1. (20 pts) (a). Find the solution of the following initial value problem.

\[ y'' + y' - 6y = 0 \quad y(0) = 2, \quad y'(0) = 9. \]

**Char poly:** \( \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) \)

\( \lambda = 2, -3 \)

**general soln:** \( y(t) = Ae^{2t} + Be^{-3t} \)

\( 2 = y(0) = A + B \) \( \implies A = 3, B = -1 \)

**Solution:** \( y(t) = 3e^{2t} - e^{-3t} \)

(b). Find the general solution to the following differential equation:

\[ y'' - 4y' + 4y = 0. \]

**Char poly:** \( \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 \)

\( \lambda = 2 \), repeated root

**general soln:** \( y(t) = Ae^{2t} + Bte^{2t} \)
2. (20 pts) (a). Newton’s law of cooling asserts that the rate at which an object cools is proportional to the difference between the object’s temperature \( T \) and the temperature of the surrounding medium \( A \). It therefore satisfies the ODE

\[
T'(t) = -k(T(t) - A).
\]

Solve for \( T(t) \) in terms of \( A, T_0 = \) the temperature of the body at time 0, and \( k = \) the proportionality constant.

Separate variables, we get

\[
\frac{dT}{T-A} = -k \, dt.
\]

Integrate both sides,

\[
\int_{T_0}^{T} \frac{ds}{s-A} = -k \int_0^t \, du
\]

\[
\ln \left| \frac{T-A}{T_0-A} \right| = -kt \quad \text{or} \quad \ln \frac{T-A}{T_0-A} = -kt.
\]

Solve for \( T \) by exponentiating, and since \( T-A \) and \( T_0-A \) have the same sign,

\[
T(t) = A + (T_0-A) e^{-kt}.
\]

(b) A murder victim is discovered at midnight and the temperature of the body is recorded at 31°C. One hour later, the temperature of the body is 29°C. Assume that the surrounding air temperature remains constant at 21°C. Calculate the victim’s time of death. (Note. The “normal” temperature of a living human being is approximately 37°C).

Let \( t=0 \) be midnight, let \( t_1 \) be the victim’s time of death. Then \( t_1 < 0 \). (The unit for \( t \) is hour)

\[
T_0 = 31, \quad A = 21, \quad \text{we have}
\]

\[
T(t) = 21 + (31-21) e^{-kt}.
\]

In particular,

\[
T(1) = 29 = 21 + (31-21) e^{-k} \Rightarrow e^{-k} = \frac{8}{10}.
\]

It implies that

\[
T(t) = 21 + 10 \cdot \left( \frac{8}{10} \right)^t.
\]

If \( T(t) = 37 \), then

\[
37 = 21 + 10 \cdot \left( \frac{8}{10} \right)^{t_1} \Rightarrow \left( \frac{8}{10} \right)^{t_1} = \frac{16}{10} = \frac{8}{5}.
\]

\[
t_1 = \frac{\ln 5}{\ln 2} = - \frac{\ln 2}{\ln 2} = -2 \quad (\text{in 2 hours})
\]

The victim’s time of death is \( \ln \frac{8}{5} / \ln 2 \) hours before midnight. (8:10pm)
(c). Consider the equation

\[ y' = f(at + by + c) \]

where \( a, b, \) and \( c \) are constants. Show that the substitution \( x = at + by + c \) changes the equation to the separable equation \( x' = a + bf(x) \). Use this method to find the general solution of the equation \( y' = (y + t)^2 \).

Since \( x = at + by + c \)

\[ \frac{dx}{dt} = a + b \frac{dy}{dt} \]

The equation \( \frac{dy}{dt} = f(at + by + c) \) becomes

\[ \frac{dx}{dt} = a + b \frac{dy}{dt} = a + bf(at + by + c) = a + bf(x) \]

Next, let \( f(at + by + c) = (y + t)^2 \).

Let \( x = y + t \).

