varphi)$ independent of choices of J, φ.

A J-hol curve contributing to $\langle \partial\alpha, \beta \rangle$



$\langle \partial\alpha, \beta \rangle := \#$ maps $u : (\Sigma, j) \to (\mathbb{R} \times Y_\varphi, J)$ such that

- J holomorphic: $du \circ j = J(u)du$.
- Asymptotic to α and β.
- "ECH index" $I = 1$.

The spectral invariants c_d

To construct spectral invariants need two ingredients:

1. $PFH(\varphi)$ has an action filtration. (twisted version)
   - $PFH^a(\varphi)$: what you see up to action level $a \in \mathbb{R}$.
2. There exist (more or less) distinguished nonzero classes $\sigma_d \in PFH(\varphi)$ for $d \in \mathbb{N}$.

Define:

$$c_d(\varphi) := \inf\{a \in \mathbb{R} : \sigma_d \in PFH^a(\varphi)\}.$$

In words: $c_d(\varphi)$ is the action level at which you first see σ_d.

Remark: d corresponds to the degree of the class.

The μ_d

The numbers $c_d(\varphi)$ as defined depend on the choice of generating Hamiltonian (because twisted PFH does). First step to remedy this: restrict to **mean-normalized** Hamiltonians, that is

$$\int_{S^2} H_t \omega = 0$$

for all t. We show this gives a well-defined invariant c_d on $\widetilde{Ham}$.

We next **homogenize** to get invariants μ_d on Ham:

$$\mu_d(\varphi) := \lim_{n \longrightarrow \infty} \frac{c_d(\tilde{\varphi}^n)}{n}$$

where $\tilde{\varphi}$ is any lift of φ.

The η_d

The μ_d are **not** in general C^0-continuous, essentially due to the mean normalization condition. To get mean normalized invariants, need a different trick.

Key computation: for even d,

$$\eta_d(\varphi) := c_d(\varphi) - \frac{d}{2}c_2(\varphi),$$

is independent of the choice of Hamiltonian for φ. We show in addition the η_d are C^0-continuous.

Thank you!

Bonus: twisted PFH.

The reference cycle

To define twisted PFH, need in addition a (trivialized) reference cycle $\gamma \subset Y_\varphi$. Using this we proceed as follows:

- A twisted PFH generator is a pair (α, Z), where α is a PFH generator and $Z \in H_2(\alpha, \gamma^d)$ is a “capping".
- The differential counts $I = 1$ curves C from (α, Z) to (β, Z'): that is $[C] + Z' = Z \in H_2(\alpha, \gamma^d)$.
- The action is given by: $\mathcal{A}(\alpha, Z) = \int_Z \omega_\varphi$.

We produce γ by trivializing Y_φ via

$$S^1 \times S^2 \longrightarrow Y_\varphi, \quad (t, x) \longrightarrow (t, (\varphi_H^t)^{-1}(x)),$$

and taking γ to be the push-forward of the invariant cycle over p_-.

Bonus II: comparison with the work of Polterovich-Shelukhin.

- Polterovich-Shelukhin approach uses (orbifold) Lagrangian spectral invariants on the symmetric product of S^2.
- PFH is also conceptually related to the symmetric product (cf Hutchings: “https://floerhomology.wordpress.com/2013/07/18/symmetric-products-i/). Rough idea:
  - degree d PFH orbit set "=" fixed point of the induced map on the d-fold symmetric product
  - holomorphic curve counted by PFH differential "=" section of bundle of symmetric products $\mathbb{R} \times Y_{S^d \varphi} \longrightarrow \mathbb{R} \times S^1$.
- Potentially very interesting to compare our approaches.
- In fact, their invariants seem to agree with ours for monotone twists.