Lecture 2. Complex Manifolds

Definition 2.1. A complex $n$-dim manifold $M$ is a smooth manifold such that there exits a collection of charts $\varphi_\alpha : U_\alpha \to V_\alpha$ (from open in $M$ onto open in $\mathbb{C}^n$) covering $M$ so that the change of charts $\varphi_\beta \circ \varphi^{-1}_\alpha$ are holomorphic on the overlap.

Note: All manifolds used are assumed Hausdorff, and have a countable basis of topology. We call $\varphi_\alpha$ ”holo charts” (induce ”holo local coord”).

Note: There could be be different complex structures on the same smooth manifold (change of charts from one to another may not be holo).

Definition 2.2. A map $f : M \to \mathbb{C}$ is holomorphic if $f \circ \varphi^{-1}_\alpha : U_\alpha \to \mathbb{C}$ is holo, i.e. holo in each chart. Similarly, a map $f : M \to N$ between cx manifolds is holomorphic if it is holomorphic in each pair of charts. ** diagram.**

Note: There are no non constant holo fcn on a compact connected manifold. – by maximum principle, which extends to higher dim (work coordinate-wise).

Definition 2.3. A biholomorphism is a bijection $f : M \to N$ such that both $f$, $f^{-1}$ are holo.

Example 2.4. $\mathbb{C}^n$ (affine space) or any (fin dim) cx v. space. Note: $D$ is not biholo to $\mathbb{C}$ (Liouville). Worse: in $n \geq 2$, the ball is not biholo to polydisk (so Riemann Mapping Thm does not extend!) Proof is subtle: they do not have the same autom group (fixing $0$). (one has abelian Lie alg, the other one does not, see Pb 1.1.16).

Example 2.5. If $\Lambda$ is a $2n$ dim lattice in $\mathbb{C}^n$ then $\mathbb{C}^n/\Lambda$ is a complex manifold, diffeo to $(S^1)^{2n}$; eg $\Lambda = \mathbb{Z}^{2n}$. Not all biholo. eg. in $n = 1$, take $\Lambda = \mathbb{Z} \oplus \mathbb{Z} \tau$ ($\tau \notin \mathbb{R}$) get elliptic curve. These come in a 1-dim family.

Example 2.6. Complex projective plane $\mathbb{CP}^n$. Geom: space of complex lines in $\mathbb{C}^{n+1}$. Alg: $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*$ with quotient topology (compact, connected). Points are equivalence classes $[z] = [z_0, \ldots, z_n]$ under scalar multiplication by $\lambda \in \mathbb{C}^*$:

$$(z_0, \ldots, z_n) \cong (\lambda z_0, \ldots, \lambda z_n)$$

Charts: for each $i = 0, \ldots, n$, let

$$U_i = \{ [z_0, \ldots, z_n] \in \mathbb{CP}^n \mid z_i \neq 0 \}$$

and $\varphi_i : U_i \to \mathbb{C}^n$ be

$$\varphi_i([z_0, \ldots, z_n]) = \left( \frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{n}}{z_i} \right)$$

(leave out $z_i/z_i$). Transition maps are easy to write down and check bijective and holomorphic: add $w_j = 1$, then divide all the coordinates by $w_i$, and skip $w_i/w_i$.

Note: Complement of $U_i$ is closed, $H_i = \{ [z] \mid z_i = 0 \}$ called the hyperplane at infinity in the $i$’th coordinate. It is biholomorphic to $\mathbb{CP}^{n-1}$.

Special case: $n = 2$: $\mathbb{CP}^1$ is diffeo to $S^2$ (and hyperplanes are points 0 and $\infty$).

Note: Can do the same for any (fin dim) cx v. space $V$, get $\mathbb{P}(V)$.

HW: Grassmanian: complex $k < n$ planes in $\mathbb{C}^n$.

Example 2.7. (Hopf manifold) Fix $\lambda > 1$, let $\mathbb{Z}$ act on $\mathbb{C}^n \setminus 0$ by multiplication by $\lambda^k$. The action is free and discrete, so the quotient is a complex manifold, diffeo to $S^1 \times S^{2n-1}$. This is an example of a compact complex manifold which does not admit a kahler structure.
Submanifolds (locally linear model);

**Example 2.8.** Zero locus $Z(f)$ of a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is a submanifold if 0 is a regular value. - IFThm extends to holomorphic setting.

**Note:** A regular point is a point where $df$ is onto (transversality). If $f : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic, this is equivalent to the fact that the complex Jacobian $J_C f = (\frac{\partial f}{\partial z})$ is onto. If $f : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic, a regular point is a point the (complex) Jacobian $J_C f$ is onto is called a regular point.

**Example 2.9.** Take homogenous poly $P$, and look at its zero $Z(P)$ in $\mathbb{C}^{n+1}$ descends to vanishing locus $V(P)$ – get subset of $\mathbb{CP}^n$ called analytic var; IFThm (extends in the complex setting) implies that if 0 is a regular value then $V(P)$ is $(n-1)$ dim complex submanifold of $\mathbb{CP}^n$.

E.g. 3 var: degree 1 get a line; deg 3 get a cubic curve in $\mathbb{CP}^2$. Smooth for almost all coef (Sard theorem, or use properties of holomorphic functions)

**Note:** zero locus of poly: smooth except where det$dP = 0$ – poly in coef; iterate.

2.1. **Varieties.** – did very briefly

- defn : an algebraic variety in $\mathbb{CP}^n$ is the zero set of a (finite) collection of homog poly $f_k : \mathbb{C}^{n+1} \to \mathbb{C}$.

- Defn: an analytic subvar of a complex manifold is a CLOSED set which locally is the zero locus of a holomorphic function; smooth pt if can be cut transv. – allows us to deal with sing.

- Skipped: WPThm – locally the zero locus of any holomorphic function is the same as the zero locus of a Weiestrass poly

**Example 2.10.** Complex Lie Groups: these are both groups and complex manifolds and group operations are holomorphic (i.e. multiplication and inverse). eg. $GL(n, \mathbb{C}), SL(n, \mathbb{C}), Sp(n, \mathbb{C})$ but not $U(n)$ (no compact cx other than tori)

2.2. **Differential Calculus on Complex Manifolds.** Next we describe an alternative definition of complex manifolds: an almost complex structure which is integrable. In the process, we also define Dolbeault cohomology.

If $X$ is a complex manifold, $TX$ is a complex vector bundle (and in fact holomorphic– to be defined later).

**Definition 2.11.** An almost complex structure on a real manifold $M$ is a vector bundle endomorphism $J \in End(TM)$ such that $J^2 = -id$.

**Note:** In particular, $TX$ is a complex vector bundle.

**Note:** If $X$ is a complex manifold, multiplication by $i$ in each coordinate chart induces a well-defined almost complex structure on $X$. (i.e. independent of chart). If $z = (z_1, \ldots, z_n)$ are holomorphic coordinates, write $z_k = x_k + iy_k$ get real coordinates $(x, y)$ in which

$$J \left( \frac{\partial}{\partial x_k} \right) = \frac{\partial}{\partial y_k}, \quad J \left( \frac{\partial}{\partial y_k} \right) = -\frac{\partial}{\partial x_k}.$$ 

This is well defined, independent of chart.

**Note:** As we shall see below, not all almost complex structures come from a complex structure (i.e. can be integrated).

**Note:** If $J$ is an almost complex structure, then its e-values are $\pm i$ and we can diagonalize $J$, after we complexify $TM$, i.e. we consider $TM^C = TM \otimes \mathbb{R} \mathbb{C}$;
Lemma 2.12. Let $T^{1,0}M$ (resp $T^{0,1}M$) the $i$ eigenbundle (resp $-i$ eigenbundle) of $J$. Then $TM = T^{1,0}M \oplus T^{0,1}M$ and

$$T^{1,0}M = \{X - iJX | X \in TM\} \quad T^{0,1}M = \{X + iJX | X \in TM\}$$

Proof. Define $T^{1,0}M$ and $T^{0,1}M$ by the display eq. Extend $J$ to the complexification as a $\mathbb{C}$-linear endomorphism, i.e. by $J(X + iY) = JX + iJY$ for all $X, Y \in T_pM$. Then

$$J(X - iJX) = JX - iJ^2X = JX + iX = i(X - iJX).$$

The other statement follows using $2X = (X - iJX) + (X - iJX)$. \qed
Lecture 3. Tensor Calculus and Integrability

This week: integrability, \((p, q)\)-forms, Dolbeault cohomology.

**Note:** (Lin alg) For us, a complex structure on a real vector space \(V\) is simply an \(\mathbb{R}\)-linear endomorphism \(J \in \text{End}(V)\) such that \(J^2 = -\text{id}\). If \((V, J)\) is a complex vector space, its "complex conjugate" is \(V = (V, -J)\). The \(\pm i\)-eigenspaces of \(J\) canonically decompose the complexification \(V \otimes_{\mathbb{R}} \mathbb{C}\) as a direct sum

\[
V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}\ 	ext{ where } V^{1,0} = \overline{V^{0,1}} \cong (V, J)
\]

or more precisely the maps \(X \mapsto \frac{1}{2}(X \mp iJX)\) induce two natural \(\mathbb{C}\)-linear isomorphisms

\[
(V, J) \cong (V^{1,0}, i) \quad \text{and} \quad (V, J) \cong (V^{1,0}, -i).
\]

** Should have talked about projections \(T_{\mathbb{C}}M \to T^{1,0}M\) etc. ** Extends to \(V^*\) and tensors, etc.

**Note:** The complexification of a real vector space is naturally a complex vector space together with a complex conjugation \(X + iY \mapsto X - iY\). If \(V\) is a complex vector space, its complexification is naturally a quaternionic vector space \(\mathbb{H} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}\).

Assume \(J\) is an almost complex structure (i.e. \(J\) is a complex structure on \(TM\)) then the \(\pm i\)-eigenbundles \(J\) give a natural decomposition

\[
TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M \quad \text{where } T^{1,0}M = \overline{T^{0,1}M} \cong (TM, J)
\]

which extends to differential forms. Denote by \(\Lambda^1 M = T^* M\) the space of 1-forms, \(\Lambda^1_{\mathbb{C}} M\) its complexification, which decomposes into

\[
\Lambda^{1,0} M = \{ \eta \in \Lambda^1_{\mathbb{C}} M \mid \eta(X) = 0 \ \forall X \in T^{0,1} M \} = \{ \tau - i\tau \circ J \mid \tau \in \Lambda^1 M \}
\]

\[
\Lambda^{0,1} M = \{ \eta \in \Lambda^1_{\mathbb{C}} M \mid \eta(X) = 0 \ \forall X \in T^{1,0} M \} = \{ \tau + i\tau \circ J \mid \tau \in \Lambda^1 M \}.
\]

Using \(\Lambda^k(E \oplus F) \cong \bigoplus \Lambda^i E \otimes \Lambda^{k-i} F\), this extends to a decomposition of (complexified) \(k\)-forms into \((p, q)\) forms where \(p + q = k\):

\[
\Lambda^k C M \cong \bigoplus_{p+q=k} \Lambda^{p,q} M \quad \Lambda^{p,q} M = \Lambda^p(T^{1,0} M)^* \otimes \Lambda^q(T^{0,1} M)^*
\]

(Smooth) sections of \(\Lambda^{p,q}\) are denoted \(A^{p,q} = \Gamma(\Lambda^{p,q})\) (the notation \(\Omega\) in cx geometry is reserved for holomorphic sections -to be defined).

**Definition 3.1.** Assume \(M\) is a complex manifold. Its holomorphic and respectively antiholomorphic tangent bundle is

\[
\tau M = T^{1,0} M \quad \text{while} \quad \overline{\tau} M = T^{0,1} M
\]

We denote \(\Omega_M = \Lambda^{1,0} M\) and similarly \(\Omega^p_M\) the space of \((p,0)\)-forms.

**Note:** In holomorphic local coordinates \((z_1, \ldots, z_n)\) on \(M\), \(\left\{ \frac{\partial}{\partial z_k} \right\}_k\) is a local basis for \(\tau M\) while \(\left\{ \frac{\partial}{\partial \overline{z}_k} \right\}_k\) is a local basis for \(\overline{\tau} M\). Similarly, \(\{dz_k = dx_k + idy_k\}\) is a local basis for \(\Lambda^{1,0} M\) while \(\{dz_k = dx_k - idy_k\}\) is a local basis for \(\Lambda^{0,1} M\). A \((p, q)\) form can be written as

\[
\eta = \sum_{i,j} \eta_{i,j} dz^i \wedge d\overline{z}^j \quad \text{where}
\]

\[
dz^i \wedge d\overline{z}^j = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge \overline{z}_{j_q}
\]
with \( i_1 < \cdots < i_p \) and \( j_1 < \cdots < j_q \) is a local basis of \((p, q)\) forms. The coef are smooth \( \mathbb{C}\)-valued functions.

3.1. **Integrable almost complex structures.** If \( J \) is an almost complex structure, the Nijenhuis tensor is defined by the formula

\[
\]

**Note:** If \( J \) is coming from a complex structure, then \( N_J \equiv 0 \): check on local basis \( \left\{ \frac{\partial}{\partial x^k}, \frac{\partial}{\partial y^k} \right\} \) coming from real/imag part of holomorphic coordinates. The converse is the famous Newlander-Nirenberg Theorem. When \( J \) is real analytic, it follows from Frobenius Theorem. The hard part is to show that the condition \( N_J \equiv 0 \) implies \( J \) is real analytic, a regularity result. In fact,

**Proposition 3.2.** Assume \( J \) is an almost complex structure on a (real) manifold \( M \). TFAE:

(a) \( J \) is a complex structure
(b) \( N_J = 0 \)
(c) \( T^{0,1} \) is an integrable distribution, i.e. \([X,Y] \in T^{0,1}M \) for all \( X, Y \in T^{0,1}M \).
(d) \( d(A^{1,0}) \subset A^{2,0} \oplus A^{1,1} \).
(e) \( d(A^{p,q}) \subset A^{p+1,q} \oplus A^{p,q+1} \) for all \( p, q \).

**Proof.** Easy parts first: (b) \( \iff \) (c) Take \( X, Y \) local vector fields on \( M \), regard them in \( T^{0,1}M \) and let \( Z = [X + iJX, Y + iJY] \); calculate \( Z - iJZ = N[X,Y] - iJN[X,Y] \). So \( Z \in T^{0,1} \iff N_J(X,Y) = 0 \).

(c) \( \iff \) (d) take \( \omega(1,0) \) form. The \((0,2)\) comp of \( d\omega \) vanishes iff it evaluates as 0 on two \((0,1)\) vectors \( Z, W \). Calculate

\[
d\omega(Z,W) = Z\omega(W) - W\omega(Z) - \omega([Z,W]) = -\omega([Z,W])
\]

So this part vanishes for all \( \omega(1,0) \) forms iff \([Z,W] \in T^{0,1}M \) for all \( Z, W \in T^{0,1}M \).

(d) \( \iff \) (e) use complex conj and Leibnitz rule

Finally, we also saw that (a) \( \implies \) (b). The implication (c) \( \implies \) (a) is NN thm; under the extra assumption that \( J \) is real analytic follows from Frobenius theorem. \( \square \)

**Remark 3.3.** Any almost complex structure on a Riemann surface (i.e. a 2 real-dimensional manifold) is integrable (by dim reasons: \( N_J(X,X) = N_J(X,JX) = 0 \) for all \( X \)). Special case: any closed surface, eg. \( M = S^2 \).

**Note:** For homological reasons (involving properties of characteristic classes), the only spheres that could admit almost complex structure are \( S^2 \) or \( S^6 \). One can construct a (non-integrable) almost complex structure on the unit sphere \( S^6 \) using the cross product in \( \mathbb{R}^7 \), regarded as imaginary octonions \( J_p(v) = p \times v \) for all \( p \in S^6 \) and \( v \in TS^6 \) i.e. \( <p,v> = 0 \). It is still an open conjecture whether \( S^6 \) admits a complex structure.

**Remark 3.4.** For an almost complex manifold, the top Chern class is the Euler class, so it would be 2 for even dim spheres. If a spheres \( S^{4k} \) were to have an almost complex structure then the Pontriagin class \( p_k \) would be \( 2c_{2k} = 4 \) contrad since \( TS \) is stably trivial. To rule out spheres \( S^{2n} \) for \( n \geq 4 \) use Bott integrality result that \( c_n(E)/(n-1)! \in H^{2n}(S^{2n}) \) must be an integral class for any ex v. bd on the sphere.

**Remark 3.5.** (c) implies that when \( G \) is a complex Lie group, then its Lie algebra is a complex Lie algebra. We also expect that if \( M \) is a compact complex manifold, its automorphism group \( \text{Aut} M \) is a complex Lie group. Its Lie algebra \( T_{id}\text{Aut}(M) \) (i.e. space of infinitesimal automorphisms) should correspond to "holomorphic" v. fields.
**Definition 3.6.** Assume $M$ is a complex manifold. A holomorphic vector field is a vector field $Z = X - iJX \in \Gamma(T^{1,0}M)$ such that the flow of $X$ consists of holomorphic maps, i.e. $L_X J = 0$.

**Note:** As we shall see, an equivalent description of holomorphic vector fields is as holomorphic sections of $\tau M$ (a holomorphic vector bundle). For now, that means that in local holomorphic coordinates $Z = \sum k a_k(z) \frac{\partial}{\partial z_k}$ where the coefficients $a_k$ are holomorphic.

### 3.2. Dolbeault cohomology.

Assume $M$ now is a complex manifold. Then (e) provides a decomposition
\[
d = \partial + \bar{\partial}
\]
where $\partial : A^{p,q} \to A^{p+1,q}$ and $\bar{\partial} : A^{p,q} \to A^{p,q+1}$ into two differential operators (i.e. satisfy Leibnitz rule:
\[
\partial(\omega \wedge \eta) = \partial \omega \wedge \eta + (-1)^{p+q=deg \omega} \omega \wedge \partial \eta
\]
with the following properties
\[
\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial \bar{\partial} + \bar{\partial} \partial = 0
\]

In holo coordinates, $\partial = \sum dz_k \wedge \frac{\partial}{\partial z_k}$ i.e. if $\eta = \sum \eta_I J dz^I \wedge dz^J$, just differentiate the coef. Draw the Hodge diamond. Top $A^{0,0}$, left $\partial$, right $\bar{\partial}$.

**Definition 3.7.** The $(p,q)$-Dolbeault cohomology group $H^{p,q}(M)$ is the $q$-homology group of the complex $(A^{p,q}, \bar{\partial})$ i.e. the space of $\bar{\partial}$-closed modulo $\bar{\partial}$-exact $(p,q)$ forms.
Plan: finish Dolbeault and discuss holo v. bd; next: add metrics.

Recall: $H^{p,q}(M)$ is the group of $\partial$-closed $(p, q)$-forms modulo $\partial$-exact ones. The Hodge numbers are $h^{p,q}$ the ranks of these.

**Example 4.1.** A $\partial$-closed $(0, 0)$-form is a holomorphic function. Complex manifolds have lots of locally defined holomorphic functions, thus $H^{0,0}(M)$ could be infinite dimensional! Unlike the real case, if $M$ is compact and connected, the only holo functions are constant (by maximum principle), i.e. $H^{0,0}(M) = \mathbb{C}$.

**Note:** If $f : M \to N$ is a holomorphic map between complex manifolds, it induces a (natural) homomorphism $f^* : H^{p,q}(N) \to H^{p,q}(M)$ on Dolbeault cohomology.

Get invariant of the COMPLEX structure – Hirzebruch asked if $h^{p,q}$ are homeo invar (need to preserve orient); contraexamples exist, but not easy to construct (two surfaces that are homeo, but not diffeo and have different hodge numbers!); (in 1 and 2 dim, $h^{p,q}$ are determined by top invar plus orientation=$b_2^+$); (and invar under defm for Kahler ones)

**Note:** Does not work for almost complex manifolds $M$: could define a differential operator $\overline{\partial} : A^{p,q} \to A^{p,q+1}$ but $\overline{\partial}^2 = 0 \iff J$ is integrable! (explained below)

Why Dolbeault/sheaf cohomology? - invariant of the complex structure; also recipient for local deformations/obstructions to existence of a certain structure eq cx structure (or ML theorem etc).

Poicare Lemma; Digression: say we want to solve the equation

$$\overline{\partial} \beta = \alpha$$

on a complex manifold. Local obstruction: $\partial \alpha = 0$ (closed). Poincare Lemma: if $\overline{\partial} \alpha = 0$ then locally can always solve the equation; globally maybe not: local solutions may not patch together. – best kept track by Cech cohomology (with coef in a v. bundle/sheaf).

Clearly global obstruction: $[\alpha] = 0$ in $H^{p,q}$. Hodge theory will say that if the homological obstruction vanishes, we can always solve this eq globally. So to solve this type of eq, there are two powerful approaches:

- Sheaf theory: work with locally defined holo functions, forms, etc; worry about how they patch.
- Sobolev spaces: work with globally defined non-holomorphic functions, forms, etc; worry about regularity.

**Lemma 4.2** (Dolbeault Lemma). Any $\overline{\partial}$-closed $(0,1)$-form is locally $\overline{\partial}$-exact.

In fact, $H^{p,q}(\text{polydisk}) = 0$ for all $q > 0$.

**Proof.** (Outline– see §1.3 in Huybrechts) In 1-variable, if $g$ is a smooth function on a neighborhood of the CLOSED unit disk $\overline{D}$, then

$$f(z) = \frac{1}{2\pi i} \int_D \frac{g(w)}{w-z} dw \wedge d\overline{w}$$

is a smooth function on $D$ and satisfies $\overline{\partial} f = g(z) d\overline{z}$ (a version of Cauchy integral formula).

Proof: best to chop it off by taking out nbd of $z$ as well as nbd of $\partial D$. Note that in 1-var $\beta = g(z) d\overline{z}$ is always $\overline{\partial}$-closed.

\[\text{same is true for polycylinders (products of disks and } \mathbb{C}\text{'s) or product of disks and punctured disks.}\]
In several variables, use also induction on the number of $dz_i$'s involved: if $\alpha = \sum_{j \leq k} \alpha_j dz_j$ is $\bar{\partial}$-closed, then its coefficients must be holomorphic in the remaining variables $z_{k+1}, \ldots, z_n$. "Integrate" the coef of $dz_k$ w.r.t. $z_k$ to find a function $\beta$ such that $\alpha - \bar{\partial}\beta$ is a linear combination of $dz_1, \ldots, dz_{k-1}$. (This works on a neighborhood of a closed polydisk.)

This argument extends to $(0,q)$ forms by filtering $A^{0,q}$ by $A^{0,q}_k$ generated by $dz_1, \ldots, dz_k$; reduce it to proving that if $\alpha \equiv 0 \mod A_k$ is $\bar{\partial}$ closed, then there exits a $\beta$ such that $\alpha = \bar{\partial}\beta \mod A_{k-1}$. The result on $(0,q)$ implies automatically the one for $(p,q)$, as the $dz_I$ terms come along unchanged.

Finally, to get the vanishing of cohomology, exhaust $D$ by closed polydisks, and use a limiting argument; to get uniform convergence, reduce it to an extension argument: can arrange so that the extensions either agree on smaller closed polydisk or else differ by at most $2^{-m}$ so we get uniform convergence.

\[\Box\]

**Remark 4.3.** Given a diff equation $Df = g$, one way to solve it is to find a kernel $k(x,y)$ in two variable, i.e. a solution of $D_x k(x,y) = \delta_{xy}$ the "delta function". Then $f(x) = \int k(x,y)g(y)dy$ satisfies

\[D_x f = \int D_x k(x,y)g(y)dy = \int \delta_{x,y}g(y)dy = g(x).\]

In our case, $k(z,w) = \frac{1}{2\pi i} \frac{dw}{w-z}$ is the Cauchy kernel, which satisfies $\bar{\partial}_z k = 0$ on $\mathbb{C}^*$.

Poincare Lemma has the following conseq used when discussing Kahler forms:

**Lemma 4.4** (Local $i\partial\bar{\partial}$-Lemma). Assume $M$ is a complex manifold and $\omega$ is a real $2$-form on $M$ which is of type $(1,1)$. Then $d\omega = 0$ iff locally $\omega = i\partial\bar{\partial}u$ for some locally defined real function $u$.

**Proof.** If $\omega = i\partial\bar{\partial}u$ then $d\omega = 0$ since $d(i\partial\bar{\partial}) = i(\partial + \bar{\partial})\partial\bar{\partial} = 0 + 0$.

Conversely, if $\omega$ is a closed real form of type $(1,1)$, then from the $d$-Poincare Lemma $\omega = d\tau$ for some real $1$-form. Decompose $\tau$ into $(1,0)$ and $(0,1)$, where $\tau^{1,0} = \bar{\tau}^{0,1}$ since real. Decomposing into types $d\tau$ we get

\[\omega = d\tau = \bar{\partial}\tau^{0,1} + (\partial\tau^{0,1} + \bar{\partial}\tau^{1,0}) + \partial\tau^{1,0}\]

so $\bar{\partial}\tau^{0,1} = 0$, and $\omega = \partial\tau^{0,1} + \bar{\partial}\tau^{1,0}$. But by $\partial$-Poincare Lemma, $\tau^{0,1} = \bar{\partial} f$ so by complex conj $\tau^{1,0} = \bar{\partial} f$. Plugging in,

\[\omega = \partial\tau^{0,1} + \bar{\partial}\tau^{1,0} = \partial\bar{\partial} f + \bar{\partial}\partial f = i\partial\bar{\partial}(2\text{Im } f)\]

A holomorphic $p$-form is a $(p,0)$-form $\eta$ such that $\partial\bar{\eta} = 0$, i.e. whose coefficients are holomorphic in local (holo) coordinates. Equivalently, it is a holomorphic section in a holomorphic bundle, defined next.

4.1. **Holomorphic vector bundles.** Two defn: one is a bundle for which the local trivializations are holo; the other is a complex v. bd with a $\bar{\partial}$-operator on its sections such that $\bar{\partial}^2 = 0$.

