Recommended Problems #3

Problem 1. Show that the two definitions of a Hamiltonian action are equivalent, i.e. show that given a smooth $G$-action on $M$, $H : g \to C^\infty(M)$ is a co-moment map iff its ”dual” $\mu : M \to g^*$ is a moment map.

Problem 2. Consider the $S^1$ action on $(\mathbb{C}^n, \omega_0)$ defined by
\[ t(z_1, \ldots, z_n) = (t^{k_1}z_1, \ldots, t^{k_n} z_n) \]
where $k_i$ are fixed integers (called the weights of the action).

(a) show that this action is Hamiltonian with moment map $\mu(z) = -\frac{1}{2} \sum_{i=1}^n k_i |z_i|^2$, and that the level sets $\mu^{-1}(c)$ are coisotropic submanifolds (away from the critical points), preserved by the $S^1$ action. Describe the kernel of the restriction of $\omega_0$ to $\mu^{-1}(c)$ (and identify it with the tangent space to the orbits of the $S^1$ action).

(b) assume next for simplicity all the weights are equal to 1. Show that $\omega_0$ descends to the quotient $\mu^{-1}(c)/S^1 \cong \mathbb{CP}^{n-1}$ as a multiple of the Fubini-Study form. How does the symplectic area of the generator relate to $c$?

(c) consider the case $n = 2$ and $k_1 = 1, k_2 = k \geq 2$. Show that the quotient is now is a ”teardrop”, i.e. is only homeomorphic to $\mathbb{CP}^1$, but it has an orbifold singularity at one of the points (where the local model is $\mathbb{C}/\mu_k$ where $\mu_k \subset S^1$ is the group of $k$’th roots of unity). What happens if instead $k_1 = k_2 \geq 2$?

Problem 3. Consider the $T^n = (S^1)^n$ action on $\mathbb{C}^n$ defined by
\[ (t_1, \ldots, t_n)(z_1, \ldots, z_n) = (t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n) \]
where $k_i$ are fixed integers. Show that:

(a) the map $\mu(z) = -\frac{1}{2} (k_1 |z_1|^2, \ldots, k_n |z_n|^2)$ is a moment map for this action.

(b) show that the (regular) level sets $\mu^{-1}(c)$ are Lagrangian submanifolds and describe what happens to them as $c$ approaches a critical value.

(c) assume next that all weights are equal to 1, and consider the following two subgroups:
\[ H = \{(t, t, \ldots, t) \in T^n \mid t \in S^1 \} \quad \text{and} \quad G = \{(t_1, \ldots, t_n) \in T^n \mid t_1 \cdot \ldots \cdot t_n = 1 \}. \]
Describe the moment maps for these two subactions. Show that the $T^n$ action descends to an action of $G$ on the quotient $\mu^{-1}(c)/H \cong S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$ (see Problem 2).

Problem 4. Consider next the $T^n$ action on $\mathbb{CP}^n$ defined by
\[ (t_0, \ldots, t_n)[z_0, \ldots, z_n] = [t_0z_0, t_1z_1, \ldots, t_n z_n] \]
where $T^n = \{(t_0, \ldots t_n) \in (S^1)^{n+1} \mid t_0 \cdot \ldots \cdot t_n = 1 \}$. 

(a) describe the fixed points of this action.

(b) show that this action is Hamiltonian, and that the image $\Delta$ of the moment map $\mu$ is the convex hull of the images of the fixed points ($\Delta$ can be identified with the standard simplex in $\mathbb{R}^{n+1}$).
(c) show that the regular level sets $\mu^{-1}(c)$ are Lagrangian tori; describe the level sets corresponding to $c$ in the boundary of $\Delta$ and observe that $\mu : \mathbb{CP}^n \to \Delta$ is a singular Lagrangian fibration.

(d) extend the discussion above to the induced action $T^n$ action on $\mathbb{CP}^n$ defined by

$$(t_0, \ldots, t_n)[z_0, \ldots, z_n] = [t_0^{k_0} z_0, t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n]$$

for other weights $k_i \in \mathbb{Z}$ in Problem 3. Is there any difference between the case when the weights are relatively prime or not?

**Problem 5.** Consider $U(n)$ the unitary group. Show that $\langle A,B \rangle = \text{trace}(A^*B)$ defines a bi-invariant metric on $U(n)$ and use it to identify $u(n)^* \cong u(n)$ (=the space of skew hermitian matrices $h$ i.e. such that $h + h^* = 0$). Show that the $U(n)$ actions below are Hamiltonian:

(a) the usual action on $\mathbb{C}^n$ has moment map $\mu(z) = \frac{i}{2} zz^*$.

(b) the action on the space $M_{k\times n}$ of $k \times n$ matrices has $\mu(B) = \frac{i}{2} BB^*$.

(c) the action on $M_{n\times n}$ by conjugation has $\mu(B) = \frac{i}{2} [B, B^*]$.

(d) for $n = 1$, $U(1) = S^1$ and show that this agrees with the moment map (up to the correct factors) in Problem 1.

(e) when restricted to the maximal subtorus $K = T^n = U(1)^n$ of $U(n)$ consisting of diagonal matrices, show that the moment map $\mu_K$ agrees with the one in Problem 3(c).

(f) describe the orbits of the coadjoint action of $U(n)$ and their moment map.

**Problem 6.** Assume $G$ acts on $(M, \omega)$ with moment map $\mu : M \to g^*$. Show that

$$\{f \circ \mu, g \circ \mu\} = \{f, g\} \circ \mu \quad \forall f, g \in C^\infty(g^*)$$

i.e. that the induced map $\mu^* : C^\infty(g^*) \to C^\infty(M)$ defined by $\mu^*(f) = f \circ \mu$ respects the Poisson structures.

**Problem 7.** (Coadjoint orbits) Consider the coadjoint action of $G$ on its dual Lie algebra $g^*$. Show that the formula

$$\omega_\eta(X_v, X_w) = \langle \eta, [v, w] \rangle \quad \forall \eta \in g^*, \forall v, w \in g$$

induces a (canonical) symplectic form on any coadjoint orbit $O_\eta \subseteq g^*$. Show that the coadjoint action of $G$ on $O_\eta$ is Hamiltonian with moment map $\mu : O_\eta \hookrightarrow g^*$ the inclusion.

Intrinsically identify the coadjoint orbits in the case $G = U(n)$ (see problem 4).

**Problem 8.** Show that an effective Hamiltonian $T^n$ action on a symplectic manifold $M^{2n}$ gives rise to an integrable system, i.e. the moment map $\mu : M \to \mathbb{R}^n$ gives rise to a (possibly singular) Lagrangian fibration.