\[ dF = \frac{xdx + ydy}{\sqrt{x^2 + y^2}} \]

12. With \( P = \frac{x}{\sqrt{x^2 + y^2}} \) and \( Q = \frac{-y}{\sqrt{x^2 + y^2}} \), we compute

\[ \frac{\partial P}{\partial y} = \frac{-2xy}{(x^2 + y^2)^{3/2}} = \frac{\partial Q}{\partial x}, \]

so the equation is exact. To find the solution we integrate

\[ F(x, y) = \int P(x, y) \, dx \]
\[ = \int \frac{x}{\sqrt{x^2 + y^2}} \, dx \]
\[ = \sqrt{x^2 + y^2} + \phi(y). \]

To find \( \phi \), we differentiate

\[ Q(x, y) = \frac{\partial F}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \phi'(y). \]

Thus \( \phi' = 0 \), so we can take \( \phi = 0 \). Hence the solution is \( F(x, y) = \sqrt{x^2 + y^2} = C \).

22. \( -x/y + \ln x = C \)

43. (a) The curves are defined by the equation \( F(x, y) = x/(x^2 + y^2) = c \). Hence the orthogonal family must satisfy

\[ \frac{dy}{dx} = \frac{\partial F}{\partial y} / \frac{\partial F}{\partial x} = \frac{2xy}{x^2 - y^2}. \]

(b) The differential equation is homogeneous. Solving in the usual way we find that the orthogonal family is defined implicitly by

\[ G(x, y) = \frac{y}{x^2 + y^2} = C. \]

The original curves are the solid curves in the following figure, and the orthogonal family is dashed.

Since the general solution is \( y(t) = t + 2Ct^2 \), every solution satisfies \( y(0) = 0 \). There is no solution with \( y(0) = 2 \). If we put the equation into normal form

\[ \frac{dy}{dt} = \frac{2y - t}{t}, \]

we see that the right hand side \( f(t, y) = (2y - t)/t \) fails to be continuous at \( t = 0 \). Consequently the hypotheses of the existence theorem are not satisfied.

21. (a) If

\[ y(t) = \begin{cases} 0, & \text{if } t \leq t_0 \\ (t - t_0)^3, & \text{if } t > t_0, \end{cases} \]

then

\[ y'(t_0^+) = \lim_{t \to t_0^+} \frac{y(t) - y(t_0)}{t - t_0} = \lim_{t \to t_0^+} \frac{(t - t_0)^3 - 0}{t - t_0} = \lim_{t \to t_0^+} (t - t_0)^2 = 0. \]

On the other hand,

\[ y'(t_0^-) = \lim_{t \to t_0^-} \frac{y(t) - y(t_0)}{t - t_0} = \lim_{t \to t_0^-} \frac{0 - 0}{t - t_0} = 0. \]

Therefore, \( y'(t_0) = 0 \), since both the left and right-hand derivatives equal zero.

(b) The right hand side of the equation, \( f(t, y) = 3y^{2/3} \), is continuous, but \( \partial f/\partial y = 2y^{-1/3} \) is not continuous where \( y = 0 \). Hence the hypotheses of Theorem 7.16 are not satisfied.
\textbf{\Sect{2.7}}

31. Notice that \( x_1(t) = t^2 \) is a solution to the same differential equation with initial value \( x_1(0) = 0 < 1 = x(0) \). The right hand side of the differential equation, \( f(t, x) = x - t^2 + 2t \) and \( \partial f / \partial x = 1 \) are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for \( x \) and \( x_1 \) cannot cross. Hence we must have \( t^2 = x_1(t) < x(t) \) for all \( t \).