Assume $M$ is a complex manifold. A **holomorphic vector bundle** is a complex vector bundle $\pi : E \to M$ for which local trivializations can be chosen so that the transition functions are holomorphic (holomorphic local trivializations). That is, there exits covering by holo charts $U_\alpha$ of $M$ and local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ (diffeos compatible with the
projections) such that the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{C}^r)$ are holomorphic, where

$$\varphi_\alpha \circ \varphi^{-1}_\beta(z, e) = (z, g_{\alpha\beta}(z)e)$$

*** from $\beta$ to $\alpha$ *****, and $r = \text{rank } E$.

**Example 4.5.** The trivial bundle $M \times \mathbb{C}^r$. The holomorphic tangent bundle $\tau M$ is holomorphic; cotangent bundle $\Omega_M = (\tau M)^* = \Lambda^{1,0}$ (its dual) and more generally the bundle $\Lambda^{p,0} = \Lambda^p(\Omega_M) = \Omega^p_M$ are holomorphic bundles: holomorphic coordinates provide local trivializations (using basis $\frac{\partial}{\partial z}$); change of holomorphic coordinates changes this basis by multiplication by a matrix with entries are holomorphic functions. For example, chain rule implies that the change of basis

$$dz_I = \sum_J \frac{\partial z_I}{\partial w_J} dw_J$$

has holomorphic coefficients (polynomials in entries of complex Jacobian $\frac{\partial z}{\partial w}$ of change of variables).

The **canonical line bundle** is $K_M = \Omega^n_M = \wedge^n \Omega_M$ where $n = \text{dim } M$.

**Note:** (Skip) A complex (or holomorphic) bundle is uniquely determined (up to isomorphism) by its transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$ which are smooth (respectively holomorphic) and subject to the following constraints: $g_{\alpha\alpha} = 1$ and cocycle condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on overlaps.
Lecture 5. **Holomorphic vector bundles**

REVIEW Ch 4-6 of Moroianu: principal v. bd, connections, curvature.

5.1. **Examples.** Holomorphic vector bundles: tangent $\tau M$, cotangent $\Omega M$, $\Lambda^p, 0 = \Omega^p_M$ (called the bundle of holo $p$-forms) but NOT $\Lambda^{p,q}$ for $q \neq 0$.

**Note:** the total space of a holomorphic v bd is a complex manifold; converse is also true, see HW problem.

Careful: isom as cx v bd NOT the same as isom as holo v. bd: there is a nontrivial holo line bundle on an elliptic curve which is trivial as a complex vector bundle.

**Principle:** natural constructions on real v. bds extend to cx v. bd and holo v. bds: e.g.

- direct sum $E \oplus F$,
- tensor product $E \otimes F$,
- exterior powers $\Lambda^k E$,
- determinant line bundle $\det E = \wedge^r E$ where $r = \text{rank} E$,
- dual v. bd $E^*$
- pull back bundle $f^* E$ under HOLO $f : N \to M$; eg restrict to cx smfld $E|_V = \iota_V^* E$, $\iota_V : V \hookrightarrow M$.

If $V$ is a complex submanifold in $M$, the normal bundle $N$ to $V$ in $M$ is holomorphic (defined as a quotient of two holo bds/coker of a CONSTANT rank map), i.e. SES

$$0 \to TV \to TM|_V \to N \to 0$$

Ker and cokernel if constant rank (else get sheaves).

**CAREFUL:** such SES of cx v. bds always SPLITS – uses orthogonal proj wrt some metric – but HOLO splitting may NOT exist. – discuss later the obstruction and example: rank 2 nontrivial bundle $0 \to \mathbb{C} \to E \to \mathbb{C} \to 0$ on an elliptic curve). However, SES of holo v. bd behaves well under taking det, i.e. $\det E = \det F \otimes \det G$.

Get adjunction formula: if $V$ is a cx smfld of $M$ with normal bundle $N$ then

$$\det TM|_V = \det TV \otimes \det N_D$$

(usually stated using the duals, the canonical bundles).

**Example 5.1.** The **tautological line bundle** $\tau$ on $\mathbb{CP}^n$. Geom: its fiber at $l \in \mathbb{CP}^n$ is the complex line $l \subset \mathbb{C}^{n+1}$. Alg: let $\tau$ be the subset of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ defined by

$$\tau = \{(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid z \in l\}$$

with the projection $\pi$ onto the first factor (the fiber of $\pi$ at $l$ is naturally isomorphic to $l$). This is a holomorphic line bundle: use the standard charts $U_i$ on $\mathbb{CP}^n$ and local trivializations $\psi_i : \pi^{-1} U_i \to U_i \times \mathbb{C}$ defined by

$$\psi_i(l, z) = (l, z_i)$$

The transition functions on $U_i \cap U_j$ – from $U_j$ to $U_i$ – are simply

$$g_{ij}([z_0, \ldots, z_n])(v) = \frac{z_i}{z_j} v$$

for all $v \in \mathbb{C}$

which are holomorphic. e.g.: when $n = 2$ we have 2 charts $z = [z, 1]$, $w = [1, w]$ on the base with $w = 1/z$. The transition function is $g(w)(v) = w^{-1}v$, $w \in \mathbb{C}^*$ (from $w$-chart to $z$ chart).

*** picture ***
Example 5.2. One can show that $K_{CP^n}$ is isomorphic to the $(n+1)$ tensor product of $\tau$—see Moroianu. e.g. $n = 2$ (HW problem) need to change basis from $dz$ to $dw = dz^{-1} = -z^{-2}dz$, so bundle is $\tau^2 = \tau \otimes \tau$, see HW problem. (Note: can absorb - sign by changing the trivialization on the $w$ side; for line bds transition fcn of tensor prod is simply product of transition fcn). More on this later, and relation to Cech cocycles.

Remark 5.3. A complex v. b is uniquely determined (up to isom) by the transition functions $g_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \rightarrow GL(r, \mathbb{C})$. These satisfy the Cech cocycle condition

$$g_{\alpha \beta}g_{\beta \gamma} = g_{\alpha \gamma}$$

and determine a class $[g_{\alpha \beta}]$ in Cech cohom of $M$ with coef in $G = GL(r, \mathbb{C})$. This is a cohomology theory based on covers of $M$.

5.2. Dolbeault Cohomology with Coef in $E$. Assume $E \rightarrow M$ is a holo v. bd. Consider $\Lambda^{p,q}(E) = \Lambda^{p,q}M \otimes E$ the space of $(p,q)$ forms on $M$ with values in $E$. Space of (smooth) sections denoted $A^{p,q}E = \Gamma(M; \Lambda^{p,q}E)$, and comes with a $\partial_E$ operator defined in a local holomorphic trivialization/frame $\{e_k\}$ of $E$ by

$$\partial_E \left( \sum_k \sigma_k \otimes e_k \right) = \sum_k (\partial \sigma_k) \otimes e_k$$

$$A^{p,0}E \xrightarrow{\partial} A^{p,1}E \xrightarrow{\partial} A^{p,2} \xrightarrow{\partial} \ldots \tag{5.1}$$

Since $\partial_E^2 = 0$, can define Dolbeault cohomology groups $H^{p,q}(M; E)$ on $M$ with values in $E$:

$$H^{p,q}(M; E) = H^q(A^{p,\bullet}E, \partial_E) = \frac{\partial_E - \text{closed } (p,q) \text{ forms}}{\partial_E - \text{exact } (p,q) \text{ forms}}$$

As before, this is an invariant (up to isom) of the holomorphic v. bd $E \rightarrow X$.

A holomorphic section $s$ of $E \rightarrow M$ is a section $s : X \rightarrow E$ (i.e. $\sigma \circ s = id$) which is holomorphic i.e. $\partial_E s = 0$. Equivalently, if $\{e_k\}$ is a local holomorphic frame for $E$, $s = \sum \sigma_k e_k$ where coef $\sigma_k$ are holomorphic functions.

The space of holomorphic sections of $E$ is $H^{0,0}_\partial(M; E)$, also denoted $H^0(M; E)$. (skip—notation comes from Cech cohomology—see HW). Holo sections of $\Lambda^{p,0} = \Omega^p_M$ are $H^{p,0}(M) = H^0(M; \Omega^p)$. Locally $E$ has many holo sections (e.g holo fcn) but may not patch globally.

Note: if $E = M \times \mathbb{C}$ we get $H^{p,q}(M; \mathbb{C}) = H^{p,q}(M)$.

Example 5.4. The tautological bundle has no holo sections; its dual has an $n+1$-dimensional space of sections, they are homog linear functions on the coordinates.

Note: By Poincare $\partial$-Lemma, on a polydisk $U$ (i.e. locally), the cohom of (5.1) vanishes except in degree 0, where $H^{p,0}(U, E)$ is the space of holo $p$-forms on $U$ with values in $E$; e.g. $p = 0$ these are holo sections of $E$. Globally they could all be nontrivial and keep track of obstruction to solving $\partial_E^2 = 0$.

Note: (skip?) The Poincare $\partial$-Lemma (and Leray Thm) implies Dolbeault Theorem

$$H^{p,q}(M) \cong H^q(M; \Omega^p_M); \quad H^{p,q}(M; E) \cong H^q(M; E \otimes \Omega^p_M).$$

(the RHS is Cech cohomology; Poincare Lemma implies $A^{p,q}$ is an acyclic resolution of $\Omega^p_M$; Leray Thm about acyclic covers implies this calculates Cech cohom).
5.3. **Integrability of complex vector bundles.** Equivalently, a holomorphic vector bundle is a complex vector bundle $E \rightarrow M$ together with a differential operator $\overline{\partial} : A^{p,q}E \rightarrow A^{p,q+1}E$ satisfying Leibniz rule and such that $\overline{\partial}^2 = 0$.

It is enough to have it from sections of $E$ to $(0,1)$-forms with values in $E$ (then extend it on $A^{p,q}(E)$ by Leibniz rule).

**Theorem 5.5.** Assume $E \rightarrow M$ is a complex v. bd on a complex manifold. A holomorphic structure on $E$ is uniquely determined by a $\mathbb{C}$-linear operator $\overline{\partial}_E : A^0(E) \rightarrow A^{0,1}(E)$ satisfying Leibniz rule and the integrability condition $\overline{\partial}_E^2 = 0$.

**Proof.** (Outline) We want to show that $\overline{\partial}_E^2 = 0$ iff locally one can chose a frame $\{e_k(z)\}$ of $E$ consisting of local holomorphic sections of $E$ (i.e. solutions of $\overline{\partial}_E \sigma = 0$). One direction is clear. For the other one, the argument goes as follows: choose some local basis/frame, show that (after possibly shrinking) can change basis to one consisting of holomorphic sections; these define the holomorphic trivializations.

Choose a local frame $\{e_k\}$ of $E$ over $U$, so

$$\overline{\partial} e_k = \sum_j \tau_{kj} \otimes e_j$$

where the coef $\tau$ are local $(0,1)$ forms on $U$, i.e. $\tau = (\tau_{kj})$ is a $\mathfrak{gl}_r(\mathbb{C})$-valued $(0,1)$ form. We want to find a change of basis i.e. a function $f : U \rightarrow GL_r(\mathbb{C})$ such that

$$\overline{\partial} f + f \tau = 0.$$  

(5.3)

If we can do that, the sections $\sigma_l = \sum_k f_k e_k$ are holomorphic: $\overline{\partial}_E \sigma_l = \sum \overline{\partial} f_k e_k + f_k \overline{\partial} e_k = 0$, thus give the desired trivialization of $E$ (so change of trivialization are holo).

But that is yet another PDE, and we are looking for local solutions; there is an obstruction to solving it, but if obstruction vanishes, we can locally solve.

The condition $\overline{\partial}_E^2 = 0$ and (5.6) implies that

$$\overline{\partial} \tau = \tau \wedge \tau \quad \text{i.e.} \quad \overline{\partial} \tau_{ij} = \sum_l \tau_{il} \wedge \tau_{lj}.$$  

(5.4)

(RHS is not zero unless rank=1). But this is precisely the obstruction to (locally) solving (5.8) (ex). If $n = 1$ (over RS) the obstruction vanishes by dim reasons $A^{0,2} = 0$. If also $r = 1$ (line bundle) (5.8) easy to solve (locally). We could look for soln $f = \exp F$ where $F : U \rightarrow \mathfrak{gl}_r(\mathbb{C})$ solves

$$\overline{\partial} F + \tau = 0.$$  

BUT THIS WORKS only if $\tau \wedge \tau = 0$ (obstruction to this eq is $\overline{\partial} \tau = 0$).

To deal with the general case, use a linear version of NN integrability Thm on $U \times \mathbb{C}^n = V$ to LOCALLY solve this. (see Moroianu, or Kobayashi ”Differential Geometry of Complex Vector Bundles”, Prop 1.3.7.)

**Note:** Should remind you of curvature, connection 1-form; principal v. bd– brush up see Ch 4-6 of Moroianu.

5.4. **Correction:** **Integrability of complex vector bundles.** Last result, integrability of $E \rightarrow M$ was rushed, and I made two mistakes:

1. Leibnitz rule on $(p,q)$-forms has a SIGN in it!! i.e.

$$\overline{\partial}_E(\omega \otimes \sigma) = \overline{\partial} \omega \otimes \sigma + (-1)^{p+q} \omega \otimes \overline{\partial}_E \sigma$$
where $\omega$ is $(p,q)$ form on $M$ and $\sigma$ a section of $E$. So

$$\overline{\partial}_E^2 = 0 \iff \overline{\partial} \tau - \tau \wedge \tau = 0 \quad \text{since } \tau \text{ is a } (0,1)-\text{ form.}$$

(2) took illegal shortcut in the last proof, which works ONLY if $\tau \wedge \tau \equiv 0$ (which may NOT hold in general). Recall that we wanted to (locally) solve

$$\overline{\partial} f + f \tau = 0; \quad \text{the obstruction is } \overline{\partial} \tau - \tau \wedge \tau = 0. \quad (5.5)$$

If we look for solution $f = \exp(F)$ then (5.5) becomes

$$\overline{\partial} F + \tau = 0; \quad \text{obstruction is } \overline{\partial} \tau = 0.$$ 

This is OK only if $\tau \wedge \tau \equiv 0$. In general, $\tau$ is a $r \times r$ matrix whose entries are $(0,1)$ forms on $U$, so $\tau \wedge \tau$ maybe nonzero.

Note: There are two special cases when $\tau \wedge \tau \equiv 0$:

(a) if the base $M$ is 1-dim (a Riemann surface) since $(0,2)$-forms vanish by dim reasons.

(a) if $E$ is a LINE bundle, since $\tau_{11} \wedge \tau_{11} = 0$, or more generally when $\tau$ is a diagonal matrix (i.e. $E$ is a direct sum of line bundles, and $\overline{\partial}_E$ respects this decomposition).

In such cases, the shortcut proof works. If the coef $\tau$ were real analytic, the linear equation (5.5) can be solved by power series, e.g Frobenius Thm. (In fact, $\overline{\partial} f + f \tau = 0$ is an involutive system iff $\overline{\partial} \tau - \tau \wedge \tau = 0$). In the general case, one can use (a linear version of) the Newlander-Nirenberg integrability Thm (regularity is slightly easier to prove for the linear PDE).

The proof in Moroianu’s book (inspired by Prop 1.3.7 in Kobayashi’s book) uses instead the usual NN Thm on $U \times \mathbb{C}^r$. The basic idea behind it is as follows: regard $\overline{\partial} + \tau$ as a twisted $\overline{\partial}$-operator on $V = U \times \mathbb{C}^r$ (a trivial bundle over $U$). Equation (5.5) intrinsically means that that are looking for a gauge transformation $f$ of the bundle which converts $\overline{\partial} + \tau$ to the usual $\overline{\partial}$ ($\tau$ is essentially a connection 1-form). It is enough to find a coordinate change $\varphi$ on the total space which is the identity along the zero section $U \times 0$ and converts $\overline{\partial} + \tau$ to the usual $\overline{\partial}$. For that, define an almost complex structure on $V$ by prescribing the desired basis of $\Lambda^{0,1}V$ (depending on $\tau$), and show it is integrable because $\overline{\partial}_E^2 = 0$ (see complete proof below).

**Theorem 5.6.** Assume $E \to M$ is a complex v. bd. on a complex manifold. A holomorphic structure on $E$ is uniquely determined by a $\mathbb{C}$-linear operator $\overline{\partial}_E : A^0(E) \to A^{0,1}(E)$ satisfying Leibnitz rule and the integrability condition $\overline{\partial}_E^2 = 0$.

**Proof.** (Outline) We want to show that $\overline{\partial}_E^2 = 0$ iff locally one can chose a frame $\{e_k(z)\}$ of $E$ consisting of local holomorphic sections of $E$ (i.e. solutions of $\overline{\partial}_E \sigma = 0$). One direction is clear. For the other one, the argument goes as follows: choose some local basis/frame, show that (after possibly shrinking) we can change basis to one consisting of holomorphic sections; these define the holomorphic trivializations (by Leibnitz rule).

Choose a local frame $\{\sigma_k\}$ of $E$ over $U$, so

$$\overline{\partial}_E \sigma_k = \sum_{j=1}^r \tau_{kj} \otimes \sigma_j \quad \text{i.e. } \overline{\partial} \sigma = \tau \otimes \sigma \quad (5.6)$$

where $\sigma$ is a column vector $(\sigma_1, \ldots, \sigma_r)^T$, the coef $\tau_{kj}$ are local $(0,1)$ forms on $U$, so $\tau = (\tau_{kj})$ is a $\mathfrak{gl}(r, \mathbb{C})$-valued $(0,1)$ form on $U$.
We know $\overline{\partial}^2 = 0$, thus $\overline{\partial}^2 \sigma_k = 0$ which by (5.6) and Leibnitz rule gives
\[
\overline{\partial} \tau - \tau \wedge \tau = 0 \quad \text{i.e.} \quad \overline{\partial} \tau_{ij} = \sum_{l=1}^{r} \tau_{il} \wedge \tau_{lj}, \tag{5.7}
\]

**Careful:** $\tau \wedge \tau$ may NOT be zero in general.

We want to find a change of basis i.e. a function $f : U \to GL(r, \mathbb{C})$ such that
\[
\overline{\partial} f + f \tau = 0 \quad \text{i.e.} \quad \overline{\partial} f_{ij} + \sum_{l} f_{il} \tau_{lj} = 0. \tag{5.8}
\]

If we can do that, then the sections
\[
e_{l} = \sum_{k} f_{lk} \sigma_{k}
\]
are holomorphic, i.e. $\overline{\partial} E e_{l} = 0$ (by Leibnitz rule). These give the desired ”holo” trivialization/frame of $E$ (by Leibnitz rule, the change of such ”holo” frames is holo).

But (5.8) yet another PDE, and we are looking for local solutions; there is an obstruction to solving it (coming from $\overline{\partial}^2 f = 0$), but if obstruction vanishes, we can locally solve it. In fact, the obstruction is nothing but (5.7):
\[
\overline{\partial} f = -f \tau \quad \implies \quad -\overline{\partial}^2 f = \overline{\partial} (f \tau) = \overline{\partial} f \wedge \tau + f \overline{\partial} \tau = f(-\tau \wedge \tau + \overline{\partial} \tau)
\]

To show (5.8) has a local solution, use Newlander-Nirenberg integrability theorem on $V = U \times \mathbb{C}^r$ to find the desired change of coordinates $f$. Denote by $\{z_{\alpha}\}_{\alpha=1}^{n}$ the local (holo) coordinates on $U$, and use $\{\sigma_{k}\}_{k=1}^{r}$ as coordinate (functions) on the fiber $\mathbb{C}^r$. Let $T$ denote the (real) subbundle of $\Lambda^1 \mathbb{C} V$ with basis
\[
\{dz_{\alpha}, \eta_{k} = d\sigma_{k} - \tau_{kj} \sigma_{j}\} \tag{5.9}
\]
and consider the (twisted) almost complex structure on $V$ for which $T$ is the bundle of $(1,0)$-forms (the $+i$-e-bundle of $J$); intuitively $dz_{\alpha} = \partial z_{\alpha}$ since $z_{\alpha}$ are holomorphic coordinate functions, and if we regarded $\sigma_{k}$ as functions of $z$, $\eta_{k} = \partial_{\alpha} \sigma_{k}$ by (5.6) which is what we want.

Condition (5.7) implies that the ”$(0,2)$ part” of $d\eta_{k}$ vanishes (i.e. $d\eta_{k}$ is a section of $T \otimes \Lambda^1 \mathbb{C} V$), thus this almost complex structure is integrable:
\[
d\eta = d(\sigma - \tau \sigma) = -(d\tau) \sigma + \tau \wedge d\sigma = -(\partial \tau + \tau \wedge \tau) \sigma + \tau \wedge d\sigma = -(\partial \tau) \sigma + \tau \wedge (\tau \sigma + d\sigma)
\]
which is clearly a linear combination of (5.9) with coef in $\Lambda^1 \mathbb{C} V$. Of course, $d(dz) = 0$.

Therefore we can complete the holomorphic coordinates $\{z_{\alpha}\}$ on $U \times 0$ with another set $\{u_{k}\}$ in the ”normal” direction to get holomorphic coordinates on $V$ (near 0). Denote by $\varphi(z, \sigma) = (z, u)$ the (smooth) change of coordinates between $(z_{\alpha}, u_{k})$ and $(z_{\alpha}, \sigma_{k})$. Its linearization $\varphi^*$, restricted to the zero section, gives the change of basis between
\[
\{dz_{\alpha}, d\sigma_{k} - \tau_{kj} \sigma_{j}\} \quad \text{and} \quad \{dz_{\alpha}, du_{k}\}
\]
Its associated matrix has the form $\left(\begin{array}{cc}I_{n} & 0 \\ \ast & f\end{array}\right)$, where $f$ is precisely the desired solution of (5.8). \[\square\]
Lecture 6. Hermitian Bundles and Connections

Next: add (hermitian) metric and a connection on the bundle; try to chose them "compatible" with each other. There is an obstruction to doing so (vanishes if the bundle is holomorphic); get the Chern connection.

6.1. Connections. Recall: If $M$ is a real manifold, and $E \to M$ is a $(cx)$-vector bundle, a connection on $E$ is a $(cx)$-linear differential operator

$$\nabla : \Gamma(E) \to \Lambda^1(E) = \Gamma(\Lambda^1 \otimes E)$$

(from smooth sections to smooth 1-forms with values in $E$) satisfying Leibnitz rule:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma \quad \forall f \in C^\infty(M), \, \sigma \in \Gamma(E)$$

For each $v \in TM$, $\nabla_v \sigma$ is the (covariant) derivative of $\sigma$ in the direction $v$.

Extend $\nabla$ to $\Lambda^p(E)$ by Leibnitz rule i.e.

$$\nabla(\eta \otimes \sigma) = d\eta \otimes \sigma + (-1)^p \eta \wedge \nabla\sigma$$

The curvature operator $R\nabla = \nabla^2$ is the $\text{End}(E)$-valued 2-form defined by

$$R\nabla(\sigma) = \nabla(\nabla\sigma) \quad \forall \sigma \in \Gamma(E)$$

In local coordinates: on $U \subset M$ choose local frame of $E$, i.e. $r$ sections $\{e_k\}$ of $E$ that restrict to a basis in each fiber. Then any section $\sigma = \sum_k \sigma_k e_k$ has coeff $\sigma_k$ in this frame.

Regard $e = (e_1, \ldots, e_r)^T$ as a column vector; for any row of coord fncs $\sigma = (\sigma_1, \ldots, \sigma_k)$, get a local section

$$\sigma \cdot e = \sum_k \sigma_k e_k.$$ 

The connection 1-form $\tau = (\tau_{kl})$ keeps track of the coeff of $\nabla$ in this basis:

$$\nabla e = \tau \otimes e \quad \text{i.e.} \quad \nabla e_k = \sum_l \tau_{kl} \otimes e_l$$

Leibnitz rule becomes in this basis

$$\nabla = d + \tau \quad \text{i.e.} \quad \nabla(\sum_k \sigma_k e_k) = \sum_k (d\sigma_k + \sum_l \sigma_l \tau_{lk}) \otimes e_k$$

The curvature matrix $\Theta = (\Theta_{kj})$ keeps track of the coeff of $R$ in this basis, i.e. $R(e_k) = \Theta_{kj} e_j$. It is given by

Cartan Str eq $\Theta = d\tau - \tau \wedge \tau$ \quad i.e. \quad $\Theta_{kj} = d\tau_{kj} - \tau_{kl} \wedge \tau_{lj}$

because

$$R(e_k) = \nabla(\tau_{kl} \otimes e_l) = d\tau_{kl} \wedge e_l - \tau_{kl} \wedge \nabla e_l$$

HW: Calculate how change of trivialization (aka gauge transformation) $g : U \to \text{End}(E)$ changes the connection 1-form/curvature: if change frame from $e$ to $e' = g(e)$ then

$$\tau'_{g(e)} = g\tau e g^{-1} + (dg) \cdot g^{-1} \quad \Theta'_{g(e)} = g\Theta e g^{-1}$$

Recall $\theta$ is NOT a globally defined 1-form, but the difference between two connections IS a globally defined 1-form on $M$. The curvature is intrinsic: globally it is

$$R \in \Lambda^2(M; \text{End}(E)) = \Lambda^2(M; E \otimes E^*)$$

Works equally well for $E \to M$ complex vector bundle over a real manifold, just use $cx$-valued smooth functions $f$, or else require the connection to be $\mathbb{C}$-linear.
6.2. **Hermitian Vector bundles.** This is a cx v bd + herm metric.

**Definition 6.1.** A hermitian v bd is a cx v bd $E \rightarrow M$ together with a hermitian metric $h$ i.e. smoothly varying hermitian inner prod on the fibers of $E$: this means $h_x : E_x \times E_x \rightarrow \mathbb{C}$ for $x \in M$ satisfies

(i) $h$ is $\mathbb{C}$-linear in the FIRST coord (thus anti cx-lin in the second i.e. sesquilinear)

(ii) $h(u, v) = \overline{h(v, u)}$;

(iii) (positive definite) $h(u, u) > 0$ for all $u \neq 0$.

Smoothly varying– either coef $h_{ij} = h(e_i, e_j)$ in (smooth) local trivializations are smooth or equiv $h(u, v) \in C^\infty$ for all $u, v$ smooth sections of $E$.