Then \( y' = (y + t)^2 \) is equivalent to

\[ x' = y' + 1 = (y + t)^2 + 1 = x^2 + 1 \]

This is a separable equation.

\[ \frac{dx}{1 + x^2} = dt \]

\[ \int \frac{dx}{1 + x^2} = \int dt \Rightarrow \arctan x = t + C \]

\[ x = \tan(t + C) \]

Since \( y = x - t \), \( y = \tan(t + C) - t \). This is the general solution of \( y' = (y + t)^2 \).
3. (15 pts) FACT: The function \( y = (x^2 + 1)e^{-x^2} \) is a solution to the ordinary differential equation

\[
(x^2 + 1)y' + 2x^3y = 0.
\]

(a). Using the above fact, find the general solution \( y(x) \) to the ordinary differential equation

\[
(x^2 + 1)y' + 2x^3y = 2(x^3 + x)e^{-x^2}.
\]

Let \( y_h = (x^2 + 1)e^{-x^2} \) and \( y = Vy_h \). Then plugging in,

\[
(x^2 + 1)y_h' + (x^2 + 1)Vy_h' + 2x^3 Vy_h = 2(x^3 + x)e^{-x^2}
\]

\[
\Rightarrow (x^2 + 1)y_h' + V[(x^2 + 1)y_h' + 2x^3 y_h] = 2(x^3 + x)e^{-x^2}
\]

\[
\Rightarrow (x^2 + 1)y_h' = 2(x^3 + x)e^{-x^2} \Rightarrow y_h' = \frac{2(x^3 + x)}{(x^2 + 1)y_h} e^{-x^2}
\]

\[
\Rightarrow V' = \frac{2x}{x^2 + 1} \Rightarrow V(x) = \ln(x^2 + 1) + C
\]

So

\[
y(x) = (x^2 + 1)e^{-x^2} \left( \ln(x^2 + 1) + C \right)
\]

(b). Find the solution \( y(x) \) to the equation from part (a) satisfying the initial value

\[
y(0) = 1.
\]

\[
1 = y(0) = (0 + 1)e^0 \left( \ln(1) + C \right) = C
\]

\[
y(x) = (x^2 + 1)e^{-x^2} \left( \ln(x^2 + 1) + C \right)
\]
4. (15 pts) Suppose that the functions $u$ and $v$ are solutions to the linear, homogeneous equation

$$y'' + p(y)y' + q(t)y = 0$$

in the interval $(\alpha, \beta)$. Prove that the Wronskian of $u$ and $v$ is either identically equal to zero on $(\alpha, \beta)$, or it is never equal to zero there.

Proposition 1.26 pp. 142-143.
5. (20 pts) (a). Show that the following equation is exact and solve it.

\[(2x + y)dx + (x - 6y)dy = 0\]

**Solution:**

\[\frac{\partial}{\partial y}(2x + y) = 1 = \frac{\partial}{\partial x}(x - 6y)\]

implies that the equation is exact. The solution is

\[x^2 + xy - 3y^2 = C\]
(b). Suppose that \((x+y)dx+2xdy = 0\) has an integrating factor that is a function of \(x\) alone (i.e \(\mu = \mu(x)\)). Find the integrating factor and use it to solve the differential equation.

\[ \text{Solution:} \]

Suppose \(\mu(x)\) is an integrating factor. We need

\[ \mu(x)(x+y)dx + \mu(x)(2x)dy = 0 \]

to be exact. So we have

\[ \frac{\partial}{\partial y}(\mu(x)(x+y)) = \frac{\partial}{\partial x}(\mu(x)(2x)) \]

i.e.

\[ \mu(x) = \mu'(x) \cdot 2x + \mu(x) \cdot 2 \]

We have solution for this ODE,

\[ \mu(x) = x^{-1/2} = \frac{1}{\sqrt{x}} \]

Now the equation becomes

\[ (\sqrt{x} + y/\sqrt{x})dx + 2\sqrt{x}dy = 0 \]

Therefore we can get the solution

\[ 2\sqrt{xy} + \frac{2}{3}x^{3/2} = C \]
6. (10 pts) The undamped system

\[
\frac{2}{5}x'' + kx = 0 \quad x(0) = 2, \quad x'(0) = v_0
\]

is observed to have period \(\pi/2\) and amplitude 2. Find \(k\) and \(v_0\).

**Solution:**

\[\omega_0 = \frac{2\pi}{\sqrt{k/5}} = 4, \text{ so } \frac{k}{25} = \omega_0^2 = 16 \text{ and therefore } k = \frac{32}{5}.\]

The equation becomes \(x'' + 16x = 0\), with general solution \(x(t) = c_1 \cos 4t + c_2 \sin 4t\). But we know that \(x(0) = 2 = \text{amplitude, hence } x(t) = 2 \cos 4t\) and therefore \(v_0 = x'(0) = 0\).