Problem 1. Consider the flow \( \{ \varphi_t \}_{t \in [0,1]} \) generated by a time dependent symplectic vector field \( X_t \) on a symplectic manifolds \((M, \omega)\). For each loop \( \gamma : S^1 \to M \), consider the surface \( \Gamma \) swept by \( \gamma \) under the flow, i.e. \( \Gamma(t,s) = \varphi_t(\gamma(s)) \) for any \( t \in [0,1] \) and \( s \in S^1 \).

(a) Show that the symplectic area of \( \Gamma \) is the flux of \( \{ \varphi_t \} \) through \( \gamma \)

\[
\langle \text{Flux}(\{ \varphi_t \}), [\gamma] \rangle = \int_{[0,1] \times S^1} \Gamma^* \omega
\]

(b) conclude that the flux though \( \gamma \) depends only on the homotopy class of \( \gamma \) and on the homotopy class of the path \( \{ \varphi_t \} \) relative its endpoints.

(c) Show that if \( \text{Flux}(\{ \varphi_t \}) = 0 \in H^1_R(M) \) then \( \varphi_t \) is isotopic with fixed endpoints to a path of Hamiltonian diffeomorphisms \( \psi_t \). Hint: need \( \text{Flux}(\{ \psi_t \}_{t \in [0,T]} ) = 0 \) for all \( T \in [0,1] \).

Problem 2. Assume \( N \) is a coisotropic submanifold of \((M, \omega)\). Use Frobenius integrability theorem to show that \( (TN)^\omega \) defines an integrable foliation on \( N \) whose leaves are isotropic submanifolds of \( M \).

Hint: check that for any vector fields \( X, Y \in (TN)^\omega \) their Lie bracket \( [X,Y] \in (TN)^\omega \).

Problem 3. Assume \((M, \omega)\) is a symplectic manifold.

(a) Show that if \( X, Y \) are symplectic vector fields, then \([X,Y] \) is a Hamiltonian vector fields with Hamiltonian function \( \omega(X,Y) \); in particular, \( \mathcal{X}_{\text{ham}}(M) \) is a Lie algebra with \([\cdot,\cdot];\)

(b) the Poisson bracket of two smooth functions \( f, g \in C^\infty(M) \) is defined by

\[
\{f,g\} = \omega(X_f, X_g) = -\mathcal{L}_{X_f}g
\]

where \( X_f \) denotes the Hamiltonian vector field defined by \( f \). Show that \((C^\infty(M),\{\cdot,\cdot\})\) is a Lie algebra that is \{\cdot,\cdot\} is: (i) \( \mathbb{R} \)-bilinear (ii) skew-sym \( \{g,f\} = -\{f,g\} \) and (ii) satisfies Jacobi identity: \( \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0; \)

(c) show that \( X_{\{f,g\}} = -[X_f,X_g] \) so the map \( f \mapsto -X_f \) is a surjective Lie algebra homomorphism; what is its kernel?

(d) show that \( \{f,g\} = 0 \iff \text{flow of } X_f \text{ preserves level sets of } g \iff \text{flow of } X_g \text{ preserves level sets of } f; \)

(e) consider a smooth function \( f : M \to \mathbb{R}^k \) where \( f = (f_1, \ldots, f_k) \) and \( \{f_i, f_j\} = 0 \) for all \( i, j \). Show that any regular level set of \( f \) is a coisotropic submanifold, the vector fields \( X_{f_i} \) are tangent to it, and moreover span the tangent space to its isotropic foliation.

Hint: \( \iota_{[X,Y]}\alpha = \mathcal{L}_X \iota_Y \alpha - \iota_Y \mathcal{L}_X \alpha = [\mathcal{L}_X, \iota_Y] \alpha \) for any form \( \alpha \).

Problem 4. Show that if \( \lambda \) is an eigenvalue of a symplectic matrix, then \( 1/\lambda, \bar{\lambda} \) and \( 1/\bar{\lambda} \) are also eigenvalues (with same multiplicity, and in fact with same Jordan form).

Problem 5. Show that \( U(n) \) is the maximal compact subgroup of \( \text{Sympl}(2n) \) by showing that any compact subgroup \( G \) can be conjugated into \( U(n) \).

Hint: Show that there exists a symplectic, symmetric, positive definite matrix \( P \) which is also \( G \)-invariant, i.e. \( A^T P A = P \) for all \( A \in G \).
**Problem 6.** Show that the maps below are isomorphisms

\[ \pi_1(\text{Sympl}(2n)) \longrightarrow \pi_1(U(n)) \xrightarrow{\det} \pi_1(S^1) = \mathbb{Z} \]

where the first map is induced by \( A \to (AA^T)^{-1/2}A \).

**Problem 7.** Assume \( W \) is a linear subspace of \( \mathbb{C}^n \) with the standard hermitian structure.

(a) show that \( W \) is Lagrangian iff \( W^\perp \omega = iW \);

(b) if \( W \) is Lagrangian then \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( W \) iff \( \{e_1, \ldots, e_n\} \) is a unitary basis of \( \mathbb{C}^n \);

(c) conclude that \( U(n) \) acts transitively on the Lagrangian Grassmanian \( \Lambda(n) \) with isotropy subgroup \( O(n) \) and thus

\[ \Lambda(n) \cong_{\text{homeo}} U(n)/O(n) \]

(d) finally show that the map below is an isomorphism

\[ \pi_1(\Lambda(n)) = \pi_1(U(n)/O(n)) \xrightarrow{\det^2} \pi_1(S^1) = \mathbb{Z} \]

(e) what changes if instead we were looking at oriented Lagrangian subspaces?

**Problem 8.** Consider the tautological bundle \( \tau_n \) over \( \Lambda(n) \): the fiber of \( \tau \) at \( W \in \Lambda(n) \) is \( W \). Show that the complexification of \( \tau_n \) is a trivial complex bundle.