**Note:** $h$ defines an anti-$\mathbb{C}$-linear isomorphism $h : E \rightarrow E^*$ or equiv a cx. v. bd isomorphism $E \cong E^*$

The real part of $h$ is a (Riemannian) metric $g = \text{Re } h$ on $E_\mathbb{R}$ satisfying

$$g(Ju, Jv) = g(u, v).$$

(i.e. $J$ isometry; $g$ is called compatible with $J$). The imaginary part is a nondeg 2-form (actually “positive”), i.e.

$$h(u, v) = g(u, v) - i\omega(u, v) \quad \text{where} \quad \omega(u, v) = g(Ju, v) = -\text{Im } h$$

In particular, any one of $(h, g, \omega)$ determines the other two. Confusingly enough, BOTH $h$, $g$ and even $\omega$ are often called the hermitian metric/form; $\omega$ is called a ”positive” because

$$\omega(u, Ju) > 0$$

(not yet symplectic, because not yet CLOSED)

**Example 6.2.** Standard metric on $\mathbb{C}^n$ is

$$h = \sum_k dz_k \otimes d\overline{z}_k = \sum_k [(dx_k)^2 + (dy_k)^2] - i \sum_k dx_k \wedge dy_k$$

descends to a metric on the tangent bundle to the torus. (holo trivial)

A frame $\{e_k\}$ in $E$ is called unitary if it is a o.n. frame in each fiber, i.e. $h(e_i, e_j) = \delta_{ij}$. By Gramm-Schmidt unitary frames always exist (locally).

**Note:** Using partitions of unity, any complex vector bundle on a real manifold admits a hermitian metric (take any $g$, “symmetrize” it wrt $J$) and also a connection $\nabla$.

**Note:** Metrics (and connections) behave well under $E^*$, $E \oplus F$, $E \otimes F$ (Leibnitz rule on coef), pullbacks, etc...

**Example 6.3.** Special case: a hermitian metric on a cx mfld $M$ is one on its (holo) tangent bundle $E = \tau M = T^{1,0}M$. This gives one on $E^* = \Lambda^{1,0}M$ and so on $\Lambda^{p,q}M$.

If $E \rightarrow M$ a holo v. bd and we have a metric both on $M$ and on $E$ we get one on $\Lambda^{p,q}E$. Same for connections.

**Definition 6.4.** A connection $\nabla$ on $E$ is compatible with a herm str $h$ (i.e. $\nabla$ is a hermitian connection) iff

$$d(h(u, v)) = h(\nabla u, v) + h(u, \nabla v).$$

i.e. $h$ is $\nabla$-parallel (regarded as a $\mathbb{C}$-valued REAL bilinear form).
From now on, $M$ is a complex manifold, $E \to M$ is a cx v. bd. Fix a $\mathbb{C}$-linear connection $\nabla$. Using proj $\Lambda^1(E) \to \Lambda^{1,0}(E) \oplus \Lambda^{0,1}(E)$ to decompose
$$\nabla = \nabla^{1,0} \oplus \nabla^{0,1}.$$ These are two $\mathbb{C}$-linear differential operators, satisfying Leibnitz rule
$$\nabla^{0,1} : \Gamma(E) \to \Lambda^{0,1}(E) \quad \text{extended to} \quad \nabla^{0,1} : \Lambda^{p,q}(E) \to \Lambda^{p,q+1}(E)$$ $$\nabla^{1,0} : \Gamma(E) \to \Lambda^{1,0}(E) \quad \text{extended to} \quad \nabla^{1,0} : \Lambda^{p,q}(E) \to \Lambda^{p+1,q}(E)$$ We saw before that $\overline{\partial}E = \nabla^{0,1}$ defines a holomorphic structure on $E$ iff $(\overline{\partial}E)^2 = 0$.

**Definition 6.5.** A conn $\nabla$ on a holo bd $E$ is called compatible with holo str if
$$\nabla^{0,1} = \overline{\partial}E.$$ Note: Note that for any connection on a cx v bd on cx manifold
$$(\nabla^{0,1})^2 = (R\nabla)^{0,2}$$ is the (0, 2) part of the curvature of $\nabla$, because
$$R\nabla = (\nabla^{1,0} + \nabla^{0,1})^2 = (\nabla^{1,0})^2 + (\text{cross terms}) + (\nabla^{0,1})^2$$ which is precisely the decomposition into $(2,0)$, $(1,1)$, $(0,2)$ part!!

**Proposition 6.6.** Assume $E \to M$ is a holomorphic vector bundle, and $h$ a hermitian metric on $E$. Then there exits a unique connection $\nabla$ on $E$ (called the Chern connection) compatible both with $h$ and the holomorphic structure on $E$.

The curvature $F$ of the Chern connection is of type $(1,1)$ and
$$\nabla = \overline{\partial}E + h^{-1} \circ \overline{\partial}E \circ h.$$ Proof. Regarding $h : E \to E^*$, it is $\nabla$-parallel ($\nabla h = 0$) iff
$$\nabla(h(\sigma)) = (\nabla h)(\sigma) + h(\nabla \sigma)$$ so $\nabla \circ h = h \circ \nabla$ But $h$ is anti-complex linear, so $\nabla_Z(h(\sigma)) = h(\nabla_Z \sigma)$ for all $Z \in T_C M$. Taking $Z \in T^{1,0}M$ gives
$$\nabla^{1,0} \circ h = h \circ \nabla^{0,1}$$ Since $\nabla^{0,1} = \overline{\partial}E$, this uniquely determines $\nabla$. The $(0, 2)$ part of the curvature vanishes since $\overline{\partial}E^2 = 0$; the $(2, 0)$ because $(\overline{\partial}E^*)_2 = 0$.

Note: In a frame $\{e_\alpha\}$ of $E$, $\nabla$ is determined by its connection 1-form $\theta = (\theta_{ij})$
$$\nabla e_i = \sum \theta_{ij} e_j$$ If $\{e_\alpha\}$ is a HOLO frame, then $\theta$ is type $(1,0)$. In fact, in terms of the coef $h_{ij}$ compatibility means
$$\partial h = \theta h \quad \text{and} \quad \overline{\partial} h = h \overline{\theta} \quad \text{i.e} \quad \theta = \partial h \cdot h^{-1}.$$ If instead $\{e_k\}$ is a UNITARY frame, then $\theta$ is skew hermitian:
$$0 = d(e_i, e_j) = \theta_{ij} + \overline{\theta}_{ji}$$ This suffice to do Hodge theory on a holo v bd $E$ – talk about harmonic solutions, and the Hodge $*$-operator which will induce Serre duality (version of PD) in the Hodge diamond complex. However, we next discuss a very special class of herm metrics on a cx mfld, called Kahler metrics.
Lecture 7. Kahler metrics

Last time: if $E \to M$ holo v. bd+ h herm connection determ unique Chern conn (if holo str on $E$ is fixed). Applic:

- (next week): Hodge theory for Dolbeaut with values in a holo v bd $E$ (over COM-PACT $M$)- harmonic soln, fin dim, version of PD (Serre dual) induced by Hodge-* operator.
- later: fix herm metric and vary the holo str- by var Chern conn (easier)– understand families of cx str on same cx v. bd. – fin dim families. – stability etc

This week: restrict to $E = TM$ the tangent bundle $E = TM$ of a cx manifold. Discuss a very special class of h. metrics on $M$, called Kahler metrics. Will have special prop, not always exits.

Many equivalent defn: (a) it is a hermitian metric so that the hermitian form $\omega = -\operatorname{Im} h$ is $d$-closed, i.e. locally has a potential $\omega = i\partial \bar{\partial} u$; (b) $h$ osculates to order two to the euclidian one (c) the Chern connection is the same as the Levi-Civita etc.

**Definition 7.1.** A hermitian metric on a cx mfld is Kahler if $\omega = -\operatorname{Im} h$ is $d$-closed, or equivalently locally has a potential i.e. $\omega = i\partial \bar{\partial} u$ (by Poincare Lemma).

**Example 7.2.** (Flat, euclidian) $\mathbb{C}^n$ is Kahler; (HW) global potential $\omega = \frac{i}{2} \partial \bar{\partial} (|z|^2)$.

**Example 7.3.** $\mathbb{CP}^n$ with the Fubini-Study form, defined by $\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log(1 + |z|^2)$ for $z \in \mathbb{C}^n = \mathbb{CP}^n \setminus H$ (the hyperplane at infinity), naturally extends to the compactification.

Better formula: if $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is the projection,

$$\pi^* \omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log(|z|^2) \quad z \in \mathbb{C}^{n+1} \setminus 0$$

Why positive? Clearly homogenous -by $U(n + 1)$ - and calculate:

$$\partial \bar{\partial} \log(1 + |z|^2) = \frac{\sum dz_i \wedge d\bar{z}_i}{1 + |z|^2} + \frac{(\sum \bar{z}_i dz_i) \wedge (\sum z_i d\bar{z}_i)}{(1 + |z|^2)^2}$$

At $z = 0$ 2nd term drops out, get positive.

**Note:** (HW) One can show that $\operatorname{Aut}(\mathbb{CP}^n) = \operatorname{PGL}(n + 1, \mathbb{C})$ i.e. biholo= invertible lin transf on $\mathbb{C}^{n+1}$ up to scale ($\mathbb{C}^*$ action). Those that also preserve $\omega_{FS}$/metric (isometries) correspond to $U(n + 1)$, and these act transitively. (**review holonomy! ***)

Also, $\int_{\text{line}} \omega_{FS} = 1$, thus $[\omega_{FS}] \in H^2(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}$ is dual to $l \in H_2(\mathbb{CP}^n, \mathbb{Z})$.

**Note:** Restriction of Kahler forms to complex submanifolds is Kahler ($d\omega|_M = 0$), so any complex submanifold $M$ of $\mathbb{CP}^n$ has Kahler form $\omega_{FS}|_M$ (even integral coef). e.g.: transverse intersections of hypersurfaces (cut out by homog poly, or holo fcn).

**Example 7.4.** On the unit ball in $\mathbb{C}^n$ could take $\omega = i\partial \bar{\partial} \log(1 - |z|^2)$. “negative curvature”

**Example 7.5.** Hopf mfds are cx but NOT Kahler for $n \geq 2$. Why not?

**Note:** Compact Kahler manifolds have $H^{2i}(M) \neq 0$ (and also $H^{i,i}(M) \neq 0$, but harder to prove) for all $i \leq \dim$. Why? $\omega$ gives a nonzero class there ($d$ and $\partial$-closed). Why nonzero? nondeg implies $\wedge^n \omega$ volume form thus $[\omega]^n \neq 0$ in deRham (also true in Dolbeault, eg use Hodge theory).
Note: A non-degenerate $d$-closed 2-form $\omega$ is called a *symplectic form* (so Kahler forms are symplectic); symplectic manifolds also have $H^{2i}(M) \neq 0$ (for same reason). Therefore $S^{2n}$ cannot be symplectic nor Kahler for $n \geq 2$. (only $S^6$ might have cx str, but it won’t be Kahler).

Note: There is parallel story symplectic/Kahler similar to alm cx/cx (difference is integrability cond). Alm cx str always exits on sympl mflds, but may not be integrable.

7.1. Equivalent descriptions. We next give equiv descriptions of the Kahler condition.

**Theorem 7.6.** Assume $h$ is a hermitian metric on a cx mfld $(M, J)$. TFAE:

(a) the hermitian form is $d$-closed (ie. satisfies $d\omega = 0$)

(b) locally it has a potential $\omega = i\partial \bar{\partial} u$.

(c) there exits holomorphic coord $z$ in which $h$ osculates to order 2 to the standard hermitian form, i.e. coef matrix

$$ (h_{ij}) = \text{Id} + O(|z|^2) \quad \text{i.e. "holomorphic normal coord" (7.1)} $$

or equiv exits unitary coframe $\varphi_i$ near $z_0$ such that $d\varphi_i(z_0) = 0$ (note: $h = \varphi_i \otimes \bar{\varphi}_i$).

(d) the Chern connection on $TM$ agrees with the Levi-Civita connection.

**Proof.** (Outline) (a) $\iff$ (b) by Poincare Lemma.

(c) $\iff$ (a) $\implies$ clear; $\iff$ solve PDE: find $f$ a holo coord change to modify the 1-jet of $h$ at 0. Brute force; In fact, quadratic change of coordinates suffices- so lin alg. (a) by linear change can arrange 0’th order term $h_{ij}(0) = \text{Id}$, so

$$ h_{ij} = \delta_{ij} + \sum a_{ijk} z_k \quad \text{linear in } z + O(|z|^2) $$

(coef of $z$ are determined because $h$ hermitian); $d\omega = 0$ implies coef $a$ are sym in $i \leftrightarrow k$. Take $w_j = z_j + \frac{1}{2} \sum a_{ijk} z_i z_k$ and calc.

(a) $\iff$ (d) basically brute force calc – use 2 Lemmas below, see Moroianu.

Main ideas:

- both compatible with herm metric (one is for $h$, the other one only for $g$);
- LC is torsion free; for Chern: torsion=0 iff $d\omega = 0$.
- $J$ is parallel wrt Chern; for LC: $\nabla J = 0$ iff $N_J = 0$ and $d\omega = 0$.

Recall: torsion $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$; curvature $R_{XY} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

If $\nabla$ is torsion free, and $\omega$ is a $k$-form,

$$ d\omega(X_0, \ldots, X_k) = \sum (-1)^i (\nabla_{X_i} \omega)(\ldots \hat{X}_i \ldots) $$

(i) LC connection $\nabla$ is the unique torsion free connection st $\nabla g = 0$

(ii) Chern conn is unique such that $\nabla h = 0$ and $R_{0,2} = 0$; moreover:

- $\nabla g = \nabla \omega = 0$ thus $\nabla J = 0$ since $\omega = g(J\cdot, \cdot)$;
- the curvature $R$ is (1,1)-form, its torsion $T$ has vanishing (1,1) part so essentially $(2,0)$. (ess. diff between $\nabla \omega$ and $d\omega$).

Specifically, prove two Lemmas:

**Lemma 7.7.** Assume $h$ is a hermitian metric on an almost complex manifold $(M, J)$, and $\nabla$ the LC connection.

(a) then

$$ \tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J) Y $$
is a connection which preserves $g$ and $J$, but has torsion $T = -\frac{1}{4}N_J$.
(b) $\nabla J = 0$ iff $N_J = 0$ and $d\omega = 0$ (i.e. $J$ is LC-parallel iff $h$ is Kahler).

Proof. (a) Use (1) Leibnitz rule (2) $g(J\cdot,J\cdot) = g$ and (3) $\nabla J$ and $J$ anti commute (since $J^2 = -\text{id}$).
(b) $(\implies)$ if $\nabla J = 0$, since $\nabla g = 0$ and $\omega = g(J\cdot,\cdot)$ then $\nabla \omega = 0$ thus $d\omega = 0$. Also since $\nabla J = 0$ then LC=$\tilde{\nabla}$ so torsion=0 = $N_J$.
$(\impliedby)$ follows from (a).  \qed
Today: Ricci curvature for Kahler metrics + more properties.

We have leftover Lemma to show that $h$ Kahler $\iff$ LC= Chern. So far we proved $\iff$: since for Chern $\nabla J = 0$ and for LC

$$\nabla J = 0 \iff (d\omega = 0 \text{ and } N_J = 0) \iff \text{Kahler}$$

For $\implies$: use uniqueness of Chern: (a) both compatible with metric, and complex i.e. $\nabla J = 0$ (since Kahler, else LC only REAL); (b) remains to check the formula for $\overline{\partial}$-operator associated to LC.

**Lemma 8.1.** Assume $M$ is a complex manifold, $h$ a hermitian metric and $\nabla$ its LC. Then its $\overline{\partial}$ operator, regarded as $\overline{\partial}^\nabla: \Gamma(TM) \to \Lambda^{1,0}(TM)$ is given by

$$\overline{\partial}^\nabla X = \frac{1}{2} (\nabla_v X + J \nabla_{Jv} X - J(\nabla_v J)X).$$

In particular, the Chen connection is equal to the LC connection iff $h$ is Kahler.

**Proof.** (Motivation) This formula is the linearization of the holo map equation at the identity! i.e. holo v. fields $X$ are infinitesimal automorphisms (biholom). The holo map eq: look for $\varphi: M \to M$ such that

$$0 = \partial \varphi = \frac{1}{2} (d \varphi + J \varphi \circ d \varphi \circ J \varphi) \iff \phi^* J = J$$

Careful: if $J$ is not parallel, $J$ depends on point. Linearize in $\varphi$ i.e. let $X = \frac{d}{dt}|_{t=0} \varphi_t$; get lin eq in $X$ (use either parallel transport wrt $\nabla$ or Lie derivative ** review **)

$$0 = \partial^\nabla X \iff 0 = \mathcal{L}_X J \overset{\text{def}}{=} \frac{d}{dt}|_{t=0} \varphi_t^* J$$

Recall: using flow eq, *** picture ****

$$T_{id}\text{Diff}(M) = \mathcal{X}(M) = \Gamma(TM)$$

as (infinite dim) Lie algebras. Here we show that

$$T_{id}\text{Aut}(M) = \mathcal{X}_{\text{holo}}(M) = \text{holo sections of } TM$$

i.e. holo vector fields are infinitesimal automorphisms.

**Note:** A Killing vector field is an infinitesimal isometry, i.e. linearize eq $g(\varphi, \varphi) = g$. HW: calc this lin; calc the Killing v.f. for $\mathbb{CP}^n, T^n$; calc infinitesimal autom (biholo).

**Note:** The RHS is a 1st order diff operator (in $X$) whose symbol is $\overline{\partial}$, and with a $0$'th order term involving $\nabla J$ ($0$'th order means no derivatives in $X$ i.e tensorial in $X$).

BACK to proof: To compare LC with Chern, regard LC as a connection on $T^{1,0}M \cong TM$ (connections pull back) using the (ex v. bd) isom

$$\psi : TM \to T^{1,0}M \quad \psi(X) = \frac{1}{2}(X + iJX) = Z$$

whose inverse $Z \mapsto X$ is called "take the real part of $Z". Denote RHS by $\overline{\partial}^\nabla$. Check it satisfies Leibnitz since

$$(\overline{\partial} f)(X) = d_Z f \quad \text{for all } f : M \to \mathbb{C}.$$
Show that $Z$ is holo section of $\tau M$ iff its real part $X$ solves $\bar{\partial}_\nabla X = 0$, using the fact that if $Z$ holo section iff $X$ is an infinitesimal autom, i.e. $\mathcal{L}_X J = 0$ (proved in the process of $N_J = 0$).

Note: If $J$ is almost complex and $h$ hermitian metric, there exits a unique connection $\nabla$ so that $\nabla h = 0$, $\nabla J = 0$ and its torsion has vanishing (1,1) part; this agrees with Chen if $J$ integrable. (HW)

8.1. Curvature Tensors. Recall: various curvatures tensors: Riemannian curvature tensor

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) \quad R_{ijkl} = g_{im}R^m_{jkl}$$

Ricci tensor (symmetric)

$$\text{Ric}(X,Y) = \text{Tr}(u \mapsto R(u,X)Y) = \sum_k R(e_k,X,Y,e_k) \quad R_{ij} = R^k_{ikj}$$

on an o.n frame, and scalar curvature

$$S = \text{tr}(\text{Ric}) = g^{ij}R_{ij}$$

For a KAHLER manifold, $\nabla = \text{LC}=\text{Chern}$; define

$$\rho(X,Y) = \text{Ric}(JX,Y)$$

called the Ricci curvature FORM of a Kahler metric. Its main properties (proved below):

- closed, agrees with cx trace of $iR_{\nabla}$.
- geom interpretation: $[\rho] = 2\pi c_1(M)$ (via Chern-Weil theory, later):

$$c_1(M) = \frac{i}{2\pi}[R_{\nabla}] = \frac{1}{2\pi}[\rho]$$

(take is as defn for now of RHS). – show indep of metric.
- (later) related to $M$ having $U(n)$ holonomy ** review parallel transport;
- Kahler-Einstein metrics:

(Einstein eq) $\text{Ric} = \lambda g$ become (KE eq) $\rho = \lambda \omega$

recall that $\omega$ was ”positive defn” i.e. $-\omega(J\cdot,\cdot) > 0$ so $\rho$ must be definite.
- $\text{Ricci-flat if } \rho = 0$; Calabi-Yau Thm (later)

Note: In general, if $E \to M$ is a cx v bundle, we can also we take the (cx) trace of the curvature $R_{\nabla}$ of a (cx) connection using $\text{tr}_C : \text{End}_C(E) \to \mathbb{C}$ to get

$$R_{\nabla} \in \Lambda^2(M;\text{End}E) \mapsto \text{tr}(R_{\nabla}) \in \Lambda^2(M)$$

Proposition 8.2. For a Kahler manifold:

(a) $\text{Ric}(JX,JY) = \text{Ric}(X,Y)$;
(b) $\text{Ric}(X,Y) = \frac{1}{2}\text{tr}(R(X,JY) \circ J)$
(c) $\rho$ is real, (1,1) and closed i.e. $d\rho = 0$
(d) in fact $-i\rho$ is the trace of curvature of the Chern connection on $E = TM$, i.e.

$$\rho = i\text{tr}_C(R_{\nabla}), \quad \text{where} \quad R_{\nabla} \in \Lambda^{1,1}(\text{End}_C(TM)) \xrightarrow{\text{tr}} \Lambda^{1,1}$$

(e) in HOLO coordinates, $\rho = -i\bar{\partial}_J \log \det(H)$ matrix assoc to herm metric.
(f) intrinsically $\frac{1}{2\pi}\rho$ represents the first Chern class $c_1(TM) = -c_1(K_M)$ where $K_M = \det C T^*M$ is canonical bundle:

$$c_1(TM) = c_1(\text{det}_C TM) = \frac{i}{2\pi}[\text{tr}_C(R_{\nabla})] = \frac{1}{2\pi}[\rho]$$
Proof. (a) uses the fact that $J$ is parallel, and an isometry + prop of $R$; say calc in o.n basis. First, $R_{\nabla}$ ex-linear so 


substitute and take trace.

(b) also uses 1st Bianchi (-due to Ricci– cyclic in 1st 3; fix last) to move entries around:

$$\text{Ric}(X, Y) = \sum \text{R}(e_k, X, JY, J\epsilon_k) = -\sum \text{R}(X, JY, e_k, J\epsilon_k) - \sum \text{R}(JY, e_k, X, J\epsilon_k)$$

$$= \text{mess} - \text{Ric}(X, Y) \quad \text{after insert J and swap coord}$$

(c) uses (a) for type and (b) use formula for $d\omega$ in terms of $\nabla \omega$ and 2nd Bianchi $\nabla R_{\nabla} = 0$.

(d) since $E = TM$ is a holo v bd, and $\text{LC} = \text{Chern}$, get its curvature is (1,1); equality follows from (b), using an unitary frame $\{e_k\}$ to give a o.n frame $\{e_k, Je_k\}$, and the relation between the real trace and cx trace of a skew hermitian matrix:

$$i\text{tr}_C(A_C) = \text{tr}_R(A \circ J)$$

for all $A = -A^T \iff A \in \mathfrak{u}(n) \iff iA$ is hermitian. (proved using spectral theory i.e. $\text{tr}_C : \mathfrak{u}(n) \to i\mathbb{R}$). Recall: in a unitary frame, the matrix of curvatures was skew hermitian.

$$\text{exp} : \mathfrak{u}(n) \to U(n) \text{ with inverse log and}$$

$$\text{log det} A = \text{tr} \text{ log } A$$

**Note:** Intrinsically, in a UNITARY frame, both connection/curvature matrix are SKEW hermitian (in Lie alg $\mathfrak{u}(n)$), denoted:

$$\tau \in \Lambda^1(\text{AdE}) \quad \Theta \in \Lambda^2(\text{AdE})$$

in a unitary frame, you are considering the curvature $F_{\nabla}$ of associated connection in the principal $U(r)$ bundle (of unitary frames ** review *** see also holonomy).

(e) brute force calc in coord! really follows from (d) and Chern-Weil theory. Use formula for curvature in terms of Christofel in terms of metric; in real setting, you may remember

$$\Gamma^i_{ki} = \frac{1}{2} g^{im} \frac{\partial g_{im}}{\partial x_k} = \frac{1}{2} g^{-1} \frac{\partial g}{\partial x_k} = \frac{\partial \log \sqrt{\det g}}{\partial x_k}$$

$$\text{Ricci} = \text{comb of lin terms in } \partial \Gamma + \text{quadratic in } \Gamma.$$ 

In cx setting, great convention: use indices $h_{\alpha \beta}$ for coef of $dz_\alpha d\bar{z}_\beta$ to write

$$h_{\alpha \beta} = \overline{h_{\beta \alpha}} = h_{\bar{\alpha} \bar{\beta}}$$

(one of each kind). Christoffel symbols behave well under cx conj,

$$\Gamma^C_{AB} = \Gamma^C_{\bar{A}\bar{B}}$$

and all mixed indices vanish – because $T^{1,0}$ is $\nabla$-parallel – only $\Gamma^\gamma_{\alpha \beta}$ and its conj left. Usual formulas for them wrt metric. Then move to curvature, many terms vanish, few remain: for Riemann tensor: type (2,2) (2 bars, 2 unbars). Keep going...

$$\text{Ric}_{\gamma \bar{\beta}} = -\frac{\partial \Gamma^\alpha_{\bar{\beta} \gamma}}{\partial z_{\bar{\beta}}} \quad \Gamma^\alpha_{\bar{\beta} \gamma} = h^{\alpha \delta} \frac{\partial h_{\alpha \delta}}{\partial z_{\bar{\beta}}} = \partial_{z_{\bar{\beta}}} \log \text{deth}$$

no quadratic part in Ricci (because of type). ** must be something intrinsic behind these: Chern-Weil.

(f) from previous calc and Chern Weil theory/properties of Chern classes (prove later).  \hfill \Box
Lecture 9. Hodge Theory

2 miracles: (a) the equation is elliptic (elliptic regularity); (b) natural Hodge $\ast$-operator that allows us to go back. When Kahler more miracles: up to factor of 2, Laplacian for $d, \partial$ and $\bar{\partial}$ is the same! So same harmonics.