\textbf{\Sect{2.9}}

16. (i) In this case, \( f(y) = 2y - 7 \), whose graph is shown in the next figure.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig1.png}
\end{center}

(ii) The phase line is easily captured from this figure, and is shown in next figure.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig2.png}
\end{center}

(iii) The phase line in the second figure indicates that solutions decrease if \( y < 7/2 \) and increase if \( y > 7/2 \). This allows us to easily construct the phase portrait shown in the \( ty \) plane in the next figure. Note the unstable equilibrium solution, \( y(t) = -2 \), and the stable equilibrium solution, \( y(t) = 3 \).

\textbf{\Sect{2.9}}

18. (i) In this case, \( f(y) = 6 + y - y^2 \) factors as \( f(y) = (2 + y)(3 - y) \), whose graph is shown in the next figure.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig3.png}
\end{center}

(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig4.png}
\end{center}

(iii) The phase line in the second figure indicates that solutions decrease if \( y < -2 \), increase for \( -2 < y < 3 \), and decrease if \( y > 3 \). This allows us to easily construct the phase portrait shown in the \( ty \) plane in the next figure. Note the unstable equilibrium solution, \( y(t) = -2 \), and the stable equilibrium solution, \( y(t) = 3 \).
26. Separating variables,

\[ \frac{dy}{dt} = (3 + y)(1 - y) \]

\[ \frac{dy}{(3 + y)(1 - y)} = dt. \]

A partial fraction decomposition allows us to continue.

\[ \int \left[ \frac{1}{4(3 + y)} + \frac{1}{4(1 - y)} \right] dy = dt \]

\[ \ln |3 + y| - \ln |1 - y| = 4t + C \]

\[ \ln \frac{3 + y}{1 - y} = 4t + C \]

\[ \frac{3 + y}{1 - y} = e^{4t} \]

\[ \frac{3 + y}{1 - y} = Ae^{4t} \]

with \( y(0) = 2 \),

\[ \frac{3 + 2}{1 - 2} = Ae^{4(0)} \implies A = -5 \]

and

\[ \frac{3 + y}{1 - y} = -5e^{4t} \]

\[ 3 + y = -5e^{4t} + 5ye^{4t} \]

\[ 3 + 5e^{4t} = y(5e^{4t} - 1) \]

\[ y = \frac{3 + 5e^{4t}}{5e^{4t} - 1}. \]

Multiply top and bottom by \( e^{-4t} \).

\[ y = \frac{3e^{-4t} + 5}{5 - e^{-4t}}. \]

Thus,

\[ \lim_{t \to \infty} y = \frac{0 + 5}{5 - 0} = 1. \]

Using qualitative analysis, plot the graph of the right-hand side of

\[ \frac{dy}{dt} = (3 + y)(1 - y) \]

\[ f(y) = (3 + y)(1 - y) \]

Note the equilibrium points at \( y = -3 \) and \( y = 1 \). Moreover, note that between \(-3\) and \(2\), solutions increase to the stable point at \( y = 1 \). Thus,

\[ \lim_{t \to \infty} y(t) = 1. \]

28. We have the equation \( x' = f(x) = x(x - 1)(x + 2) \).

The equilibrium points are at \( x = 0, 1, \) and \(-2\), where \( f(x) = 0 \). We have \( f'(x) = 3x^2 + 2x - 2 \).

Since \( f'(0) = -2 < 0 \), \( x = 0 \) is asymptotically stable.

Because \( f'(1) = 3 > 0 \), \( x = 1 \) is unstable.

Finally, because \( f'(-2) = 2 > 0 \), \( x = -2 \) is also unstable.

4. First, solve the differential equation \( y'' + 2y' + 2y = \sin 2\pi t \) for the highest derivative of \( y \) present in the equation.

\[ y'' = -2y' - 2y + \sin 2\pi t \]

Next, set \( v = y' \). Then

\[ v' = y'' = -2y' - 2y + \sin 2\pi t \]

\[ = -2v - 2y + \sin 2\pi t. \]

Thus, we now have the following system of first order equations.

\[ y' = v \]

\[ v' = -2v - 2y + \sin 2\pi t \]