There is a parallel story for deRham, (even for noncompact manifold – but not fin dim!), Dolbeault, and Dolbeault with values in $E$ holo v. bd.

\[
A^k(M) \xrightarrow{d} A^{k+1} \quad A^{p,q} \xrightarrow{\overline{\partial}} A^{p,q+1} \quad A^{p,q}(E) \xrightarrow{\bar{\partial}_E} A^{p,q+1}(E)
\]

Could also do $\partial$-Dolbeault cohomology for $A^{p,q} \xrightarrow{\partial} A^{p+1,q}$, but that’s nothing new by cx conj.

In all these cases we have a chain complex with a (1st order) differential operator $D$ st. $D^2 = 0$; define (co)homology of this complex by looking at $D$-closed/ $D$-exact forms. For each $\alpha \in A^{p,q}$ get an equivalence class (Dolbeault cohomology)

\[
[\alpha] = \alpha + \partial A^{p,q-1}
\]

Is there a “best” representative for $[\alpha]$?

Want to prove (if $M$ compact):

- the cohomology groups are finite dimensional
- there is a canonical representative for $[\alpha]$:– after we make extra choice, that of a metric! – harmonic form
- bonus: prove a version of Poincare duality/Serre duality in cohomology

Wedge product and integration on an $n$-dimensional compact $M$ defines a product

\[
(\alpha, \beta) = \int_M \alpha \wedge \beta
\]

which gives pairings

\[
A^k \otimes A^{n-k} \rightarrow \mathbb{R} \quad A^{p,q} \otimes A^{n-p,n-q} \rightarrow \mathbb{C} \quad A^{p,q}(E) \otimes A^{n-p,n-q}(E^*) \rightarrow \mathbb{C}
\]

where in the last part, evaluate on the $E$ part using $E \otimes E^* \rightarrow \mathbb{C}$. We will prove these descend to NONDEGENERATE pairings in cohomology giving rise to isomorphism (PD, Serre dual)

\[
H^k(M) \cong H^{n-k}(M)^* \quad H^{p,q}(M,E) \cong H^{n-p,n-q}(M,E^*)
\]

Motivation/Plan: (for Dolbeault cohom) find best representatives for $[\alpha] \in H^{p,q}(M)$ where $\overline{\partial}$-closed/ $\overline{\partial}$-exact. Introduce hermitian metric; this gives an $L^2$-inner product on $A^{p,q}$ by

\[
\langle \alpha, \beta \rangle = \int_M \langle \alpha(z), \beta(z) \rangle d\text{vol}
\]

thus a norm $\|\alpha\|$. Minimize the norm

\[
\|\alpha + D\varphi\| \quad \text{as } \varphi \text{ varies}
\]

argue there exist a unique representative, obtained by orthogonal projection! Uniqueness is easy, existence: hard part. – variational principle, calculate Euler-Lagrange equations.

If $\alpha$ has minimum norm, then minimize on lines ** picture **

\[
0 = \left. \frac{d}{dt} \right|_{t=0} \|\alpha + tD\varphi\|^2 = \langle \alpha, D\varphi \rangle + \langle D\varphi, \alpha \rangle
\]

therefore

\[
\langle \alpha, D\varphi \rangle = 0 \quad \text{for all } \varphi
\]

(9.1)
Such a representative is unique (if it exists!) since
\[ \| \alpha + D\varphi \|^2 = \| \alpha \|^2 + \| D\varphi \|^2 + 0 > \| \alpha \|^2 \quad \text{if } D\varphi \neq 0 \]

**Note:** for hermitian metric use \( t \in \mathbb{C} \): for \( t \in \mathbb{R} \) get Real part of inner product vanishes, then use it to get imaginary part as well.

Can rewrite equation above as
\[ D^*\alpha = 0 \]
where \( D^* \) is the FORMAL adjoint \( D^* \) of \( D \) defined by the equality
\[ \langle D\alpha, \beta \rangle = \langle \alpha, D^*\beta \rangle \]
for all (compactly supported) sections \( \alpha, \beta \). (in the sense of distributions).

**Note:** Here \( D : \Gamma(E) \to \Gamma(F) \) could any linear differential operator, \( E, F \to M \) vector bundles; fix metric on \( E, F \) and \( M \), this defines inner product on space of sections, thus we can use formula above to define its formal adjoint \( D^* \) (depends on choice of metrics!).

**Note:** formal since \( D \) unbounded from \( L^2 \) to \( L^2 \); use Sobolev spaces.

**Note:** In our case we can write down a formula for \( D^* \) (explain later): use integration by parts, Stokes theorem (no boundary term), and Hodge-*-operator. Get:
\[ \begin{array}{c}
A^{p,q} \xrightarrow{\overline{\partial}} A^{p,q+1} \\
\star \downarrow \quad \quad \downarrow \star \\
A^{n-q,n-p} \xleftarrow{\partial} A^{n-q-1,n-p}
\end{array} \]

Summarize: Finding best representative for a \( D \)-cohomology class boils down to solving a pair of equations
\[ D\alpha = 0, \quad D^*\alpha = 0 \iff \Delta \alpha = 0 \]
where
\[ \Delta = DD^* + D^*D \]
is the Laplacian associated to \( D \).

why?
\[ \langle \Delta \alpha, \alpha \rangle = \langle DD^*\alpha, \alpha \rangle + \langle D^*D\alpha, \alpha \rangle = \| D\alpha \|^2 + \| D^*\alpha \|^2 \]

**Definition 9.1.** A form \( \alpha \) is called harmonic if \( \Delta \alpha = 0 \):
\[ \mathcal{H} = \{ \alpha \mid \Delta \alpha = 0 \} = \{ \alpha \mid D\alpha = 0, D^*\alpha = 0 \} \]

We get a version of harmonic forms for each choice of \( D = d, \overline{\partial}, \partial \) on \( M \) and even \( \overline{\partial}_E \) for \( E \) holomorphic. Main properties:
- minimal norm repr of \([\alpha]\) is harmonic
- \( \Delta \) is formally self adjoint and (semi)-positive definite.

from these uniqueness of representative immediate;
existence of harmonic representative: show equation
\[ \Delta \alpha = 0 \]
has solutions. idea: show it has weak solutions (in the sense of distributions, use Sobolev spaces), then prove regularity result: any weak soln is smooth and is actually a strong soln.

The later part uses the fact that \( \Delta \) is an elliptic operator and spectral theorem for elliptic self-adjoint operators. **Note:** harmonics = \( \ker \Delta = \coker \Delta \) since \( \Delta \) is (formally) self adjoint; so we are looking for 0-eigenspaces.
Theorem 9.2 (Hodge). Assume $M$ is compact cx mfld, fix a hermitian metric. Let

$$\mathcal{H}^{p,q}(M) = \{ \alpha \in A^{p,q}(M) \mid \Delta_\overline{\partial} \alpha = 0 \} = \text{Ker } \Delta.$$

Then

- dim $\mathcal{H}^{p,q}(M)$ is finite.
- there is a well defined orthogonal projection $h : A^{p,q} \to \mathcal{H}^{p,q}$, descends to isom $H^{p,q} \cong \mathcal{H}^{p,q}$.
- there exists a unique operator $G : A^{p,q} \to A^{p,q}$ (called Green's operator) such that:
  $$G|_\mathcal{H} = 0, \text{ commutes with } \overline{\partial} \text{ and } \overline{\partial}^* \text{ and }$$
  $$\text{id} = h + \Delta G$$
- In particular, there is an orthogonal decomposition
  $$A^{p,q} = \mathcal{H} \oplus \overline{\partial} A^{p,q-1} \oplus \overline{\partial}^* A^{p,q+1}$$
  given by $\alpha = h(\alpha) + \overline{\partial}(\overline{\partial}^* G \alpha) + \overline{\partial}^*(\overline{\partial} G \alpha)$.

Note: This implies that the equation (in $\alpha$)

$$\Delta \alpha = \eta \quad \text{has a solution iff } \Delta \eta = 0$$

if so, then it has a unique solution $\alpha = G(\eta)$ which is perpendicular to harmonics.

Equivalently, if restrict to $\mathcal{H}^+$, $\Delta$ becomes invertible and its inverse is $G$ (a smoothing operator: gains 2 derivatives). Spectral Theorem: e-spaces of $\Delta$ (2nd order operator)

$$\Delta \varphi = \lambda \varphi$$

are finite dimensional, and e-vectors are smooth (regularity). (Spectrum is discrete, gives $L^2$-decomposition into e-spaces).

9.1. Hodge star operator. This is lin alg. Let $M$ an (oriented) manifold with metric $g$.
Define rescaled inner product on $A^k$

$$\langle \alpha, \beta \rangle = \frac{1}{k!} \int_M \langle \alpha, \beta \rangle d\text{vol}$$

Note: Factorial is needed so that if $e_i$ is o.n. basis of $V$ then $e_I$ is o.n. basis of $\wedge^k V$. Recall:

$$d\text{vol} = \sqrt{\det(g_{ij})} dx_1 \wedge \cdots \wedge dx_n = \frac{\omega^n}{n!}$$

Define

$$* : \Lambda^p \to \Lambda^{n-p} \quad \text{by } \langle \alpha, \beta \rangle = \langle \alpha, *\beta \rangle \quad \text{i.e. } \frac{1}{k!} \int_M \langle \alpha, \beta \rangle d\text{vol} = \int_M \alpha \wedge *\beta$$

2 pairings here: inner product and integral of wedge product; $*$ converts one into the other.

Note: in an o.n basis $\{e_i\}$, if $I^c$ denotes the complement of $I$,

$$*e_I = \pm e_{I^c} \quad \text{where } \pm = \text{sign}(I, I^c) \text{ as a permutation of the indices}$$

Lemma 9.3. The Hodge star (depends on metric!) has the following prop:

- $*1 = d\text{vol} \quad *d\text{vol} = 1$
- $\langle \omega, \eta \rangle = \langle *\omega, *\eta \rangle$ i.e. $*$ is an isometry
- $*^2 = (-1)^{k(n-k)}$ on $\Lambda^k$, thus self adjoint up to sign.

If $M$ is a complex manifold $* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}$ (note swap)
**Lemma 9.4.** With this, $d^* : A^{k+1} \to A^k$ is

$$d^* = -(-1)^n k \ast d \ast \quad \ast \mathbb{J}^* = -\ast \partial^*$$

**Proof.** (Stokes and formal manipulation):

\[
\langle d\alpha, \beta \rangle_{d\text{vol}} = d\alpha \wedge (\ast \beta) = d(\alpha \wedge \ast \beta) + (-1)^k \alpha \wedge d \ast \beta
\]

\[
\langle \alpha, \ast d \ast \beta \rangle_{d\text{vol}} = \alpha \wedge \ast^2 d(\ast \beta) = \pm \alpha \wedge d(\ast \beta)
\]

\[\square\]
Lecture 10. Hodge Theorem and Serre duality

If $M$ is a complex manifold, fix hermitian metric $h = \langle \cdot, \cdot \rangle_{\mathbb{C}}$. Extend Hodge $*$-operator to complex valued forms as a $\mathbb{C}$-linear operator. Then:

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle_{\mathbb{C}} d\text{vol}$$

wrt hermitian inner product. Check it takes $*$ : $\Lambda^{p,q} \rightarrow \Lambda^{n-q,n-p}$ note swap. Also $\omega^n = n! d\text{vol}$.

**Note:** Better behaved is the ANTI-complex linear extension (compose with complex conj)

$$\bar{*} : \Lambda^{p,q} \rightarrow \Lambda^{n-p,n-q}$$

$$\bar{*} \phi = * \phi = * \phi$$

**Lemma 10.1.** When $M$ is EVEN dimensional

- $*^2 = \bar{*}^2 = (-1)^k$ on $\Lambda^k$
- the adjoints are given by (thus 1st order diff)

$$d^* = - * d * \quad \bar{\partial}^* = - * \bar{\partial} * \quad \partial^* = - * \partial *$$

- Hodge $*$ preserves harmonics:

$$* \Delta_d = \Delta_d *; \quad \bar{\partial} \Delta_{\bar{\partial}} = \Delta_{\bar{\partial}} \bar{\partial}$$

- $D^2 = (D^*)^2 = 0$ and Laplacian for $D$ commutes with both $D$ and $D^*$.

**Proof.** Immediate (HW) e.g.: prove $d^* = - * d *$ using Stokes (int by parts)

$$\langle d\alpha, \beta \rangle_{\text{dvol}} = d\alpha \wedge (*\beta) = d(\alpha \wedge *\beta) - (-1)^k \alpha \wedge d * \beta$$

$$\langle \alpha, * d * \beta \rangle_{\text{dvol}} = \alpha \wedge *^2 d(*\beta) = (-1)^k \alpha \wedge d(*\beta)$$

Therefore

$$\langle d\alpha, \beta \rangle_{\text{dvol}} + \langle \alpha, * d * \beta \rangle_{\text{dvol}} = d(\alpha \wedge *\beta)$$

so by Stokes these are adjoints.

Careful: complex conj switches $\partial$-harmonics with $\bar{\partial}$ ones!! (as does usual Hodge $*$!)

$$\mathcal{H}^{p,q}_{\partial} = \mathcal{H}^{q,p}_{\bar{\partial}}$$

Note $\partial$!

In general, a $\bar{\partial}$-closed form $\alpha$ may not be $\partial$-closed, etc.

**Example 10.2.** When $M = \mathbb{R}^n$ with flat metric we get $\Delta_d = - \sum_k \frac{\partial^2}{\partial x_k^2}$. Note - sign.

For $\mathbb{C}^n$ get $\Delta_d = 2 \Delta_{\bar{\partial}} = 2 \Delta_\partial$. (extends to Kahler – note 2nd order)

In general (HW), Laplacian on functions is (where $g = \det(g_{ij})$)

$$\Delta f = - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{ij} \frac{\partial f}{\partial x_i} \right)$$

extends to forms

Symbol of $\Delta$ is $\sigma(\xi) = -|\xi|^2 id$ thus ELLIPTIC (so elliptic estimates+regularity).

**Proof.** (Outline of pf of Hodge Thm) Key: show $\Delta$ invertible perp to ker=coker, i.e. eq

$$\Delta \alpha = \eta \quad \text{has a solution iff } \Delta \eta = 0 \quad (\text{obstr to solving}) \quad (10.1)$$

if so, then it has a unique solution $\alpha = G(\eta)$ which is perpendicular to harmonics.

But easier to invert $Id + \Delta$ since positive definite (so ker=coker=0), still SAdj so e-val.

Key: basic elliptic estimate: shortcut for $\Delta$: take modified (Dirichlet) scalar product

$$\langle \eta, \eta \rangle_{D} = \langle D\eta, D\eta \rangle + \langle D^* \eta, D^* \eta \rangle + \langle \eta, \eta \rangle = \langle \eta, (id + \Delta) \eta \rangle$$
complete $C^\infty$ to get Hilbert space. Show this norm is uniformly equivalent to the usual Sobolev space $W^{1,2}$ i.e. Garding inequality (other way is clear):

$$\langle \eta, \eta \rangle_D \geq c \|\eta\|_{1,2}^2 \quad \implies \quad \| (I + \Delta)\eta \|_{0,2} \geq c \|\eta\|_{1,2}. \quad (10.2)$$

Therefore $(I + \Delta)$ invertible (inverse is a smoothing operator). Done. (more details:)

Recall: Sobolev spaces $W^{k,p}$ for sections in some bundle:

- means $k$ derivatives in $L^p$ ($p = 2$ to get Hilbert space) i.e. $\|\eta\|_{k,2}^2 = \int_M |\nabla^k \eta|^2 + |\nabla \eta|^2$

for SOME choice of metric and connection (all uniformly equiv if $M$ compact).

- Sobolev Embeddings (Rellich Lemma) $W^{k,2} \rightarrow W^{k-1,2}$ compact embedding (i.e. any bounded seq has conv subseq; use for minimizing seq/seq of soln).

- $\cap_k W^{k,2} = C^\infty$ smooth; actually $W^{k,2} \rightarrow C^r$ for $k - r > n/2$.

Basic elliptic estimate (10.2) implies:

- regularity: if $\alpha \in H^0$ is a weak soln of (10.1) and $\eta \in H_k$ then $\alpha \in H_{k+2}$ (gains 2)
- bootstrap get smooth e-vectors of $I + \Delta$; no zero e-val (pos defn)!
- $(I + \Delta)$ is invertible: $(I + \Delta)^{-1}: H^0 \rightarrow H^1$ bounded thus compact from $H_0 \rightarrow H_0$
- look at e-spaces - Spectral Thm for $(I + \Delta)^{-1}$ Sadj, positive definite, compact op.
- of course, it is a Fredholm op so finite dimensional e-space.

Note: In fact, $G$ is an integral operator, Delta functions.

10.1. **Serre Duality.** If $M$ compact and connected, have intrinsic pairing

$$A^k \times A^{n-k} \rightarrow \mathbb{R} \quad A^{p,q} \times A^{n-p,n-q} \rightarrow \mathbb{C} \quad \text{by} \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$$

Stokes $\implies$ descends to pairing in deRham/Dolbeault cohom. Get Poincare/Serre duality: this pairing is NONDEG.

**Theorem 10.3** (Serre duality). There is a canonical $\mathbb{C}$-linear isomorphism:

$$H^{p,q}(M) \cong H^{n-p,n-q}(M)^*$$

So $h^{p,q} = h^{n-p,n-q}$. Careful: cx conj $\mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}}$. Hermitian metric $\mathbb{C}$-lin $E \cong E^*$

**Proof.** Choose metric, get harmonics, ANTI-cx-linear Hodge-* op preserved harmonics

$$\tilde{*} : \mathcal{H}^{p,q} \rightarrow \mathcal{H}^{n-p,n-q} \quad \text{and} \quad \int \alpha \wedge \tilde{\alpha} = \|\alpha\|^2 > 0 \quad \text{so nondeg.}$$

Remains true even if $E$ holo v. bd; then

$$A^{p,q}(E) \times A^{n-p,n-q}(E^*) \rightarrow \mathbb{C}$$

using wedge on forms and eval $E \otimes E^* \rightarrow \mathbb{C}$ on sections of $E$. This induces

**Theorem 10.4** (Kodaira-Serre duality). There is a canonical $\mathbb{C}$-linear isomorphism:

$$H^{p,q}(M, E) \cong H^{n-p,n-q}(X, E^*)^*$$
Why? Hermitian metric on $E$, regard as an ANTI-cx lin isom $h : E^* \to E$. With Hodge $*$ on $M$ it gives: ANTI-cx linear isom

$$\tilde{s}_E : A^{p,q}(M, E) \to A^{n-p,n-q}(E^*) \quad (\tilde{s}_E)(\omega \otimes \sigma) = (\tilde{s}_E)\omega \otimes h(\sigma)$$

descends to ANTI-cx-linear isom (not intrinsic!)

$$\tilde{s}_E : \mathcal{H}^{p,q}(M, E) \to \mathcal{H}^{n-p,n-q}(M, E^*)$$

**Careful:** In general,
- the 2 Laplacians for $d, \bar{\partial}$ are unrelated (as are DeRham and Dolbeault)
- wedge of two harmonic forms may not be harmonic!
- restriction of a harmonic form to submanifold may not be harmonic

All these (and many more) miracles hold for Kahler manifolds.

10.2. **Kahler identities.** Assume $M$ is Kahler; $\omega = -\text{Im} \ h$ the Kahler form, closed $(1,1)$ form wrt all $d, \partial, \bar{\partial}$.

Goal: compare deRham with Dolbeault, and get more properties of cohom. – involves a LOT of calculation. – and introduce more operators! (these work in gen)

Recall: $d = \partial + \bar{\partial}$, which anti commute (and square to 0). Introduce: twisted differential

$$d^c = i(\bar{\partial} - \partial) \quad \text{so} \quad \bar{\partial} = d - id^c \quad \text{real/imag part}$$

- $d, d^c$ are REAL operators (other two are complex) and

$$dd^c = -d^c d = 2i\partial\bar{\partial} \quad \text{Poincare} \ i\partial\bar{\partial} \text{ lemma!}$$

- can also take adjoints $d^{c*}$; Note: the adjoints of $\partial, \bar{\partial}$ and of $d, d^c$ also anticommute.

If Kahler, also have the Lefschetz operator $L$ and its adjoint $\Lambda = L^*$ (both REAL op)

$$L : A^{p,q} \to A^{p+1,q+1} \quad L\eta = \omega \wedge \eta \quad \Lambda = L^* : A^{p+1,q+1} \to A^{p,q}$$

**Lemma 10.5** (Kahler identities). On a Kahler manifold:

(a) $[L, d^c] = d^c$, $[L, d] = 0$, adjoints: $[\Lambda, d] = -d^{c*}$, $[\Lambda, d^c] = 0$.

(b) $[L, \partial^s] = i\partial^s$, $[L, \bar{\partial}^s] = -i\partial^s$, $[L, \partial] = [L, \bar{\partial}] = 0$. (and adjoints!).

(c) $\partial \partial^s + \bar{\partial} \partial^s = 0$, and so $\partial \partial^{c*} + \partial^{c*} \bar{\partial} = 0$

(d) $\Delta_d = 2\Delta_\bar{\partial} = 2\Delta_\partial$

**Proof.** (a) 1st relation – key– brute force calc; use holo normal coord or see Moroianu for more intrinsic way - uses o.n. frame, ext prod (wedge) and its adjoint, int prod (contraction).

rest follow immediately. 2nd one since $\omega$ is closed; 3rd+4th by taking adjoints

(b) decompose (a) by type;

(c) use adjoint form of (b): replace $i\partial^s = [\Lambda, \partial]$ so

$$i\partial\partial^s = \partial(\Lambda \partial - \partial \Lambda) \quad \text{while} \quad i\partial\bar{\partial} = (\Lambda \bar{\partial} - \bar{\partial} \Lambda)\bar{\partial}$$

(d) now follows: $\Delta_d = \Delta_\bar{\partial} + \Delta_\partial$ and do same trick $i\partial\partial^s = [\Lambda, \partial]$ to show $i\Delta_\bar{\partial} = i\Delta_\partial$. \qed

**Theorem 10.6** (Hodge decomposition). On a Kahler manifold, we have decomp

$$H^k_{dR}(M; \mathbb{C}) = \bigoplus H^{p,q}(M) \quad \text{orthogonal decom for harmonics}$$

where $H^{p,q} = \overline{H^{q,p}}$ (and the Dolbeault for $\bar{\partial}$ is the same as for $\partial$).

Symmetries in Hodge diamond -Serre dual, $h^{p,q} = h^{q,p}$; Betti numbers $b_k = \sum h^{p,q}$; recall also that $h^{p,p} > 0$ if $p \leq n$. (repr by $\omega$; $\omega^n = n!\text{vol}$)

**Corollary 10.7.** If $M$ is Kahler, Betti numbers $b_{2k+1} = \text{even}$. 
Example 10.8. Hopf manifold is diffeo to $S^1 \times S^{2n-1}$ thus not Kahler if $n \geq 2$. (We already knew this since $H^2 = 0$).

Example 10.9. Kodaira (Thurston) example: twisted $T^2$ bd over $T^2$, admits cx str but $b_1 = 3$: Quotient of $(z, w) \in \mathbb{C} \times \mathbb{C}$ by group gen by

\[
\begin{align*}
  z &\mapsto z + 1, & z &\mapsto z + i, \\
  w &\mapsto w + z + 1, & w &\mapsto w - iz + i
\end{align*}
\]

$z \in T^2$ is the base, the fiber $w$ twists (dehn twist). Equivalently, quotient by $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}^2$:

\[
(j, k) \ast (j', k') = (j + j', A_j k + k') \quad A_j = \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix} \quad \text{Dehn twist}
\]
11. Hard Lefschetz Thm. Cohomology of Kahler manifold have more properties; on the way to proving Laplacians are the same, we proved prove several other important commutator identities; some give Lefschetz theorem: regard as saying that the cohomology of a Kahler manifold is an \( \mathfrak{sl}_2(\mathbb{Z}) \) representation.

Recall: on a Kahler manifold we have \( L = \text{wedge by } \omega \) - main character- its adjoint \( \Lambda = L^* \). Let \( H \) be the counting operator (shift down dimensions to center at 0):

\[
H : A^k \rightarrow A^k \quad H(\eta) = (k-n)\eta
\]

**Note**: \( \mathfrak{sl}_2(\mathbb{Z}) = 2 \times 2 \) matrices of trace 0 with usual commutator; standard generators

\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = X^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

Satisfy certain commutator relations; we shall see \( X \) corresponds to \( L \) Lefschetz operator.

**Lemma 11.1** (Kahler identities, cont). *We also have the following Kahler identities:
(a) \( [L, \Lambda] = H \), \( [H, L] = 2L \), \( [H, \Lambda] = -2\Lambda \) (i.e. \( \mathfrak{sl}_2(\mathbb{Z}) \)-representation).
(b) it implies \( [L, \Delta] = 0 \) equiv \( [\Lambda, \Delta] = 0 \) (thus \( L, \Lambda \) descend to cohom)

**Note**: \( L : H^{p,q} \rightarrow H^{p+1,q+1} \) is top: depends ONLY on Kahler class \([\omega] \in H^{1,1}(M)\)!

**Theorem 11.2** (Hard Lefschetz). If \( M \) compact Kahler, the map

\[
L^k : H^{n-k}(M) \rightarrow H^{n+k}(M)
\]

is an isomorphism

and both DeRham and Dolbeault cohom are \( \mathfrak{sl}_2(\mathbb{Z}) \)-representations.

**Example 11.3.** For \( H^*(\mathbb{C}P^1) \): only nonzero are \( H^{0,0}, H^{1,1} \) both 1-dim, and are isom by wedge by \( \omega \)! Similar for \( H^*(\mathbb{C}P^n) \) see HW.

**Corollary 11.4.** If \( M \) complex submfld of \( \mathbb{C}P^N \), \( h \) hyperplane in \( \mathbb{C}P^N \) then

\[
\cap h^k : H_{n+k}(M) \cong H_{n-k}(M)
\]

because \( \omega \) is PD to hyperplane \( h \), and PD takes wedge by \( \omega \) to intersecting by \( h \).

**Note**: Later(Skip): Lefschetz hyperplane Thm restricts cohom of smooth hypersurface \( V = M \cap H \) of a cx smfld \( M \) of \( \mathbb{C}P^M \); \( \iota^* : H^q(M) \rightarrow H^q(V) \) is an isom in \( * \leq \dim M - 2 \) and inj in \( * = n - 1 \) (so only new cohom appears in middle dim).

**Note**: There is one irred \( \mathfrak{sl}_2(\mathbb{Z}) \)-repr in each dimension, starting with the fundamental one \( \mathbb{C}^2 \); rest are sym tensor products of it: the \( n + 1 \)-dim one is

\[
V(n) = \text{Sym}^n(\mathbb{C}^2) = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n
\]

- \( V_i \) are the 1-dim e-spaces of \( H \) i.e. \( H \) has simple e-val \(-n, -n + 2, \ldots, n - 2, n\)
- \( L \) moves between them by \( L : V_k \rightarrow V_{k+2} \);
- terminates with something in \( \text{Ker } L^{n+1} \); starts with \( \text{Ker } \Lambda \);
- start at the bottom: with \( \text{Ker } \Lambda \) primitive part: if \( v \in \text{Ker } \Lambda \) is e-vect of \( H \) (called primitive) then \( L^k v \) also e-vect of diff e-value (and must terminate).

More generally, get **Lefschetz decomp** of cohom of cpt, Kahler mfld into "primitive parts":

\[
H^m = \oplus L^k P^{n-2k}; \quad P^k = \oplus P^{p,q}
\]

where \( P \) represents the **primitive part**=the intersection with \( \text{Ker } \Lambda \).
11.2. **Riemann bilinear pairing.** For a Kahler manifold, we can extend intersection pairing in middle (co)homology to a nondeg bilinear (skew sym) pairing \( Q \) on \( H^k(M) \); it can be turned into a Hermitian prod \( H Q(\alpha,\beta) = \int_M \alpha \wedge \beta \wedge \omega^{n-k} \); \( H(\alpha,\beta) = i^k Q(\alpha,\bar{\beta}) \)

The Hodge decomposition and the Lefschetz is orthogonal wrt \( H \), and definite on primitives.

11.3. **Line bundles.** Next: understand cx/holo LINE bds. Several approaches:

(a) transition functions \( \Rightarrow \) 1-Cech cocycle
*** READ Sheaves+cohom in §3 of Griffiths-Harris or Huybrechts Appendix B****

(b) zeros of generic sect/zeros and poles with multiplicity of meromorph sect \( \Rightarrow \) divisors.

(c) First Chern class=Euler class related to both, and to curvature via Chern Weil theory.

Q: Difference between holo isom and cx isom? i.e. how many different holo structures there are on the same cx line bd \( \Rightarrow \) deform of holo str/moduli spaces of holo str (end of Q).

If \( M \) is a cx mfld, let \( \text{Pic}(M) \) be the set of isom classes of holo line bds over \( M \). The tensor prod \( L_1 \otimes L_2 \) and dual \( L^* \) make it into an abelian group, called the Picard group. The Jacobian \( \text{Pic}^0(M) \) is the subgroup of topologically trivially ones.

**Example 11.5.** Goal: \( \text{Pic}(\mathbb{C}P^n) = \mathbb{Z} \), generated by the (dual of) tautological line bundle; for \( \mathbb{C}P^n \) isom of holo line bds= isom of cx line bds, i.e. \( \text{Pic}^0(\mathbb{C}P^n) = 0 \).

Why? Use Cech cohom; more precisely exp seq and Dolbeault Thm.

**Example 11.6.** For the elliptic curve \( E \), there is a 1-param fam of holo line bds are trivial as a cx line (moduli is 1-dim); in fact, goal: \( \text{Pic}^0(E) \cong E \).

Why? For any \( p \in E \), there is a holo line bd \( O(p) \) which has a holo section that has a simple zero at \( p \) and nowhere else; all have same \( c_1 \) thus isom as cx line bd. But if \( p \neq q \), \( O(p-q) \) is NOT holo trivial: else there would be a mero fcn on \( E \) which has precisely one simple zero and one simple pole; it would give rise to a degree 1 map \( f : E \to \mathbb{C}P^1 \) which would have to be an isom, impossible.

So moduli at least 1-dim (fix \( p \), let \( q \) vary); exp seq and \( h^{0,1}(E) = 1 \) \( \Rightarrow \) at most 1 dim.

Similar to: on a torus there is a 1-param family of cx str, on \( S^2 \) there is a unique one.

11.4. **Transition fcn.** Cx/Holo line bds up to isom are determ by transition fcn:

\( g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^* \)

which are smooth/holo functions, and satisfy the cocycle condition ** pic **

\( g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \) on the overlap \( U_\alpha \cap U_\beta \cap U_\gamma \)

Note: we also have \( g_{\alpha\alpha} = id \) and \( g_{\alpha\beta} = g_{\beta\alpha}^{-1} \), so more intrinsically

\( g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \)

This means that holo line bds naturally give rise to a 1-Cech cocycle \( H^1(M,O^*) \) where \( O^* \) denotes the sheaf of non vanishing holo functions. Goal:

**Theorem 11.7.** There is a natural group isom \( \text{Pic}(M) \cong H^1(M,O^*) \).

The SES \( 0 \to \mathbb{Z} \to O \xrightarrow{\exp} O^* \to 0 \) where \( f \mapsto \exp(2\pi if) \) gives rise to LES in cohom and

\( \delta^* : H^1(M,O^*) \to H^2(M,\mathbb{Z}) \quad \delta^*[L] = c_1(L) \)
will represent the first Chern class (Kodaira-Spencer Thm, proved later). As we shall see, cx line bds are classified by their Chern class (use exp seq and \(H^1(M, \mathcal{A}) = 0\)), but holo ones need finer invar: e.g. \(\mathcal{O}(p - q)\) on an elliptic curve.

On \(\mathbb{CP}^n\), \(H^1(M, \mathcal{O}) = H^{0,1}(M) = 0\) by Dolbeault Thm + HW so exp seq implies \(\text{Pic}^0 = 0\) (holo isom = cx isom).

11.5. **Sections and divisors.** With notation above, a section \(s\) in a cx line bd is a collection of functions \(s_\alpha : U_\alpha \to \mathbb{C}\) such that

\[
s_\alpha = g_{\alpha\beta} s_\beta
\]

So if \(s\) is a nowhere zero (holo) section, then the line bd is (holo) trivial (\(s\) defines a global frame, i.e. global isom with \(M \times \mathbb{C}\)). A global mero fn is a section of the trivial bundle. In fact, any collection \(s_\alpha : U_\alpha \to \mathbb{C}^*\) give rise to an exact 1-Cech cocycle, i.e. a trivialized line bundle. Different trivializations (and so autom of a line bd) are classified by \(H^0(M, \mathcal{O}^*)\) i.e. global holomorphic functions which are nowhere zero. These are constant sections \(\mathbb{C}^*\) if \(M\) is compact.

More generally, we will show that a nonzero mero section determines the holo line bd up to isom; enough to know its divisor, i.e. zeros and poles (with multipl); Kodaira vanishing implies line bds over cx smflds of \(\mathbb{CP}^N\) have meromorphic sections.

**Example 11.8.** (Skip) On a RS, for any point \(p\) can consider the holo line \(\mathcal{O}(p)\) which has a holo section with a simple zero at \(p\) and nowhere else; it is obtained by clutching two trivialization. Taking tensor products and duals get \(\mathcal{O}(\sum a_i p_i)\) where \(a_i \in \mathbb{Z}\) finite, called a divisor.

11.6. **Chern classes.** Many defns: axioms, classifying spaces, Chern Weil theory etc:

- satisfy universal properties
- top invar under isom+defm of cx v. bds
- obstructions to various Q:
  - are 2 cx v. bd isom?
  - is a bd trivial?
  - does a bd have a non-vanishing section? \(k\) lin indep sections?

Given any cx v. bd \(E \to M\) over a mfld (any top space), we can associate \(c_1(E) \in H^2(M, \mathbb{Z})\) satisfying following axioms:

(a) (natural) \(c_1(f^* E) = f^* c_1(E)\).
(b) (sum) \(c_1(E \oplus F) = c_1(E) + c_1(F)\)
(c) (normalization) \(c_1(\tau) = -1\) for the tautological line bd over \(\mathbb{CP}^1\). (i.e. \(c_1(\tau)[\mathbb{CP}^1] = -1\).)

**Note:** \(E \mapsto c_1(E)\) is a functor, and has extra properties e.g.:

- (product) for line bundles: \(c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)\)
- \(c_1(\mathbb{CP}^n) = -h\) where \(h\) is PD to the hyperplane \(H \subset \mathbb{CP}^n\)
- (use restriction to \(\mathbb{CP}^1\)); and (recall: \(\det E = \Lambda^{top} E\) is its top exterior power):

\[
c_1(\det E) = c_1(E), \quad c_1(E^*) = -c_1(E)
\]

Use the **Splitting principle:** while NOT every cx v. bd is a direct sum of line bundles, it becomes so after pulling back to some canonical manifold \(M(E)\); then we can push forward. Key: the pull back \(\pi^* E\) of a bundle \(\pi : E \to M\) over itself has a canonical section \(\sigma : E \to \pi^* E\)

\[
\sigma(x, v) = (x, v; v), \quad \text{for all } x \in M, \ v \in E_x
\]
Therefore the pullback of $E$ to $\mathbb{P}(E)$ splits off a complex line $\langle \sigma \rangle$ spanned by $\sigma$, see Bott-Tu for excellent description):

\[
\begin{array}{cccc}
\pi^*E & E & \text{so} & L(\sigma) \xrightarrow{c} \pi^*E & E \\
\sigma \downarrow & \downarrow \pi & & \downarrow \pi \\
E & \xrightarrow{\pi} M & & \mathbb{P}(E) & \xrightarrow{\pi} M
\end{array}
\]

This reduces the discussion to line bundle; for cx LINE bundles:

- $c_1(L)$ is a complete invariant: i.e. $L_1, L_2 \to M$ are isom iff $c_1(L_1) = c_1(L_2)$;
- $c_1(L) = \chi(L)$ Euler class and so is PD to the zero locus of a generic section.
- (proved below) $c_1(L) \in H^2(\mathbb{Z}; \mathbb{R})$ is repr by the curvature of a connection in $L$.

Careful: for higher rank vector bundles, Chern classes are NOT complete invariants, i.e. it could happen that the Chern classes are the same, but the bundles are not isomorphic. In that case, one could look at secondary characteristic classes (secondary obstructions).

**Note:** In the context of Chern-Weil theory, the secondary characteristic classes are called transgressive classes, or (Cheeger-)Chern-Simons classes (the Chern-Simons 3-form is the most famous example).
Lecture 12. Chern classes and Chern-Weil Theory

For higher rank bundles, we can also define higher Chern classes, but NOT complete invar: 
\( c_i(E) \in H^{2i}(M, \mathbb{Z}) \) (vanish beyond the rank); assemble in total Chern class 
\[
\begin{align*}
\chi(E) = 1 + c_1(E) + c_2(E) + \ldots
\end{align*}
\]
which is multipl 
\[
\chi(E \oplus F) = \chi(E)\chi(F)
\]
If \( E = L_1 \oplus \cdots \oplus L_r \) line bds of Chern classes \( x_i \) (called the roots) then 
\[
\chi(E) = (1 + x_1) \ldots (1 + x_r).
\]
So \( c_k \) is the \( k \)'th elem sym poly in the roots \( x_1, \ldots, x_r \) (Viete). Splitting principle allows one 
to talk about Chern roots for any \( \text{cx v. bd} \).
Better behaved (but \( \mathbb{Q} \) coef!) is the total Chern character: 
\[
\begin{align*}
\chi(L) &= e^{c_1(L)} \\
\chi(E \oplus F) &= \chi(E) \oplus \chi(F) \\
\chi(E \otimes F) &= \chi(E)\chi(F)
\end{align*}
\]
i.e. \( \chi_k \) is the "other" sym poly: sum of \( k \)'th powers of roots (over \( k! \)).
Note: (skip?) \( \chi \) is a ring homomorphisms from \( K \)-theory: \( \chi : K(M) \to H^*(M) \). \( \text{Cx v. bd} \) 
up to isom with \( \oplus \) is a semigr; make into a group by considering \( E - F = \text{virtual v. bd} \).
Note: Best approach to defn Chern classes is via the classifying space \( BG \), where \( G \) is the 
structure group of the bd: any \( G \)-bundle \( E \to M \) is the pullback of the universal bundle 
\( EG \to BG \) by the classifying map \( f : M \to BG \):

\[
\begin{array}{ccc}
E & \to & EG \\
\downarrow \pi & & \downarrow \\
M \xrightarrow{f} BG
\end{array}
\]
e.g.: for \( \text{cx LINE bds} \ EG \to BG \) is the tautological line bd \( \tau \to \mathbb{CP}^\infty \). By naturality, 
\( c_1(L) = f^*c_1(\tau) \) is the pullback of the generator \(-h \in H^2(\mathbb{CP}^\infty, \mathbb{Z})\).
Note: The str group of a rank \( r \) complex v. \( \text{bd} \) is \( G = GL(r, \mathbb{C}) \) (i.e. where transition 
fcn take values in); if we also fix unitary metric, it reduces to \( G = U(r) \) (maximal compact 
subgroup + a deformation retraction so same \( BG \)).
Recall: unique factoriz of \( A \in GL \) into unitary and positive defn: \( AA^* > 0 \) and \( s \adj etc \).
Instead: geom interpr using curvature: Chern-Weil repr theory, works for any \( G \)-bundle.

12.1. Curvature and first Chern class. For any \( \text{cx v. bd} \ E \) over real mfld \( M \), fix ANY 
connection \( \nabla \), and let 
\[
R^\nabla = \nabla^2 \in A^2(M; \text{End}(E))
\]
its curvature form. The 1st Chern FORM is 
\[
c_1(E, \nabla) = \frac{i}{2\pi} \text{tr}(R^\nabla) \in A^2(M)
\]
where we used \( \text{tr} : \text{End}(E) = E \otimes E^* \to \mathbb{C} \).

**Theorem 12.1.** The assoc \( (E, \nabla) \) cx v. \( \text{bd} \) with connection \( \rightsquigarrow c_1(E, \nabla) \) satisfies:
(a) \( c_1(E, \nabla) \) is a well defined, closed 2-form 
(b) \( c_1(E, \nabla + A) = c_1(E, \nabla) + \frac{i}{2\pi} d(\text{tr}A) \), and its class in \( H^2(M, \mathbb{R}) \) is indep of conn 
(c) (natural) \( f^*[c_1(E, \nabla)] = c_1(f^*E, f^*\nabla) \) 
(d) behaves well under \( \oplus, \otimes, \det, E^* \) e.g. \( c_1(E_1 \oplus E_2, \nabla_1 + \nabla_2) = c_1(E_1, \nabla_1) + c_1(E_2, \nabla_2) \) 
(e) for the Chern connection on \( \tau \to \mathbb{CP}^n \), \( c_1(\tau, \nabla) = -\omega_{FS} \) the Fubini-Study form.
In particular, it represents the image of the first Chern class \( c_1(E) \in H^2(M, \mathbb{Z}) \):

\[
c_1(E) = \frac{i}{2\pi} [\text{tr} (R^\nabla)] \in H^2(M; \mathbb{R})
\]

Note: axioms true at the level of forms/repr of DeRham class; BUT loose torsion;

**Proof.** (a) Well defined/closed: intrinsic construction and Bianchi identity \( \nabla (R^\nabla) = 0 \).

\[\nabla e_i = \tau_{ij} e_j \quad \Theta = d\tau - \tau \wedge \tau \quad \text{tr} \Theta = \Theta_{ii} = d\theta_{ii}\]

since \( \text{tr}(\tau \wedge \tau) = \tau_{ik} \wedge \tau_{ki} = 0 \) since both sym and skew sym (1-forms); note: closed. Both curvature and trace are intrinsic, i.e. behave well under change of trivialization/basis i.e. gauge transf \( g : U \rightarrow GL(r, \mathbb{C}) \):

\[
\tau_{g(e)} = (dg)g^{-1} + g\tau_{e}g^{-1} \quad \Theta_{g(e)} = g\Theta_{e}g^{-1} \quad \text{SAVE (12.1)}
\]

where \( e \) is the COLUMN vector with entries \( e_i \). So \( \text{tr} \Theta \) is independent of trivialization!

Note: for a LINE bundle \( \Theta = d\tau \) closed but NOT exact. CAREFUL: \( \tau \) NOT globally defn but \( d\tau \) IS; \( \tau \) behaves like \( d\theta \) in polar coord! NOT a coincidence:

\[
\tau_{g(e)} - \tau_{e} = d\log g \quad \Theta_{g(e)} = \Theta_{e} \quad \text{if} \quad g : U \rightarrow S^1, \text{regarded as} \ S^1 \text{ action on} \ e
\]

Note: For any (matrix) Lie gr \( G \), the Maurer Cartan form is defined by

\[
\tau_g = (dg)g^{-1} = d\log g \quad \forall g \in G \quad \text{so} \quad \tau \in A^1(G; g)
\]

(i.e. the logarithmic differential of the identity). One can check \( \tau \) satisfies the Maurer-Cartan eq (integrability condition):

\[
d\tau - \frac{1}{2} [\tau, \tau] = 0.
\]

Note: (skip) More gen, in a frame \( e \)** as MATRICES** Bianchi identity becomes:

\[
d\Theta_e = [\tau_{e}, \Theta_e]
\]

(b) The difference of 2 connections is a 1-form \( A = \nabla_1 - \nabla_2 \), and \( R^{\nabla_1} - R^{\nabla_2} = dA - A \wedge A \) so their traces differ by \( \text{tr} dA = d\text{tr} A \), a closed form, i.e.

\[
c_1(E, \nabla + A) = c_1(E, \nabla) + \frac{i}{2\pi} d(\text{tr} A) \quad \text{for all} \ A \in A^1(\text{End}(E)).
\]

Real coef? fix a hermitian metric and compatible connection. In a UNITARY frame, \( \tau \) is skew hermitian (PS#3)

\[
\tau_{ij} + \tau_{ji} = 0
\]

thus so is \( \Theta \); so its trace is purely imaginary. Next time: show it has \( \mathbb{Z} \) coef – using formulas above and Cech=deRham cohom.

(c) For the rest of the axioms, use PS# 3: connections and their curvature behave well under direct sum, tensor product, etc; then take trace/pass to cohom:

\[
R^{\nabla_1 \oplus \nabla_2} = R^{\nabla_1} \oplus R^{\nabla_2} \quad R^{\nabla_1 \otimes \nabla_2} = R^{\nabla_1} \otimes 1 + 1 \otimes R^{\nabla_2} \quad R^{\nabla^*} = -(R^{\nabla})^t
\]

so

\[
c_1(E^*, \nabla^*) = -c_1(E, \nabla) \quad \text{etc}
\]

Only property left is the normalization, which we prove next. \( \square \)
Note: for the CHERN connection in a HOLO frame $e$, the conn 1-form is (PS#3)

$$\tau = \partial H \cdot H^{-1} = \partial \log H$$

where $H = (h_{ij})$ is the matrix assoc to hermitian metric in the frame $e$.

**Lemma 12.2.** For the Chern connection on $\tau \to \mathbb{CP}^n$

$$c_1(\tau, \nabla) = \frac{i}{2\pi} R^\nabla = -\omega_{FS} \quad \text{thus} \quad c_1(\tau) = -h$$

where $h$ is the Poincare dual of the hyperplane $H \subset \mathbb{CP}^n$ class.

**Proof.** Recall: the tautological line $bd \tau \to \mathbb{CP}^n$ is the subset $(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}$ such that $z \in l$. * picture * The standard herm metric on $\mathbb{CP}^n+1$ induces one on $\tau$. Use formula above. For a LINE bundle, a holo frame = local holo section $\sigma \neq 0$, $H = |\sigma|^2$ its magnitude,

$$\tau = \partial H \cdot H^{-1} = \partial \log |\sigma|^2$$

$$\Theta = d\tau = i\partial\bar{\partial} \log |\sigma|^2$$

How to find such section $\sigma$? On $U_0$ i.e. $z_0 \neq 0$ take the (canonical) section

$$\sigma([1, z_1, \ldots, z_n]) = (1, z_1, \ldots, z_n) \quad \text{a point in the fiber of } \tau \subset \mathbb{CP}^{n+1} \setminus 0.$$ 

But recall that on $U_0$ the Fubini-Study form is

$$\omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log(1 + |z|^2) = -\frac{i}{2\pi} \partial\bar{\partial} \log |\sigma|^2$$

and both extend to compactification. (Note order reversal.) □

**Note:** Usually $\tau^* = O(H)$ or $O(1)$ is called the hyperplane line bundle; its tensor powers are denoted $O(k)$. Next time: we show it gen Pic($\mathbb{CP}^n$).

**Note:** The coord fcn $z_i$ on $\mathbb{CP}^n$ is a holo section of $O(1)$, vanishing on the hyperplane at infinity $z_i = 0$. In fact, $\{z_i\}$ are a basis of $H^0(\mathbb{CP}^n, O(1))$ as a vector space, i.e. any holo section $\sigma$ is a linear combination of these; therefore the zero locus of ANY holo section of $O(1)$ is a hyperplane.

12.2. **Higher Chern classes, Chern-Weil Theory.** If $E = L_1 \oplus \cdots \oplus L_r$ with the direct sum connection, its curvature is diag, so the trace is the sum of the curv of the pieces, i.e. the first symm poly in them, i.e sum of e-values. Can also take the other sym poly:

**Theorem 12.3.** The total Chern form

$$c(E, \nabla) = \det \left( I + \frac{i}{2\pi} R^\nabla \right)$$

descends to cohomology where it represents $c(E)$.

Why? think e-values by splitting principle (diag the matrix $\Theta$ after some pullback).

Better: Weil story, use repr theory; Recall: in a frame $e$ get $\Theta_e$; change frame by $g$ get:

$$\tau_{g(e)} = (dg)g^{-1} + g\theta_e g^{-1} \quad g\Theta_{g(e)} g^{-1} = \Theta_e \quad \text{(12.2)}$$

Interpret this as considering the assoc frame bundle (whose fiber at a point $x \in M$ consists of all frames $e$ of $E_x$), i.e assoc principal $G$-bundle, where $G = GL(n, \mathbb{C})$ or $U(n)$ or some other subgroup e.g. $SU(n)$.

The conn matrix and curvature take values in the Lie algebra $\mathfrak{g}$; more precisely in the Adjoint representation of $G$ on $\mathfrak{g}$ by (12.2). For sym $G$-invar tensor/multilin map

$$P : (\mathfrak{g})^k \to \mathbb{C} \quad \text{i.e.} \quad P \in \text{Sym}^k(\mathfrak{g}^*|)^G$$

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we can evaluate it on \( k \) copies of \( \frac{i}{2\pi} \Theta \) to get a closed, \( 2k \)-form on \( M \) (HW); its deRham cohom is indep of connection: any two connections are joined by a path of connections, which can be regarded as a connection in the pullback bd over \( M \times I \); use Stokes.

**Theorem 12.4.** (Chern-Weil) Any (cx) v. bd \( E \to M \), determines a Chern-Weil map taking \( G \)-invariant, symmetric poly to cohom classes:

\[
\text{Sym}^k(g^*)^G \to H^{2k}(M; \mathbb{C})
\]

*This is algebra homomorphism: i.e. behaves well under both sum and product.*

**Note:** RHS is intrinsically the deRham model of \( H^*_dR(BG; \mathbb{C}) = \text{Sym}^*(g^*)^G \) as \( G \)-equivariant cohomology of \( EG \), so Chern-Weil is \( f^* \), the pullback by classifying map \( f: M \to BG \).

**Note:** Euler class of a real v.bd is not a stable invar, but for a cx v. bd it is the top Chern class, and that is stable.

---

\[ \text{the product is wedge product on forms/cup product in cohom} \]

Lecture 13. Sheaves and Cech cohomology

** next lecture: divisors and line bds – read Appendix in Moroianu or §2.3 Huybrechts **

Today: Sheaves, Cech cohom + applications
Crash intro to both, motivated by examples (see Appendix B of Huybrechts for details).

1. Sheaves: gen of bundles, understand a bd in terms of its local sections and how they patch together; better behaved functorially, get also e.g. skyscraper sheaf supported at a point: its sections are delta-functions at that point.

2. Sheaf/Cech cohomology: natural cohom theory with ”values in a bundle/(pre)-sheaf”.
   key property: SES of coef \(\Rightarrow\) LES in cohomology.

3. Goal:
   - Dolbeaut Thm \(\check{H}^{q}(M, \Omega^{p}_{M} \otimes E) = H^{p,q}(M; E)\);
   - classify holo/smooth line bds up to isom by 1-Cech cocycle/1st Chern class
   - defm/obstr: given a cx line bd, can we put a holo str? how many?
   - calc \(\text{Pic}(\mathbb{C}P^{n})\); Lefschetz Thm on \((1,1)\) classes

13.1. Sheaves. \(M=\) cx mfd. Recall: locally lots of holo fcn, few globally (csts if \(M\) cnt).

Sheaf of holo fcn on \(M\): For each open \(U \subseteq M\), let
\[ O_{M}(U) = \text{space of holo fcn on } U; \]

usually \(U=\text{chart so get } O_{C^{n}}(U')\). Often had to shrink \(U\) so work with GERMS of holo fcn (two holo fcn near \(x\) are equiv if they agree on a smaller nbd of \(x\)); formally: take the stalk \(O_{M,x} = \bigcap_{x \in U} O_{M}(U)\) at the point \(x\).

Can do the same for \(O^{*}=\) nonzero holo fcn; more generally sheaf of holo/smooth sections of a holo/smooth bundle \(E \to M\), like those appearing in Dolbeault cohom:
\[ \Gamma(E) = \text{A}(E) \text{ smooth sect of } E \quad \text{or } \quad \Omega(E) = \text{O}(E) \text{ holo sections of } E \]
e.g. \(A^{p,q}(M; E) = \text{smooth sections of } \Lambda^{p,q}(M) \otimes E\) or \(\Omega_{M}^{p,q}\text{holo sections of } \Lambda^{p,0}(M)\).

**Definition 13.1.** A sheaf \(\mathcal{F}\) on \(M\) associates to each open \(U \subseteq M\) a ”space of sections” \(\mathcal{F}(U) = \Gamma(U; \mathcal{F})\) which is a group (vect space etc), and to each inclusion \(U \subseteq V\) of open sets a ”restriction” \(r_{U,V}: \mathcal{F}(V) \to \mathcal{F}(U)\) which is a homomorph, satisfying compatibility cond:

**picture**

(i) \(r_{U,V} \circ r_{V,W} = r_{U,W}\) for all \(U \subseteq V \subseteq W\) and \(r_{U,U} = \text{id}\);
(ii) (gluing) if two sections \(\sigma_{i} \in \mathcal{F}(U_{i})\) agree on \(U_{1} \cap U_{2}\) then there exists a section \(\sigma \in \mathcal{F}(U_{1} \cup U_{2})\) which extends both i.e. such that restriction \(\sigma|_{U_{i}} = \sigma_{i}\).
(iii) (uniqueness) if \(\sigma \in \mathcal{F}(U_{1} \cup U_{2})\) restricts to 0 on \(U_{1}\) and \(U_{2}\) then \(\sigma = 0\).

Note:
- condition (i) defines a presheaf, i.e. a functor from open subsets of \(M\) to groups/etc.
- The ”stalk”= ”fiber” at point \(x\) is \(\mathcal{F}_{x} = \bigcap_{x \in U} \mathcal{F}(U)\), i.e. germs of sect;
- can defn morphism of (pre)-sheaves; exact seq (careful: inj means what you think, but surj ONLY required for \(U\) small)

**Example 13.2.** Exponential SES:
\[ 0 \to \mathbb{Z} \to O \xrightarrow{\exp} O^{*} \to 0 \]
where \(f \mapsto \exp(2\pi i f)\). This is injective for all \(U\), but ONTO only after shrinking \(U\): locally can take log, but maybe not globally; e.g.: \(z \in O^{*}(\mathbb{C}^{*})\) is not in image, but its restr to any ball \(U \subseteq \mathbb{C}^{*}\) is. (so the image of exp is NOT a sheaf)
Careful: for sheaves naive defn of IMAGE/Cokernel gives only a presheaf! The naive defn
works fine for the kernel.

Note: Each pre-sheaf \( F \) can be converted into a sheaf \( F^+ \) (equality for sheaves). Define
\( F^+(U) \) to consist of "local sections" i.e.

\[
\text{maps } s : U \to \bigcup_{x \in U} F_x \text{ with } s(x) \in F_x \text{ (germs of sections)}
\]

and such that \( s \) is the restriction of a section of \( F \) around each point \( x \in U \).

(SKIP) If \( Y \subset X \) CLOSED subset, a sheaf on \( Y \) can be regarded as one on \( X \) by defn
\( U \mapsto F(U \cap Y) \) called extending \( F \) by 0; get a skyscraper sheaf on \( M \) supported along \( V \).

Example 13.3. (SKIP) For any \( V \) cx smfd of \( M \) get two SES of sheafs:
(a) on \( M \):

\[
0 \to I_V \hookrightarrow O_M \to O_V \to 0 \tag{13.1}
\]

where we regard \( O_V \) as a a skyscraper sheaf on \( M \), and \( J_V = \text{Ker } r \) is the ideal of holo
functions which vanish when restricted to \( V \) (sometimes denoted \( O_M(-V) \))
(b) tangent/normal sequence on \( V \) (adjunction formula)

\[
0 \to T_V \to T_M|_V \to N_XV \to 0 \tag{13.2}
\]

regarded holo v. fields.

13.2. Sheaf cohom/Cech cohom. Records failure to globally glue local sections.

Fix a (pre)-sheaf \( F \) over \( M \), and a locally finite cover \( U = \{U_\alpha\} \) of \( M \); let \( U_I = \cap_{i \in I} U_i \).
Define a Cech (co)-chain complex

\[
\tilde{\mathcal{C}}^0(U; F) \overset{\delta}{\to} \tilde{\mathcal{C}}^1(U; F) \overset{\delta}{\to} \tilde{\mathcal{C}}^2(U; F) \to \ldots
\]

where \( \tilde{\mathcal{C}}^0 = \prod_\alpha F(U_\alpha) \) and \( \tilde{\mathcal{C}}^k = \prod_{|I|=k+1} F(U_I) \) sections on \( k+1 \) overlaps. The coboundary

\[
(\delta \sigma)_I = \sum_{j=0}^{k+1} (-1)^j \sigma_{I\setminus i_j}|_{U_I}
\]

(alternating sum of restrictions on overlap) e.g.

on \( \tilde{\mathcal{C}}^0 \) get \( (\delta \sigma)_{\alpha\beta} = -\sigma_\alpha + \sigma_\beta \) on overlap \( U_{\alpha\beta} \)

on \( \tilde{\mathcal{C}}^1 \) get \( (\delta \sigma)_{\alpha\beta\gamma} = \sigma_\beta - \sigma_\alpha + \sigma_{\alpha\beta} \) on overlap.

Take (co)hom (since \( \delta^2 = 0 \)) get \( \tilde{H}^p(U; F) \). Refine the cover (get chain homotopy), and define
Cech cohom as direct limit under refinements:

\[
\tilde{H}^p(M; F) = \lim_{\longrightarrow} \tilde{H}^p(U; F)
\]

Example 13.4. \( \tilde{H}^0(M; O) \) consists of global holo fcn on \( M \); \( \tilde{H}^0(M, Z) \) consists of locally
constant integer valued functions.

Key prop:
- SES of sheaves induce LES in cohom, eg exp seq \( \implies \) LES

\[
H^1(M, Z) \to H^1(M, O) \to H^1(M, O^*) \overset{\delta^*}{\to} H^2(M, Z) \to H^2(M, O)
\]
• if \( \mathcal{F} \) = sheaf of smooth sections in some bd, can use partitions of 1 to patch local sections
  \( \implies \check{H}^q(M, \mathcal{F}) = 0 \) for all \( q > 0 \). (such a sheaf is called ACYCLIC).
  e.g. any 1-Cech cocycle \( \sigma = \{ \sigma_{\alpha\beta} \} \) is \( \delta \)-exact, i.e. \( \sigma = \delta(\tau) \) where
  \[
  \tau_\alpha = \sum_\beta \varphi_{\beta\alpha} \sigma_{\alpha\beta} \quad \text{and} \quad \{ \varphi_\alpha \} \text{ part of 1 subord to } \{ U_\alpha \}
  \]
  \( \implies \check{H}^q(M, A^{*,*}) = 0 \) for all \( q > 0 \); of course, this FAILS for sheaves of HOLO sections.
• \textbf{Leray Thm} \( \check{H}^*(\mathcal{U}; \mathcal{F}) = \check{H}^*(M, \mathcal{F}) \) if cover \( \mathcal{U} \) is acyclic for sheaf \( \mathcal{F} \) i.e. \( H^q(U_1; \mathcal{F}) = 0 \) for all \( q > 0 \) and all \( I \).
• \( \check{H}^*(M, \mathbb{Z}) = H^*(M, \mathbb{Z}) \) ”usual” cohom if \( M \) a mfd (or simplicial cx).
• \textbf{dRham Thm} \( \check{H}^q(M, \mathbb{R}) = H^q_{dR}(M) \). Use Poincare Lemma: LES is exact, i.e acyclic resol:
  \[
  0 \to C \to A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \to \ldots
  \]
• \textbf{Dolbeault Thm} \( \check{H}^q(M, \Omega^p) = H^p\Omega^q(M) \). Use Poincare \( \overline{\partial} \)-Lemma, i.e.
  \[
  0 \to \Omega^p \xhookleftarrow{} A^{p,0} \xrightarrow{\overline{\partial}} A^{p,1} \xrightarrow{\overline{\partial}} A^{p,2} \to \ldots
  \]

\textbf{Theorem 13.5} (Kodaira-Spencer). \( Cx \) line bds/isom on smooth \( M \) are classif by
\[
[g_{\alpha\beta}] \in \check{H}^1(M, A^*) \cong H^2(M, \mathbb{Z}) \quad \text{and} \quad \delta^*(L) = [c_1(L, \nabla)] = c_1(L).
\]
Holo line bds/isom on \( cx M \) are classified by \( [g_{\alpha\beta}] \in H^1(M, O^*) \), i.e. there is a natural isom
\[
\text{Pic}(M) \cong H^1(M, O^*).
\]

\textbf{Proof.} (outline) key: exp sequence for smooth/holo functions gives (compatible) LESs:
\[
\begin{array}{c}
H^1(M, \mathbb{Z}) \xrightarrow{\delta} H^1(M, O) \xrightarrow{\delta} H^1(M, O^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \xrightarrow{\delta} H^2(M, O) \\
H^1(M, \mathbb{Z}) \xrightarrow{\delta} H^1(M, A) = 0 \xrightarrow{\delta} H^1(M, A^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \xrightarrow{\delta} H^2(M, A) = 0
\end{array}
\]
Holo/smooth line bds + choice of local trivializ gives rise to 1-Cech cocycle \([g]\); show
(a) its class is indep of choice of trivializ
(b) any cocycle \([g]\) \( \in \check{H}^1(\mathcal{U}) \) represents a line bd trivialized over \( \mathcal{U} \)
(c) if the cocycle is exact then the bundle is trivial
e.g. for (c): if \( g = \delta(\varphi) \) i.e. \( g_{\alpha\beta} = \frac{\varphi_{\beta\alpha}}{\varphi_\alpha} \implies \varphi \) = nowhere zero section = global triv (line bd).

Next, \( \delta^* \) is an isom for \( cx \) line bds by the exp seq + \( H^q(M, A) = 0 \) \( \forall q > 0 \) (part of 1).
Why \( \delta^*(L) = [c_1(L, \nabla)]? \) Use the formula for the connecting homomorphism (from alg top):
\[
\delta^*(g)_{\alpha\beta\gamma} = \frac{1}{2\pi i} (\log g_{\alpha\beta} + \log g_{\beta\gamma} + \log g_{\gamma\alpha}) \in \check{O}^2(M, A^0) \tag{13.3}
\]
But in fact this a 2-Cech cocycle with \textit{integer} coef, i.e. RHS \( \in \check{H}^2(M, \mathbb{Z}) \) as can be seen by taking log (when \( U_{\alpha\beta} \) s.conn) of
\[
g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1.
\]
On the other hand, we have formulas for Chern form, curvature, conn 1-form (in terms of transition fcn)
\[
c_1(L, \nabla) = \frac{i}{2\pi} \Theta \quad \Theta_\alpha = d\tau_\alpha \quad \tau_\beta - \tau_\alpha = -d\log g_{\alpha\beta}
\]
\[ \Theta = d\tau \in \hat{C}^0(M, A^2), \quad (\tau_a) \in \hat{C}^1(M; A^1), \quad \delta(\tau) = d(\log g) \]

This fits in the double deRham/Cech complex \( \hat{C}^q(U, A^p) = \text{Cech complex with values in } p\)-forms (similar arg is used for deRham=Cech)

\[ \begin{array}{cccccc}
C & \overset{\iota}{\longrightarrow} & A^0 & \overset{d}{\longrightarrow} & A^1 & \overset{d}{\longrightarrow} & A^2 \\
\downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \\
\hat{C}^0(U, \mathbb{C}) & \overset{\log g}{\longrightarrow} & \hat{C}^1(U, \mathbb{C}) & \overset{\tau}{\longrightarrow} & \hat{C}^2(U, \mathbb{C}) \ni \delta^*(g) & \overset{\iota}{\longrightarrow} \\
\end{array} \]

since (13.3) becomes

\[ \iota(\delta^*(g)) = \delta(\log g) \]

(up to some \(2\pi i\) factor to have \(\mathbb{Z}\) coef, but which can be absorbed in defn of \(\iota : C \to A^0\)). \(\Box\)

**Corollary 13.6.** If \(H^{0,2}(M) = 0\) then any cx line bd admits a holo str. If \(H^{0,1}(M) = 0\) then this structure is unique (up to holo isom). The automorphism group of a holo line bd is \(H^0(\mathcal{O})\) (up to some \(2\pi i\) factor to have \(\mathbb{Z}\) coef, but which can be absorbed in defn of \(\iota : C \to A^0\)). The autom of holo v bd = \(H^0(M; \text{End}(E)) = H^{0,0}(M)\) for a line bd.

**Proof.** Follows immediately from the exp sequence and Dolbeault Thm: \(\hat{H}^q(M, \mathcal{O}) = H^{p,q}(M)\).

For \(\mathbb{CP}^n, H^{0,2} = H^{0,1} = 0\), and we already know \(c_1(\tau) = -h\) generates \(H^2(M; \mathbb{Z}) = \mathbb{Z}\). \(\Box\)

**Note:** Kodaira-Spencer Thm for line bds means:

- obstruction to existence of holo str = kernel of \(H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O})\)
- deformations of holo str = image of \(\exp : H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*)\)
- autom of holo v bd = \(H^0(M; \text{End}(E)) = H^{0,0}(M)\) for a line bd.

**Theorem 13.7** *(Lefschetz Thm on (1,1)-classes).* Assume \(M\) is compact Kahler. Then

\[ \text{Pic}(X) \longrightarrow H^{1,1}(M) \cap H^2(M, \mathbb{Z}) \]

is surjective. i.e. \(\forall \gamma \in H^{1,1}(M; \mathbb{Z})\) there exists a holo line bd \(L\) such that \(c_1(L) = \gamma\).

Moreover, for any real, closed, (1,1)-form \(\omega\) such that \([\omega] = c_1(L)\) there exists a hermitian metric on \(L\) such that its Chern connection has \(c_1(L, \nabla) = \omega\).

**Proof.** *(Outline)* Key: KAHLER \(\implies\) (a) deRham and Dolbeault complexes and Thms are compatible; (b) global \(\partial \bar{\partial}\) Lemma (follows from (a) by diagram chase).

More precisely, for a Kahler manifold, \(H^k(M; \mathbb{C}) = \bigoplus H^{p,q}(M)\) and diagram commutes:

\[ \begin{array}{ccc}
H^k(M, \mathbb{C}) & \overset{\iota^*}{\longrightarrow} & H^k(M, \mathcal{O}) \\
\downarrow{\text{deRham}} \cong & \cong & \downarrow{\text{Dolbeault}} \\
H^k_{dR}(M, \mathbb{C}) & \overset{\exp}{\longrightarrow} & H^{0,k}(M) \\
\end{array} \]

because

\[ \begin{array}{ccc}
A^k & \overset{d}{\longrightarrow} & A^{k+1} \\
\downarrow{\pi^{0,k}} & & \downarrow{\pi^{0,k+1}} \\
A^{0,k} & \overset{\partial}{\longrightarrow} & A^{0,k+1} \\
\end{array} \]

commutes at the level of the two resolutions. CAREFUL: it does NOT commute for \(p > 0!!\)
So for a Kahler mfd, the exp sequence becomes LES:

\[ \text{Pic}(M) = H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \xrightarrow{\pi^{0,2}} H^{0,2}(M). \]

Recall: for any cx line bd \( c_1(L, \nabla) \) is a real form. But for a real form \( \alpha \) we have \( \alpha^{0,2} = \overline{\alpha^{2,0}} \) thus \( \alpha^{0,2} = 0 \) iff \( \alpha \) is (1,1). So Ker \( \pi^{0,2} \) is \( H^{1,1}(M, \mathbb{Z}) \). But that is also the image of \( \delta^* = c_1 \), because the sequence is exact.

To prove the result at the level of forms, start with any hermitian metric \( h_0 \), and use \( \partial \bar{\partial} \) Lemma (HW4/#2) to modify it to one \( h = h_0 e^f \) with desired Chern form. Recall: for the Chern connection, in a holo frame (of the line bd)

\[ \Theta = \partial \bar{\partial} \log H \]

Look at the diff \( \alpha = \omega - c_1(L, \nabla_0) \), a \( d \)-exact real (1,1)-form, so \( \exists \) a global fcn \( f \) such that \( \alpha = \partial \bar{\partial} f \). \( \square \)

HW: Fill in these details; also explore uniqueness (i.e. how many choices?)

HW: explicitly trace the image in \( H^{0,2}(M) \) of a 2-Cech cocycle \( (z) \in \check{H}^2(M, \mathbb{Z}) \) through both de Rham and Dolbeault Thm in [13.4].

**Corollary 13.8.** (SKIP/HW) If \( M \) is compact Kahler then the Jacobian

\[ \text{Pic}^0(M) \cong H^{0,1}(M)/H^1(M, \mathbb{Z}) \]

is naturally a cx torus of dim \( b_1(M) \).

**Proof.** Look earlier in the sequence: \( \text{Pic}^0 = H^1(M, \mathcal{O})/H^1(M, \mathbb{Z}) = H^{0,1}(M)/H^1(M, \mathbb{Z}). \)

Argue that \( H^1(M, \mathbb{Z}) \hookrightarrow H^1(M, \mathbb{C}) \xrightarrow{\pi^{0,1}} H^{0,1}(X) \) is injective with discrete image, aka a lattice, thus quotient is indeed a torus. \( \square \)

**Note:** Careful: The Jacobian is always equal to the quotient \( H^1(M, \mathcal{O})/H^1(M, \mathbb{Z}) \), but in general this quotient may be terrible (topologically; locally the quotient may look fine, but globally orbits may return to be dense; typical example of ”bad quotient” is \( \mathbb{R} \mod \mathbb{Z} \oplus \alpha \mathbb{Z} \) where \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), where the image is NOT discrete).

**Example 13.9.** If \( C \) is a compact cx curve, \( \text{Pic}^0(C) = H^{0,1}(C)/H^1(C, \mathbb{Z}) \) is a \( g \)-dim torus.
Lecture 14. Divisors, line bundles and Calabi-Yau manifolds

Today we discuss two (seemingly unrelated) topics: (a) divisors and line bundles, (b) Calabi-Yau manifolds. Next time, we focus on

- holonomy ** review 5.5-5.8 of Moroianu, read p213-215 of Huybrechts **
- Kahler or Ricci-flat Kahler = CY mfdls (via Calabi-Yau thm).** review lect 8 **

14.1. **Euler class and PD.** We can also describe a cx line bd is via its Euler class, which is Poincare dual (PD) to the zero locus of a generic section (i.e. transverse to the zero section).

**Proposition 14.1.** Assume $E \to M$ is a complex vector bundle of rank $r$ over smooth mfd $M$. Then its top Chern class is the same as the Euler class:

$$c_r(E) = \chi(E) = \text{PD to zero locus of generic smooth section of } E.$$  
(i.e. a section transverse to the zero section).

(HW) Prove by hand a special case: $L=$line bundle, $s=$ holo section transverse to zero section, so zero locus $V = s^{-1}(0)$ smooth and $R^V$ is a Chern connection on $L$. Show that

$$\frac{i}{2\pi} \int_M R^V \wedge \alpha = \int_V \alpha$$  
for all $d$-closed forms $\alpha$ on $M$

**Hint:** use $R^V = \bar{\partial} \partial \log |s|^2 = \frac{1}{2} (\partial - \bar{\partial}) \log |s|^2$, Stokes theorem, and Cauchy integral formula in one variable (the normal direction to $V$).

**Note:** Can rescale normal to $V$ and make curvature ”spike up” and converge to a ”delta-function” along $V$. So $V \overset{PD}{\sim} \eta_V=$delta-form on $V$ (”restricts” to $d\text{vol}_V$ along $V$, 0 outside).

**Note:** Going back, one can construct a line bundle over $X$ with a section that vanishes transversally along $V$: assume $U$ is a tubular neighborhood of $V$, and $s$ is a smooth/holomorphic function on $U$ that vanishes transversally along $V$ (and nowhere else). We can clutch this section $s_1 : U \to \mathbb{C}$ with the constant section $s_2 : M \setminus V \to \mathbb{C}$, $s_2 \equiv 1$ on the overlap $U \setminus V$ to get a nowhere zero transition function $g_{12} = \frac{s_2}{s_1} : U \setminus V \to \mathbb{C}^*$ and therefore a smooth/holomorphic bundle $L$ with a smooth/holomorphic section $s$ whose transverse zero locus is $V$.

**Note:** The tubular neighborhood theorem implies that $s_1$ can be regarded as the canonical section of the pullback of $N_V$ over itself, and therefore $L|_V = N_V$ (adjunction formula).

**Note:** Transv is easy to achieve in smooth setting: can locally perturb section to transverse one, but NOT in the HOLO setting! Luckily, the zeros of holo fcn are well behaved (unlike zeros of smooth fcn!) – they are analytic hypersurfaces, generically smooth.

CAREFUL: Today I ignore singularities completely!! (one needs to deal with them!!)

14.2. **Line bundles on $\mathbb{CP}^n$.** Recall that holo line bundles on $\mathbb{CP}^n$ are classified by their $c_1$, i.e. $L = \mathcal{O}(d)$ where $d$ is called the degree. One can find all global holomorphic sections:

$$H^0(\mathbb{CP}^n, \mathcal{O}(d)) = \{ \text{ degree d homogenous polynomials in } z_0, \ldots, z_n \}.$$  

Of course, $\mathcal{O}(d)$ has no holomorphic sections when $d < 0$ (tautological line bundle is $\mathcal{O}(-1)$).

A degree $d$ hypersurface $V$ in $\mathbb{CP}^n$ is by definition the zero locus of a degree $d$ homogenous polynomial (equivalently of a holo section of $\mathcal{O}(d)$). It has two key properties:

- the fundamental class $[V] = dH$ in $H_{2n-2}(\mathbb{CP}^n)$.
- the normal bundle $N_V \cong \mathcal{O}(d)|_V$ (as holo line bundles, by adjunction formula).
Note: (skip?) This implies \( c_1(N_V) = \chi(N_V) \) is the Poincare dual in \( V \) of \( dH \cap V \) (i.e. \( d \) times the intersection of \( V \) with a generic hyperplane).

Geometrically: deform coef of poly to make it factor into linear terms, so \( V \) is homologous to \( d \) hyperplanes ** picture **; a small deformation of \( V \) is a section of \( N_V \), so at least its Euler class \( \chi(N_V) = c_1(N_V) \) is PD to the self intersection \( V \cdot V = dH \cdot V \).

Cup product in \( H^* \overset{PD}{\leftrightarrow} \) intersection product in \( H_{2n-*} \) (perturb to make \( \mathfrak{h} \), then intersect).

**Example 14.2.** (SKIP) The case \( \mathbb{CP}^1 \) is very simple: any degree \( d \) poly factors out, and \( V \) is the collection of zeros= unordered points (with multiplicities). In this case, a poly is determined by its zeros (up to an overall \( \mathbb{C}^* \) action) so \( H^0(\mathbb{CP}^1, \mathcal{O}(d)) \) is a \( d + 1 \) dimensional complex vector space. Therefore the map that sends a section \( s \in H^0(\mathbb{CP}^1, \mathcal{O}(d)) \) to its zero locus \( s^{-1}(0) \in \text{Sym}^d(\mathbb{CP}^1) \) (its zeros including multiplicity) gives a bijection/isom

\[
\mathbb{CP}^d = \mathbb{P}(H^0(\mathbb{CP}^1, \mathcal{O}(d))) \overset{\sim}{\rightarrow} \text{Sym}^d(\mathbb{CP}^1) = \mathbb{CP}^d
\]

where last equality is via the Viete relations.

14.3. **Divisors and Line Bundles.** In general, any (smooth) complex hypersurface \( V \) in \( M \) can be regarded as the (transverse) zero locus of a holomorphic section \( s \) in some holomorphic line bundle \( L \) (canonical up to isom), denoted \( \mathcal{O}_M(V) \); in particular, the Euler class of \( L \) is Poincare dual to \( V \). If \( M \) is compact, the section is also canonical up to scale (ratio of two holomorphic sections vanishing transversally along \( V \) is a holo function on \( M \)).

This extends to define a group homomorphism

\[
\text{Div}(M) \rightarrow \text{Pic}(M) \quad D \mapsto \mathcal{O}_M(D)
\]

where \( \text{Div}(M) \) is the collection of divisors \( D \) in \( M \), i.e. formal sums

\[
D = \sum a_i V_i
\]

where \( V_i \) are (analytic) hypersurfaces in \( M \) (possibly SINGULAR), the sum is locally finite and the coefficients \( a_i \) are integers. If all coefficients are nonnegative, the divisor \( D \) is called effective, denoted \( D \geq 0 \), and can be regarded as the zero locus (with multiplicities \( a_i \)) of the holomorphic section \( s = \prod s_i^{a_i} \) of the bundle \( \mathcal{O}(V_i)^{a_i} = \mathcal{O}(\sum a_i V_i) \).

**Example 14.3.** On a (closed) Riemann surface \( \Sigma \) (1-dim cx mfd), a divisor \( D \) is a finite sum of points \( x_i \) with integer multiplicity. An effective divisor is an unordered collection of points. As the points move around on \( \Sigma \), the holomorphic line bundles \( \mathcal{O}(\sum x_i) \) may vary. (SKIP) In fact, we get a map \( \text{Sym}^k(\Sigma) \rightarrow \text{Pic}(\Sigma) \) (related to the Abel-Jacobi map). When the genus of \( \Sigma \) is at least 1 (after choosing a base point) we get an injection \( \Sigma \rightarrow \text{Jac}(\Sigma) \), by the same argument as for elliptic curve. There is a beautiful theory involving \( \text{Sym}^k(\Sigma) \), the Jacobian \( \text{Jac}(\Sigma) \) (which is a \( 2g \)-dim complex torus by the exp seq) and theta divisors that needless to say we have to skip...

Going back, we can associate to each meromorphic section \( s \neq 0 \) in a holo line bundle a divisor \( \text{div}(s) \) keeping track of its zero locus and infinity/poles locus (including multiplicities, i.e. order of vanishing/poles). Moreover, the ratio of two meromorphic sections of \( L \rightarrow M \) is a meromorphic function on \( M \) (aka a meromorphic section of the trivial line bundle).

Two divisors \( D_1, D_2 \) are linearly equivalent \( D_1 \sim D_2 \) if \( \mathcal{O}(D_1) \cong \mathcal{O}(D_2) \) (isom as holo line bds) or equivalently \( D_1 - D_2 \) is the divisor of a global meromorphic function \( f \) (called a principal divisor).

**Note:** Meromorphic sections of \( L \) are locally meromorphic functions, i.e. ratios of local holomorphic functions – thus best described via a sheaf \( \mathcal{M}(L) \). (SKIP) In fact, \( \text{Div}(M) \cong \)
\( \tilde{H}^0(M, \mathcal{O}^*/\mathfrak{M}^*) \), but won’t need this.

**Careful!** I ignored singularities!! To deal with them, replace "transverse" by "irreducible" = smooth locus connected = local defining eq does not factor further (note: \( \mathcal{O}_{M,x} \) is UFD, noetherian etc, see Huybrechts §1.1; key: Weiestrass Preparation Thm: locally zeros of holo functions are the same as zeros of poly in one var with holo coef in rest).

**Example 14.4.** In \( \mathbb{C}^2 \), hypersurface \( y^2 = x^2 \) is sing at \( x = y = 0 \) and its irred comp are \( y = x \) and \( y = -x \); hypersurface \( y^2 = x^3 \) is also sing at 0, but is irred; ** picture **

The function \( f(x, y) = (y - x)^3(y^2 - x^3)^5 e^x \) vanishes to order 3 along \( V_1 = \{ y = x \} \) and to order 5 along \( V_2 = \{ y^2 = x^3 \} \). Its ”zero divisor” is \( \text{div } f = 3V_1 + 5V_2 \). Similarly for poles.

To summarize: We get a HUGE abelian group \( \text{Div}(M) \) generated by all (irred) analytic hypersurfaces and a natural group homomorphism

\[
\text{Div}(M) \to \text{Pic}(M) \quad \text{defined by } D \mapsto \mathcal{O}(D)
\]

which descends to the quotient by lin equiv to give an injection (group monomorphism)

\[
\text{Div}(M)/\sim \hookrightarrow \text{Pic}(M)
\]

For line bundles \( L \) with a meromorphic section \( s \neq 0 \) we can also go back, and associate the divisor of \( s \) (in a compatible fashion, i.e. \( \mathcal{O}(\text{div}(s)) \cong L \)).

**Note:** As a consequence of Kodaira vanishing we will prove that if \( M \) is projective (i.e. \( M \) is a complex submanifold of \( \mathbb{C}\mathbb{P}^n \) then ANY holo line bundle has a nontrivial meromorphic section. Therefore if \( M \) is projective, \( \text{Div}(M)/\sim \cong \text{Pic}(M) \).

**Example 14.5.** (The canonical bundle of \( M = \mathbb{C}\mathbb{P}^n \).) Consider the section

\[
s(z) = dz_1 \wedge \cdots \wedge dz_n
\]

of the line bundle \( \Omega^1_M = K_M \) of holomorphic differentials. Then \( s \) is a nowhere vanishing meromorphic section, with poles of order \( n + 1 \) along the hyperplane \( z_0 = 0 \). Therefore

\[
K_{\mathbb{C}\mathbb{P}^n} = \mathcal{O}(-(n + 1)H) = \tau^{n+1}
\]

**Note:** (HW) With more effort, one can show that there is a natural SES of holo v. bds:

\[
0 \to \mathcal{O} \to \bigoplus_{k=0}^n \mathcal{O}(H) \to T\mathbb{C}\mathbb{P}^n \to 0. \tag{14.1}
\]

**Example 14.6.** (Adjunction formulas) FINALLY, assume \( V \subset M \) is a (smooth) hypersurface. We get two adjunction formulas:

\[
N_V = \mathcal{O}_M(V)|_V \quad \text{and} \quad K_V = (K_M \otimes \mathcal{O}_M(V))|_V
\]

The first one follows directly using defining eq and transition fns (as in §25.3 in Moroianu).

The second one follows immediately from the first one by taking determinants in the SES of holo BUNDLES on \( V \):

\[
0 \to TV \to TM|_V \to N_V \to 0 \quad \text{or dually} \quad 0 \to N^*_V \to \Lambda^{1,0}M|_V \to \Lambda^{1,0}V \to 0
\]

so \( K_M|_V = K_V \otimes N^*_V \).

**Note:** (SKIP) Equivalently, 1st adjunction means: if \( s \) is a nontrivial holomorphic section cutting out \( V \) transversally, multiplication by \( s^{-1} \) defines an injective sheaf homo, thus a SES

\[
0 \to \mathcal{O}_M(-V) \to \mathcal{O}_M \xrightarrow{\cdot s^{-1}} \mathcal{O}_V \to 0 \tag{14.2}
\]
The quotient is $\mathcal{O}_V$, regarded as a skyscraper sheaf on $M$ (extend it by 0, i.e. its sections are delta-functions supported on $V$), and $r$ denotes the restriction of along $V$. Dually, the quotient can be regarded as $\mathcal{I}_V$, the ideal of holomorphic functions on $M$ which vanish when restricted to $V$. (see (13.1) and p 84 of Huybrechts)

**Example 14.7.** If $V$ is a (smooth) degree $d$ hypersurface in $\mathbb{CP}^n$ its canonical bundle

$$K_V = \mathcal{O}(d - n - 1)|_V$$

For degree $d$ curves $C$ in $\mathbb{CP}^2$, adjunction formula implies $g = (d - 1)(d - 2)/2$. So a (smooth) cubic in $\mathbb{CP}^2$ is a torus (elliptic curve).

14.4. **Calabi-Yau manifolds.** Let $M^n$ a complex mfd. Recall that the canonical bundle

$$K_M = \Omega^n_M = \Lambda^n(M) = \Lambda^n(\Lambda^{1,0}M) = \det_C(\Lambda^{1,0}M)$$

is a holomorphic line bundle. A nowhere vanishing holomorphic section $\Omega$ of it is called a complex/holomorphic volume form on $M$. Of course, $\Omega$ exists iff $K_M$ is holomorphically trivial. In particular we need $c_1(TM) = 0$, but that may not be enough.

(recall $c_1(K_M) = -c_1(\det(TM)) = -c_1(TM)$).

Note that $\Omega$ is unique up to scale if $M$ is compact (by Liouville theorem).

**Definition 14.8.** A Calabi-Yau manifold is a (compact) Kahler manifold whose canonical bundle is trivial, i.e. admits both a Kahler form $\omega$ and a holomorphic volume form $\Omega$.

Careful!! There are several (inequivalent!) defn of CY manifolds (sometimes one requires simply connected, or allows noncompact cases etc). They appear in string theory (CY 3-folds), have many nice properties; e.g. Calabi-Yau Thm $\Rightarrow$ Kahler, Ricci-flat metric.

**Example 14.9.** The tori $M = \mathbb{C}^n/\Lambda$ have a holomorphic volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$, but are not simply connected. (Intrinsically $\Lambda = H^1(M, \mathbb{Z}) \subset H^{0,1}(M)$.)

**Example 14.10.** (dim = 1) The only compact cx curves with trivial $K_M$ are tori (i.e. elliptic curves). In this case $\omega = dz \wedge d\bar{z}$ while $\Omega = dz$.

**Example 14.11.** (dim = 2) Compact cx surfaces with trivial $K_M$ are classified: they are either tori (abelian varieties) or else satisfy $H^{0,1}(M) = H^1(M, \mathcal{O}) = 0$ (thus Pic$^0(M) = 0$) and are called $K3$ surfaces. All $K3$ surfaces are diffeomorphic, simply connected and Kahler (Siu Thm), thus are Calabi-Yau manifolds. (have same Hodge diamond, $\chi = 24$, intersection pairing is unimodular = $2E_8 \oplus 3H$ where $H =$hyperbolic). Well studied, very rich properties (appear in String dualities). Kodaira-Spencer deformation theory+more $\Rightarrow$ (polarized) $K3$ surfaces come in a 20 = $h^{2,2}$ dim moduli space (and projective ones are codim 1).

Examples of $K3$’s: (generic) degree 4 hypersurf in $\mathbb{CP}^3$. Some $K3$’s are elliptic fibrations over $\mathbb{CP}^1$ with 24 nodal fibers ** picture **, some have more singular fibers e.g. $E_8$ singularity.

**Example 14.12.** (dim=3) Generic quintic in $\mathbb{CP}^4$ is a CY 3-fold. The Fermat quintic

$$z_0^5 + \cdots + z_5^5 = 0$$

is a Calabi-Yau 3-fold, but it is singular– either deform it by adding $\varepsilon z_0 \ldots z_5$ or else resolve the singularities (by blowup, see PS#2) $\rightsquigarrow$ mirror symmetry.

**Example 14.13.** Any degree $n+1$ (smooth) hypersurfaces $V$ in $\mathbb{CP}^n$ has holo trivial canonical bundle (by adjunction formula), thus carry a holomorphic volume form. They are simply connected for $n \geq 2$, thus Calabi-Yau manifolds. More generally, one can take (complete) intersections in $\mathbb{CP}^N$ of (transverse) hypersurfaces of degrees $\sum d_i = N + 1$, or even in weighted projective spaces (of appropriate degrees).
Lecture 15. SPECIAL HOLONOMY, KÄHLER-EINSTEIN METRICS

Today: switch gears, return to diff geom. Plan:

- holonomy, and its relation to the Kahler and Kahler Ricci-flat condition;
- more gen, Kahler-Einstein eq and their bundle version, Hermitian Yang-Mills eq
- emphasize trichotomy zero/negative/positive curvature; skip relation to stability

15.1. Holonomy. Recall: any connection $\nabla$ in a bundle $E \to M$ defines a parallel transport

$$P_\gamma : E_x \xrightarrow{\cong} E_y$$ (linear isomorphism)

along each path $\gamma : [0,1] \to M$ from $x$ to $y$. (Any $\sigma_0 \in E_x$ uniquely extends to a parallel section $\sigma(t)$ of $E$ along $\gamma$ i.e. to a soln of ODE $\nabla_{\dot{\gamma}(t)} \sigma(t) = 0$ with IC $\sigma(0) = \sigma_0$).

If $\gamma$ is a loop based at $x$, $P_\gamma \in GL(E_x)$, and the holonomy (sub)group

$$\text{Hol}_x(E, \nabla) = \{ P_\gamma | \gamma \text{ loop based at } x \} \subseteq GL(E_x).$$

(there is also the restricted holonomy group coming from contractible paths). Holonomy is conjugate along paths, i.e. if $\gamma$ is a path from $x$ to $y$ then

$$\text{Hol}_y(E, \nabla) = P_\gamma \circ \text{Hol}_x(E, \nabla) \circ P_\gamma^{-1}.$$

If $M$ is connected, we get a holonomy group $\text{Hol}(E, \nabla)$ (well defined up to conjugacy).

15.2. Flat Bundles. If the bundle is flat, i.e. curvature $R^\nabla = 0$, the holonomy is invariant under homotopies of paths rel end points, so descends to $\pi_1$ to give the holonomy (monodromy) representation

$$\pi_1(M, x) \to GL(E_x)$$

Note: In fact, the bundle is flat iff the restricted holonomy group is trivial. By Chern-Weil theory, the Chern classes of a flat bundle are torsion (vanish in $H^*_dR(M)$).

Note: If $M$ is simply connected, any flat (principal) $G$-bundle is trivial (i.e. has a nonzero parallel global section); otherwise, local parallel sections exist, but may not patch globally! (equiv, global ones on universal cover may not descend to quotient).

Note: Holonomy identifies the (moduli) space the flat $G$-bundles (up to isom) to representations $\pi_1(M) \to G$ (up to conjugacy):

$$\{ \text{FLAT principal } G\text{-bundles}/\text{isom} \} \xrightarrow{\cong} \{ \text{repr } \pi_1(M, x) \to G \} /\text{conj}$$

Example 15.1. If $G$ is a finite group, a principal $G$-bundle is a (regular) cover of $M$!

Example 15.2. The moduli space of flat holomorphic bundles over Riemann surface $\Sigma$ is the Jacobian $\text{Pic}^0(\Sigma) = \text{Hom}(\pi_1(\Sigma) \to S^1)$, a $2g$ real dim torus. ($G = U(1) = S^1$ is abelian).

Note: (SKIP) Atiyah-Bott studied the moduli space of flat complex bundles over a (compact) Riemann surface and related it to the moduli space of (semi)-stable holomorphic bundles.

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3also called Hermitian Einstein eq
4technically the connection $\nabla$ is flat!
15.3. **Holonomy and Kahler manifolds.** Restrict to a 2n-dim Riemannian manifold \((M, g)\), so \(E = TM\) with its Levi-Civita connection \(\nabla\). Then the parallel transport is an isometry (by compatibility with metric), thus \(\text{Hol}_x(M, g)\) is conjugate to a subgroup of \(O(2n)\).

Berger Thm classifies all possible types of holonomy for an irreducible, simply connected manifold. (Here irred means not a product of lower dim Riemannian mflds, or equiv irred representation). We can do the same for a complex manifold with a Hermitian metric. Out of Berger’s list, there are several important cases:

- \(U(n)\) which gives Kahler manifolds
- \(SU(n)\) get Calabi-Yau manifolds (Ricci flat Kahler metrics)
- \(Sp(n/2)\) get hyperKahler (holomorphic symplectic).

We only discuss only the first two. Last one is rare, but with very rich properties (e.g. \(K3\) surfaces are hyperkahler). Hyperkahler can be regarded as the quaternionic analog of Kahler, and is a special case of CY manifolds (since \(Sp(n/2, \mathbb{C}) = U(n/2, \mathbb{H}) \subset SU(n)\)).

**Note:** \(\exists!\) soln to ODE \(\implies\) **Holonomy principle:** any (pointwise) tensor on \(T_xM\) which is \(\text{Hol}_x\)-invariant can be uniquely extended to a parallel tensor field on \(M\).

**Lemma 15.3.** Assume \((M, g)\) is a real 2n dim Riemannian manifold. Then

\[(M, g)\text{ is Kahler } \iff \text{Hol}(M, g) \subseteq U(n).\]

If \(\text{Hol}(M, g) = U(n)\), \(\exists!\) integrable complex structure for which \((M, J, g)\) is Kahler.

**Proof.** \(\implies\) follows immediately because when \(M\) is Kahler then \(\nabla J = 0\) for the LC connection, thus holonomy is also complex linear, thus all together hermitian. (Note that \(GL(n, \mathbb{C}) \cap O(2n) = U(n)\)).

\(\iff\) need to show that \(\exists\) integrable complex structure on \(M\) such that \(g\) is Kahler. Start with \(J_x\) a complex structure on the fiber \(T_xM\) (so \(J_x^2 = -\text{id}\)). The assumption implies \(J_x\) is holonomy invariant. So it extends to a (Levi-Civita) parallel a section \(J \in \text{End}(TM)\). But \(J^2\) is also parallel thus \(J^2 = -\text{id}\), i.e. \(J\) is an almost complex structure on \(M\) such that \(\nabla J = 0\). But \(g\) is also parallel so the 2-form \(\omega = g(J \cdot, \cdot)\) is parallel and thus closed. So \((M, J, g)\) is Kahler (recall \(\iff\) \(\nabla J = 0\) for LC and \(d\omega = 0\)). Uniqueness: HW. \(\square\)

**Note:** We crucially used that \(\nabla\) is torsion free. Where?

15.4. **Kahler-Einstein metrics.** Assume \(M\) is Kahler. Recall that the Ricci FORM

\[\rho(X, Y) = \text{Ric}(JX, Y) = i \text{Tr}_C(R^\nabla) = -i\partial \bar{\partial} \log \det H\]

where \(\nabla = \text{Chern connection on } TM\) (which is the same as the LC conn). In particular, \(i\rho\) is the curvature of the Chern conn the canonical bundle \(K_M\),

\[-c_1(K_M, \nabla) = \frac{1}{2\pi} \rho = c_1(TM, \nabla)\]

which was stated without a proof in Lecture 8.

When \((M, g)\) is Kahler, then \(g\) is Einstein \(\iff\) \(g\) is Kahler Einstein i.e.

\[\text{Ric}(g) = \lambda g \iff \rho = \lambda \omega\]

where \(\lambda\) is a real constant (the Einstein constant). In particular, we must have

\[c_1(M) = \frac{\lambda}{2\pi} [\omega]\]

So the vanishing of the image of \(c_1(M)\) in \(H^2_{dR}(M)\) is a necessary condition for existence of a Ricci-flat Kahler metric. Amazingly, Calabi-Yau Thm will imply it is also sufficient!
Note: More generally, for a Kahler-Einstein metric, depending on the sign of $\lambda$, the Ricci form must be either $\equiv 0$, or $> 0$ or $< 0$. This imposes restrictions on the first Chern class of a Kahler-Einstein manifold.

Note: A real, $(1,1)$ form $\alpha$ is called *positive* ($\alpha > 0$) iff $\alpha(\cdot, J\cdot)$ is positive definite. e.g. $\alpha = -\text{Im } h$ where $h$ is a hermitian metric. If $\alpha$ is also closed, it must be a Kahler form!

Note: Discussion extends to special metrics on HOLO v bds over a Kahler manifold $(M, g)$.

Let $E \rightarrow M$ be a holo v. bd with a hermitian metric $h$, and curvature $R^\nabla \in \Lambda^2(M; \text{End}(E))$ of the Chern connection $\nabla$. Let $\text{tr}_g(R^\nabla) \in \text{End}(E)$ now denote the trace in the manifold direction, using the Kahler metric $g$:

$$\text{tr}_g(R^\nabla) \in \text{End}(E) \quad \text{in coordinates} \quad \text{tr}_g(R^\nabla)^\beta_\alpha = g^{ij} R^\beta_{\alpha ij}$$

A *Hermitian-Einstein metric* on $E$ is similarly a solution of the following eq:

$$\text{tr}_g(R^\nabla) = \lambda \cdot \text{id} \quad \text{(Hermitian Yang-Mills = Hermitian-Einstein eq).} \quad (15.1)$$

Note: For a nice reference, see the Atiyah-Bott paper on HYM on Riemann surfaces.

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5 CAREFUL, before we used the trace in the fiber direction!
Lecture 16. THE CALABI-YAU THEOREM

First, we have one left-over issue from last time: the Kahler Ricci-flat cond

**Lemma 16.1.** Assume $M^n$ is Kahler and simply connected. Then $M$ is Ricci-flat iff

(a) $\rho = 0 \iff$ the Chern connection on $K_M$ is flat.
(b) $M$ has a parallel complex volume form $\Omega$.
(c) $\text{Hol}(M, g)$ is a subgroup of $SU(n)$.

**Proof.** (a) is clear; (a) $\iff$ (b) since $M$ is simply connected, a line bundle is flat iff it is trivial; so $(K_M, \nabla)$ is flat $\iff$ trivial, i.e. has a nonzero parallel section $\Omega$.
(b) $\implies$ (c) $\Omega$ is a nonzero parallel section of $K_M = \det \mathbb{C} T^* M$ so holonomy must preserve the determinant.
(b) $\iff$ (c) choose a trivialization of the fiber $K_{M,x} = \det \mathbb{C}(T^*_x M)$ at $x$; by assumption it is holonomy invariant, and therefore uniquely extends by parallel transport to a global trivialization of $K_M$. In particular, we get a parallel section $\Omega$ of $K_M$. $\Box$

**Corollary 16.2.** If $\text{Hol}(M, g) \subseteq SU(n)$, $\exists$ integrable cx structure on $M$ such that $g$ is Ricci-flat and Kahler. In particular, $M$ is a Calabi-Yau manifold. If $\text{Hol} = SU(n)$, the Ricci-flat Kahler metric is unique.

**Proof.** As before, $M$ must be Kahler, have a parallel Kahler form $\omega$, and a parallel complex volume form $\Omega$. Moreover, $i\rho=$curvature of Chern connection is trivial, thus $g$ is Kahler Ricci-flat. Since $\Omega$ is parallel, then $\Omega$ is holomorphic (why?), thus $M$ is Calabi-Yau. $\Box$

16.1. **The Calabi-Yau Theorem.** We next discuss the Calabi-Yau theorem and related results about existence and uniqueness of extremal metrics: KE metrics on $M$ and HYM metrics on $E \to M$; these are metrics that minimize a certain natural functional. This serves only as motivation, we work directly with the resulting differential equation, discussing existence and uniqueness of its solutions.

**Theorem 16.3 (Calabi-Yau).** Let $M^n$ be a compact Kahler mfd with Kahler form $\omega_0$. For any closed, real $(1,1)$-form $\rho$ representing $2\pi c_1(TM)$, $\exists$! Kahler form $\omega$ on $M$ such that

- $[\omega] = [\omega_0] \in H^2_{dR}(M)$
- the Ricci form associated to $\omega$ is precisely $\rho$.

In particular, if $c_1(TM) = 0$ in $H^2_{dR}(M)$ then $M$ carries a Ricci-flat Kahler metric (in the same Kahler class).

**Note:** This was conjectured by Calabi (’56), who proved uniqueness; two decades later Yau (’78) proved existence, which is much harder. As we shall see, it boils down to $\exists!$ solution to the complex Monge-Ampere equation. This is a highly nonlinear 2nd order differential equation whose linearization is $-\Delta_\omega$.

**Corollary 16.4.** Assume $M$ is an $n$-dim compact, complex manifold. Then $M$ admits a metric with holonomy $\subseteq SU(n) \iff M$ is Kahler and has a nowhere vanishing holomorphic $n$-form $\iff M$ is a Calabi-Yau manifold.

**Note:** Calabi’s motivation was to use a variational problem on the space $K$ of Kahler forms $\omega$ cohomologous with $\omega_0$, to find the ”best” representative, i.e. one that minimizes the norm square of the curvature:

$$C(\omega) = \int_M |\text{curvature of } \omega|^2$$
Calabi showed that it does NOT matter which curvature we use in this minimization problem (scalar, Ricci or even the full Riemann tensor), the minimizer is always a KE metric.  
**Note:** This is similar to geodesics minimizing energy $E(\gamma) = \int |\dot{\gamma}|^2$. The critical points satisfy a certain diff eq (Euler-Lagrange eq). The critical points are local minima (i.e. Hessian is negative definite!), i.e. they are stable solutions. We encountered a similar situation in Hodge theory: harmonics were minimizers of norm square in their cohomology class. 

**Note:** This idea extends to give the Hermitian-Yang-Mills functional on holo vector bundles: 

$$ \mathcal{YM}(\nabla) = \int_M |R^\nabla|^2 = \int_M R^\nabla \wedge *R^\nabla $$

Its critical points are solutions of the Hermitian Yang-Mills eq. (Skip) This functional is invariant under a huge (infinite dimensional) "gauge group" $\mathcal{G} = \text{Aut}(E)$, and we are trying to find the "best" representative up to a gauge transformation, i.e. looking at the moduli space of HYM connections up to gauge.

**Example 16.5.** (SKIP/HW) For degree 0 (i.e. top trivial) holo line bundles over a compact cx genus $g$ curve $C$, the HYM eq become $R^\nabla = 0$, so we get a correspondence

$$ \xymatrix{ \{\text{moduli space of degree 0 holo line bundles}\} & \{\text{moduli space of flat } U(1)\text{-connections}\} \\
\text{exp sequence} & \text{holonomy} \\
\text{Pic}^0(C) = H^{0,1}(C)/H^1(C, \mathbb{Z}) = \mathbb{C}^g/\mathbb{Z}^{2g} & \text{diffo} \quad \text{Hom}(\pi_1(C), S^1) = (S^1)^{2g} $$

so we get two matching descriptions of the Jacobian as a $g$ cx dim torus.

(HW) Show that if $M$ is Kahler, the HYM eq on a line bundle always have a solution. 

**Note:** (SKIP) Morally speaking, in all these examples, we are doing Morse theory on an infinite dimensional space for the action functional, finding its critical points. The functionals are invariant under a huge (infinite dimensional) gauge group (of isomorphisms) so we should work "perpendicular" to it. See Atiyah-Bott paper on HYM on RS for more details.

### 16.2. The Proof of the Calabi-Yau Theorem.** (Outline) Basic idea: show that the map 

$$ \omega \mapsto \rho_\omega \quad \text{the Ricci form associated to } \omega $$

is an invertible map between 

$$ \mathcal{K} = \{\text{Kahler forms } \omega \text{ cohom to } \omega_0\} \to \mathcal{R} = \{\text{closed, real, (1,1) forms } \rho \text{ cohom to } \rho_0\}. $$

**Note:** In local holo coordinates, 

$$ \omega = h_{ij}dz_i \wedge \bar{z}_j \quad \text{while} \quad i\rho = \partial \bar{\partial} \log \det(h_{ij}) $$

so this involves solving a second order nonlinear differential equation. 

**Step 0:** Describe the map $\mathcal{K} \to \mathcal{R}$ in a more useful way. Recall: both Kahler and Ricci form are closed, real, (1,1) forms. The global $\partial \bar{\partial}$ lemma (cf PS4#2(a)) reduces both to functions: 

$$ \omega - \omega_0 = i\partial \bar{\partial}u \quad \text{while} \quad i(\rho - \rho_0) = \partial \bar{\partial}f $$

where $u$ and respectively $f$ are unique up to an additive constant. To kill this redundancy, add a normalization condition, prescribing the average value (wrt original metric $g_0$). With these identifications, the map (16.1) becomes 

$$ \omega = \omega_0 + i\partial \bar{\partial}u \mapsto \rho = \rho_0 - i\partial \bar{\partial}f $$

(16.2)
where $u$ and $f$ are related by the formula:

$$(\omega_0 + i\partial\bar{\partial}u)^n = e^f \omega_0^n.$$  \hfill (16.3)

(the complex Monge-Ampere eq, a highly nonlinear 2nd order diff eq, nonlinear in the highest order terms).

Why do we get (16.3)? Because the difference between two Ricci forms can be expressed in terms of the ratio between the two volume forms (by the local coordinates expression):

$$\rho - \rho_0 = -i\partial\bar{\partial}\log \frac{d\text{vol}}{d\text{vol}_0} = -i\partial\bar{\partial}\log \frac{\omega^n}{\omega_0^n}.$$  

Note: Specifically, if $\Omega = dz_1 \wedge \cdots \wedge dz_n$ is a local holomorphic section $K_M$, then

$$i\rho = \partial\bar{\partial}\log |\Omega|^2$$  

where  

$$|\Omega|^2 d\text{vol} = \Omega \wedge *\Omega = (-1)^{n(n+1)/2} \Omega \wedge \bar{\Omega},$$

and the RHS is intrinsic, independent of metric! Moreover, because $\omega$ and $\omega_0$ are cohomologous, the ratio of their volume forms is global, positive function i.e. $\omega^n = e^f \omega_0^n$, where the average value of $e^f$ is 1 (since $[\omega^n] = [\omega_0^n]$).

Summarizing, we are interested in solving the equation (16.3). We regard it as a map $\text{Cal} : K \to \mathcal{R}$, $\text{Cal}(u) = f = \log \frac{(\omega_0 + i\partial\bar{\partial}u)^n}{\omega_0^n}$, equal to the map (16.1) under the identifications

$$K = \{ u \in C^\infty(M) \mid \omega_0 + i\partial\bar{\partial}u > 0, \text{avg}(u) = 0 \}$$

$$\mathcal{R} = \{ f \in C^\infty(M) \mid \text{avg}(e^f) = 1 \}$$

via the global $\partial\bar{\partial}$-lemma. To prove the Calabi-Yau Theorem it suffices to prove the following:

**Theorem 16.6.** The Calabi map $\text{Cal} : K \to \mathcal{R}$ is a diffeomorphism.

In particular, for any $\rho = \rho_0 - i\partial\bar{\partial}f$, there is a unique $u \in K$ such that the Ricci form of $\omega = \omega_0 + i\partial\bar{\partial}u$ is $\rho$.

**Proof.** (sketch) First of all, $\text{Cal}$ is well defined, i.e. for any $u \in K$, there is a unique $f \in \mathcal{R}$ satisfying (16.3), because the RHS is a volume form on $M$ cohomologous with $\omega_0^n$.

**Step 1:** $\text{Cal}$ is injective. It is enough to show that if $\text{Cal}(u) = 0$ then $u = 0$ (otherwise change the reference metric). But then

$$0 = \omega^n - \omega_0^n = (\omega - \omega_0) \wedge \left( \sum_{i=0}^{n-1} \omega^i \omega_0^{n-1-i} \right) = (i\partial\bar{\partial}u) \wedge \tau$$

where the sum $\tau$ in the parenthesis is a positive $(n-1, n-1)$ real form. Then maximum principle for the equation

$$\partial\bar{\partial}u \wedge \tau = 0$$

implies $u = 0$ (this can be also shown by hand using Stokes theorem, see Moroianu p 127).

**Step 2:** $\text{Cal}$ is a local diffeo. Again, it suffices to show that the differential at $u = 0$ is invertible. This follows if we show that the linearization of (16.3) is the Laplacian $-\Delta_\tau$, which is an elliptic operator and which is invertible on the space of functions of average value zero (i.e. perpendicular to its kernel).
To calculate the linearization of Cal, fix a variation \( v \in T_0K \) in \( u \), take the path \( u_t = tv \) in \( K \) tangent to it, and let \( f_t = \text{Cal}(u_t) \). We need to calculate

\[
\text{Cal}^*(v) = \frac{d}{dt} \bigg|_{t=0} f_t \quad \text{where} \quad (\omega_0 + it\partial \overline{\partial} v)^n = e^{f_t} \omega_0^n.
\]

Differentiating both sides of the equation wrt \( t \), evaluating at \( t = 0 \), and dividing by \( n! \) gives

\[
i\partial \overline{\partial} v \wedge \frac{\omega_0^{n-1}}{(n-1)!} = \frac{df_t}{dt} \bigg|_{t=0} \wedge \frac{\omega_0^n}{n!}
\]

which can be solved using the Hodge \(*\)-operator in terms of the adjoint \( \Lambda \) of the Lefschetz operator to get

\[
\text{Cal}^*(v) = \frac{d}{dt} \bigg|_{t=0} f_t = \Lambda(i\partial \overline{\partial} v) = -\overline{\partial} \overline{\partial} v = -\Delta_{\overline{\partial}} v.
\]

**Step 3:** Cal is surjective. This is the hardest part of the argument. One first shows that Cal is proper, by using Yau’s *a priori estimates*. Yau then used a continuity argument to show surjectivity of Cal. \( \square \)

**Note:** A priori estimates are used to prove *compactness* of the space of solutions (e.g. one can pass to the limit to find a minimizer for the variational problem). To complete the argument, one can then use either (i) a continuity argument or (ii) Donaldson’s heat kernel approach (to show existence of a solution from the a priori estimates).

**Note:** This is NOT a constructive proof! We still have NO IDEA how the Ricci-flat metric on a K3 surface looks like!

16.3. **Other cases.** How about \( \exists! \) KE metrics with \( \lambda \neq 0 \)? i.e. \( \exists! \) soln of the eq

\[
\rho = \lambda \omega.
\]

There is a HUGE difference in the *analysis* between \( \lambda < 0 \) and \( \lambda > 0 \), but the preliminary set-up is the same, a minor modification of the \( \lambda = 0 \) case.

By rescaling the metric (the Ricci tensor does not change!), we can assume \( \lambda = \pm 1 \). A necessary condition for the existence of a KE metric is that \( c_1(M) \) be definite, i.e. there exits a closed, real *positive* (1,1) form \( \omega_0 \) representing \( \pm 2\pi c_1(M) \). Any such form is a Kahler form, and its Ricci form \( \rho_0 \) satisfies

\[
[\rho_0] = \pm [\omega_0]
\]

(i.e. solves the equation in cohomology). So now we start with

\[
\rho_0 = \pm \omega_0 + i\partial \overline{\partial} f \quad \text{which we want to modify to} \quad \rho = \pm \omega.
\]

by taking \( \omega = \omega_0 + i\partial \overline{\partial} u \). As before, this boils down showing that the modified Calabi map

\[
\text{Cal}^\pm(u) = \text{Cal}(u) \pm u
\]

is a local diffeo. The linearization is

\[
\text{Cal}^\pm_*(v) = -\Delta_{\overline{\partial}} v \pm v.
\]

Now we see the CRUCIAL sign difference: when \( \lambda < 0 \), this linearization is a *negative* definite s adj. operator, which is very useful, e.g. we can use the maximum principle to prove uniqueness. Note that it has no kernel, so we need not worry about the constant functions.

**Theorem 16.7** (Aubin/Yau ’78). A compact complex manifold with negative \( c_1 \) admits a unique KE metric with Einstein constant \(-1\).
Of course, the hardest part is the existence of a solution.

**Note:** The Fano (positive) case is much more delicate: there are counterexamples to the existence of a KE metric, and further topological restrictions, e.g. Matsushima Theorem: a Fano KE manifold must have a reductive\(^6\) automorphism group (see §19.2 in Moroianu).

**Example 16.8.** \(\mathbb{CP}^2\) and \(\mathbb{CP}^1 \times \mathbb{CP}^1\) admit a KE metric (the FS metric); however, \(\mathbb{CP}^2\) blown-up at one or two points cannot admit a KE metric (by Matsushima Thm).

Yau conjectured that the existence of KE metric in the Fano case is related to an appropriate stability condition, refined by Tian. Recently ('12) this conjecture was proved by Donaldson-Chern-Sun/Tian (the relevant notion is called \(K\)-poly-stability).

Finally, for the HYM problem,

**Theorem 16.9.** [Donaldson-Yau-Unlenbeck '85-'86] A holomorphic v bd \(E\) on a compact Kahler manifold admits an irreducible\(^7\) HYM metric if and only if the bundle \(E\) is stable.

In particular, there is a Kobayashi-Hitchin correspondence:

\[
\{\text{moduli space of stable bundles}\} \overset{\cong}{\longrightarrow} \{\text{moduli space of solutions to HYM eq}\}.
\]

The stability of \(E\) is numerical condition restricting what types of holomorphic sub-bundles \(F\) could \(E\) have, coming from the second fundamental form of \(F\) in \(E\) (see PS3, §6, 7).

**Definition 16.10.** A holomorphic bundle \(E\) is called stable if for any sub-bundle (sub-sheaf) \(F \neq E\), the slope \(\mu(F) < \mu(E)\).

Here the slope of a bundle is

\[
\mu(E) = \frac{\deg E}{\text{rank}E} \quad \text{where} \quad \deg E = \int_M c_1(E) \omega^{n-1}
\]

is essentially the first Chern number. The slope enters in the GIT construction of the moduli space of stable holomorphic bundles in algebraic geometry (as an analytic/algebraic variety). Unstable bundles have to be tossed out, else the quotient would not be Hausdorff. Semi-stable bundles are the "borderline case" and give rise to singularities in the moduli space.

**Note:** This notion of stability depends on the Kahler class (up to scale). There are various other notions of stability, depending exactly how one describes the quotient.

**Example 16.11.** A rank 2 holomorphic vector bundle on a Riemann surface \(\Sigma\) is stable if all its line subbundles satisfy \(c_1(F) < \frac{1}{2}c_1(E)\). In particular, \(E = L_1 \oplus L_2\) is never stable; it is semi-stable only if \(c_1(L_1) = c_1(L_2) = \frac{1}{2}c_1(E)\).

**Note:** As before, it is fairly easy to show the existence of HYM solution implies stability. The hard part is existence of a solution on a stable bundle.

**Example 16.12.** (HW) Show that any holo line bundle (over cpt Kahler mfd) has a HYM structure.

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\(^6\)i.e. its Lie algebra is a direct sum of its center and a semi simple part.

\(^7\)reducible soln have more automorphisms than expected and cause trouble in the quotient; so do semi-stable bundles.
Lecture 17. Weitzenbock Formula and Vanishing Theorems

Fix \((M, g, J)\) is a compact Kahler manifold. Assume \(E \to M\) is a holomorphic bundle with a hermitian metric \(h\) and Chern connection \(\nabla\). On the bundle \(\Lambda^{p,q}E \to M\), get two differential operators:

\[
\bar{\partial} : A^{p,q}(E) \to A^{p,q+1}(E) \quad \nabla : A^{p,q}(E) \to \Gamma(\Lambda^1_C M \otimes \Lambda^{p,q}E)
\]

The Weitzenbock formula is

\[
2(\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) = \nabla^* \nabla + \mathcal{R} \quad \text{where} \quad \mathcal{R} \in \text{End}(\Lambda^{p,q}E)
\]

(17.1)

depends only on the curvature of the Chern conn on \(E\) (and of the Levi-Civita on \(\Lambda^{p,q}_M\)).

Note: This means that the difference between the two 2nd order diff operators is a self adjoint 0'th order operator that depends only on the curvature.

Note: There is a similar identity for the usual Laplacian, called the Bochner formula, and one for the Dirac operator (in spin bundles) called the Lichnerowicz formula.

Note: When \(\eta \in \Lambda^{p,q}\) and \(M\) compact, talking the \(L^2\)-inner product with \(\eta\) gives

\[
2 \int_M |\bar{\partial} \eta|^2 + |\bar{\partial}^* \eta|^2 = \int_M |\nabla \eta|^2 + \int_M \langle \mathcal{R} \eta, \eta \rangle
\]

(17.2)

Vanishing Thm: If \(\mathcal{R} > 0\) (positive definite) on \(\Lambda^{p,q}E\) then

\[
H^{p,q}(M; E) = H^q(M, \Omega^p \otimes E) = 0
\]

Proof. Use Hodge theory: if \(\eta \in H^{p,q}(E)\), the LHS of (17.2) vanishes, while the RHS is a sum of two nonnegative terms, and therefore \(\nabla \alpha = 0\) and \(\langle R \alpha, \alpha \rangle \equiv 0\) so \(\alpha = 0\). \(\square\)

Note: If \(\mathcal{R} = 0\) on \(\Lambda^{p,q}(E)\) then all harmonics are parallel.

Note: (HW) Weitzenbock formula (+Cauchy-Schwartz 2\(\langle a, b \rangle \leq \epsilon \|a\|^2 + \epsilon^{-1}\|b\|^2\) \(\implies\) Garding’s inequality (10.2) (used in the proof of Hodge Thm).

The Weitzenbock formula follows from a (straightforward, but tedious) calculation in a parallel o.n. frame \(\{e_k\}\) on \(TM \cong T^*M\) (cf §20 in Moroianu). The section \(\mathcal{R} \in \text{End}(\Lambda^{p,q}E)\) in (17.1) is equal to

\[
\mathcal{R}(\eta) = \frac{i}{2} \tilde{R}_{je_j, e_j} \eta - \frac{1}{2} (e_j - iJe_j) \wedge (e_k + iJe_k) \tilde{\eta}(\tilde{R}_{je_j, e_k} \eta)
\]

(17.3)

where \(\tilde{R} = R \otimes 1 + 1 \otimes R^E\) is the curvature of the bundle \(\Lambda^{p,q}E = \Lambda^{p,q}_M \otimes E\) (and we used the summation convention on repeated indices). We also used the exterior product \(X \wedge \cdot : \Lambda^k \to \Lambda^{k+1}\) (i.e. wedge with the metric dual of \(X \in TM\)) and its adjoint, the interior product \(\langle X, \cdot \rangle : \Lambda^{k+1} \to \Lambda^k\)

\[
\langle X, \omega \rangle(\cdot) = \omega(X, \cdot).
\]

If we denote by \(\varphi_k = \frac{1}{2}(e_k - iJe_k)\) the corresponding frame of \(T^{1,0}M \cong \Lambda^{0,1}_M\), then

\[
\bar{\partial} = \varphi_k \wedge \nabla e_k, \quad \bar{\partial}^* = -\nabla e_k \wedge \varphi_k, \quad \nabla^* (\tau \otimes \eta) = (d^* \tau) \eta - \nabla \tau \eta
\]

\[
\nabla \nabla^* = \nabla \nabla e_k e_k - \nabla e_k \nabla e_k.
\]

We first use the \(q = 0\) case, where the Weitzenbock formula reduces to:

\[
2 \bar{\partial}^* \bar{\partial} = \nabla^* \nabla + \mathcal{R} \quad \text{where} \quad \mathcal{R} = \frac{i}{2} \tilde{R}_{je_k, e_k} \in \text{End}(\Lambda^{p,0}E).
\]

(17.4)
Theorem 17.1. Assume $M$ is compact cx mfld. If $M$ has a Kahler metric with $\text{Ric} < 0$ then $M$ has no nontrivial holomorphic vector fields i.e. $H^0(M, TM) = 0$.

In particular, the automorphism\footnote{i.e. group of biholomorphisms} group $\text{Aut}(M)$ is discrete.

Proof. If $E = T^{1,0}M$ (holo tangent space) and $p = q = 0$ then (17.4) becomes
\[ \mathcal{R}(\xi) = i\rho(\xi) = -\text{Ric}(\xi) \quad \text{for all v. fields } \xi \in \Gamma(T^{1,0}M) \]
so vanishing theorem implies the rest. Recall that infinitesimal automorphisms are holomorphic vector fields, thus $Tid\text{Aut}(M) = 0$. \qed

Example 17.2. The unit ball with $\omega = i\partial\bar{\partial}(1 - \|z\|^2)$ is KE with negative Einstein constant, thus so are compact ball quotients. Any genus $g \geq 2$ Riemann surface has this property.

Note: $\text{Aut}M$ depends only on the complex structure of $M$, not on the metric! But existence of a certain type of metric imposes restrictions on $\text{Aut}(M)$.

Note: The Matsushima theorem also uses Hodge decomposition and the Weitzenbock formula to show that for a (Fano) KE manifold, the Lie algebra $\mathfrak{g}$ of holomorphic vector fields is the complexification $\mathfrak{g} = \mathfrak{h} \otimes \mathbb{C}$ of the Lie algebra $\mathfrak{h}$ of Killing vector fields (aka infinitesimal isometries). Since isometries are a compact Lie group, both its Lie algebra and its complexification are reductive.

Note: Recall that $\text{Aut}(M)$ is a complex Lie group, typically noncompact: the only compact connected complex Lie groups must be abelian and in fact must be tori.

Theorem 17.3. Assume $M$ is compact cx manifold. If $M$ has a Kahler metric with $\text{Ric} > 0$ then $M$ has no nontrivial holomorphic $p$-forms, i.e. $H^{p,0}(M) = H^0(M; \Omega^p) = 0$ for $p > 0$.

If $M$ has a Ricci-flat Kahler metric, then any holomorphic $p$-form is parallel.

Proof. Now take $E = \mathbb{C}$ and $q = 0$, so (17.4) becomes
\[ \mathcal{R}(\eta) = \text{Ric}(\eta) \quad \text{for all } \eta \in \Lambda^{p,0} \]
and apply the vanishing theorem. \qed

Example 17.4. $\mathbb{C}P^n$ with the Fubini-Study form is a Fano KE, and we already knew that $h^{p,0} = 0$ for $p \neq 0$. The tori or CY manifolds have a Ricci-flat metric, so all their holomorphic $p$-forms are parallel. (We already knew this for the holomorphic volume form).

Definition 17.5. A holo line bundle $L$ is called negative if it has a hermitian metric with $i\text{R}^\nabla < 0$ (as a real, $(1,1)$-form) $\iff i\text{R}^\nabla(\cdot, J\cdot) < 0$ (as a sym tensor).

Note: On a compact Kahler manifold, the global $\partial\bar{\partial}$ lemma implies that $L$ is negative iff $c_1(L) < 0$, i.e. $c_1(L)$ has a negative representative. (Why? Start with any herm metric $h$ on $L$ with curvature $R^\nabla$. If $[i\text{R}^\nabla] = [i\alpha]$ then $i(R^\nabla - \rho) = i\partial\bar{\partial}f$, and therefore $\alpha$ is the curvature of the hermitian metric $he^f$.)

Note: If $L \to M$ is a positive line bundle, then $\omega = \frac{i}{2\pi} R^\nabla > 0$ is a Kahler form on $M$ representing $c_1(L)$. So existence of a positive (or negative) line bundle implies $M$ is Kahler!

Theorem 17.6. A negative holomorphic line bundle over a compact Kahler manifold has no nontrivial holomorphic sections.
Proof. Follows from (17.4) for \( p = q = 0 \) and \( E = L \) which becomes
\[
R = \frac{i}{2} R^\nabla(J e_k, e_k) = -\frac{i}{2} R^\nabla(e_k, J e_k) > 0
\]
so vanishing theorem applies. \( \square \)

Example 17.7. The tautological line bundle is negative, so it has no holomorphic sections. The same is true for \( \mathcal{O}(-k) \) with \( k > 0 \).

To get more consequences of the Vanishing Theorem, we also use an Index Theorem.

17.1. Hirzebruch-Riemann-Roch Thm. Recall: Dolbeault cohomology \( H^q(M, E) \) was the cohomology of the elliptic complex
\[
\Omega^0 \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \Omega^2 \xrightarrow{\partial} \ldots
\]
which can be rolled up to
\[
D = \overline{D} \oplus \overline{D}^* : \Omega^{0, \text{even}} \to \Omega^{0, \text{odd}}
\]
This is a first order elliptic differential operator (whose square\(^9\) is the Laplacian, of index
\[
\text{ind } D = \dim \ker D - \dim \text{coker } D.
\]
Hodge theory implies that the index is equal to the holomorphic\(^10\) Euler characteristic of \( E \):
\[
\chi(M, E) = \sum_{k=0}^m (-1)^k \dim H^k(M, E) = \sum_{k=0}^m (-1)^k \dim \mathcal{H}^k(M, E) = \text{ind } D
\]

Note: Recall that the index of a Fredholm operator is invariant under deformation (through Fredholm operators). In particular, as we vary the complex structure on \( E \), \( \ker D \) and \( \text{coker } D \) may jump up, but the index stays constant!

The HRR Thm is a special case of the Atiyah-Singer index Thm: The index of an elliptic operator \( D \) is topological, and can be explicitly calculated in terms of the symbol of \( D \). For \( D = \overline{D} \oplus \overline{D}^* \) it gives a universal formula in the characteristic classes of \( M \) and \( E \).

Theorem 17.8 (Hirzebruch-Riemann-Roch). Assume \( E \) is a holomorphic vector bundle over a compact, complex manifold \( M \). Then the holomorphic Euler characteristic is equal to
\[
\chi(M, E) = \int_M Td(M) \text{ch}(E)
\]
where \( Td(M) \) is the Todd class of \( TM \), and \( \text{ch}(E) \) is the total Chern character of \( E \).

Note: Todd class is multiplicative, while the total Chern character is additive. If \( E = L_1 \oplus \cdots \oplus L_r \) with Chern roots \( x_i = c_1(L_i) \) then
\[
\text{ch}(E) = \sum_{i=1}^r e^{x_i} \quad \text{while} \quad Td(E) = \prod_{i=1}^r \frac{x_i}{1 - e^{-x_i}}
\]

Note: Todd class involves Bernoulli numbers:
\[
\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{k=1}^\infty (-1)^{k+1} \frac{B_k t^{2k}}{(2k)!} = 1 + \frac{t}{2} - \frac{t^2}{12} + \ldots
\]

\(^9\) technically \( \Delta = D^* D \) (we could enlarge the domain of \( D \), but that would change the index!)
\(^10\) also called Poincare-Euler characteristic
Where is this generating function coming from? Hirzebruch explained it is uniquely determined by requiring: (i) $Td$ is multiplicative (and $Td(O) = 1$) and (ii) $Td(CP^n)[CP^n] = 1$ for all $n$ (because we know $\chi(CP^n, O) = 1$). Combined with the SES (14.1), this means we are looking for a formal power series $f(x)$ such that the coeff of $x^n$ in $f(x)^{n+1}$ is 1. By Lagrange inversion formula, the only solution is $f(x) = x/(1 - e^{-x})$.

**Example 17.9.** Over a complex curve $C$ of genus $g$,

$$\chi(C, E) = \int_C \left( 1 + \frac{1}{2}c_1(TC) + \ldots \right) \left( \text{rank}E + c_1(E) + \ldots \right) = c_1(E) + \text{rank}E(1 - g).$$

because $c_1(TC) = \chi(C) = 2 - 2g$. So HRR generalizes Riemann-Roch formula: for a degree $d = c_1(E) \in H^2(C, \mathbb{Z}) = \mathbb{Z}$, rank $r$ holo bundle:

$$h^0(C, E) - h^1(C, E) = d + r(1 - g).$$

In particular, any holo line bundle of degree $d \geq g$ must have a nontrivial holo section, i.e. $h^0(C, L) > 0$. Its zero divisor will be $d$ points (with multiplicity). This shows that the map

$$\text{Sym}^d(C) \to \text{Pic}^d(C) \quad \{x_i\} \mapsto O \left( \sum_{i=1}^{d} x_i \right)$$

is onto.

Recall $\text{Pic}^d(C) \cong \text{Pic}^0(C) = H^{0,1}(C)/H^1(C, \mathbb{Z})$ is a $g$ cx dim torus, and $\text{dim Sym}^g(C) = g$. For $g = 1$, we get an isom $\text{Pic}^0(C) \cong \mathbb{C}$ as a complex group. (HW) How about genus 2?

(HW) Calculate $\chi(S, E)$ for a surface $S$, as a formula involving $\text{rank}E$, $\text{dim} M$, and their 1st and 2nd Chern classes. For the trivial line bundle over a surface $S$

$$\chi(S, O) = \int_S \frac{c_2(S) + c_1^2(S)}{12}$$

so we get a divisibility property (Noether’s Thm).

**Example 17.10.** For example, if $c_1^2(S) = 0$ (as is the case if $S$ is an elliptic surface, i.e. $S \to \mathbb{P}^1$ is an elliptic fibration with some singular fibers) then $c_2(S) = \chi(S)$ (usual Euler characteristic) has to be divisible by 12. For a K3 surface, $\chi(S, O) = 2$ because $c_1(S) = 0$ and $c_2(S) = \chi(S) = 24$ (the elliptically fibered one has 24 simple nodal fibers).
Lecture 18. **Kodaira Vanishing and Kodaira Embedding**

Correction: Todd class is **MULTIPLICATIVE**, not additive! i.e.

\[ Td(E) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}} \]

if \( E \) has Chern roots \( x_1, \ldots, x_r \), eg. if \( E = L_1 \oplus \cdots \oplus L_r \) line bundles with \( x_i = c_1(L_i) \).

**Theorem 18.1** (Kobayashi). A compact Kahler manifold with \( \text{Ric} > 0 \) is simply connected. In particular, if \( M \) is Fano (i.e. \( c_1(M) > 0 \)) then \( M \) is simply connected.

**Proof.** Myer’s Theorem implies that \( M \) has finite fundamental group, so the universal cover \( \tilde{M} \) is also compact. By the vanishing results in previous lecture, integrating both sides we get

\[ \omega \]

After some calculation (using crucially that \( \omega \) is a Kahler form \( K \text{ahler form} \)) the metric, and it is crucial that we choose the “right” metrics!

\[ \text{Kodaira’s Theorem implies that} \]

\[ \text{Ric} > 0 \implies 0 = h^{0,0} = h^{0,p} \ (\text{since Kahler}) \text{ for } p \neq 0 \text{ and of course } h^{0,0} = 1. \]

The same argument works for \( \tilde{M} \) to give

\[ \chi(M, \mathcal{O}) = \chi(\tilde{M}, \mathcal{O}) = 1. \]

But if \( \pi : \tilde{M} \to M \) is a degree \( k \) cover, then \( T\tilde{M} = \pi^*TM \), while \( \pi_*[\tilde{M}] = [M] \), and so

\[ Td(\tilde{M}) = \pi^*Td(M) \quad \int_{\tilde{M}} Td(\tilde{M}) = \int_{\tilde{M}} \pi^*Td(M) = k \int_M Td(M). \]

HRR \( \implies \chi(\tilde{M}, \mathcal{O}) = k\chi(M, \mathcal{O}). \) Therefore \( k = 1 \), i.e. \( M \) was already s.conn.

The last statement follows from Calabi-Yau Thm: if \( i\alpha > 0 \) represents \( 2\pi c_1(M) \) then we can find another Kahler metric \( \omega_1 \) whose Ricci form is \( \alpha > 0 \), thus Ricci tensor \( \text{Ric} > 0. \)

**Theorem 18.2** (Kodaira vanishing). If \( L \) is a positive holomorphic line bundle over a compact Kahler \( M^n \) then \( H^{p,q}(M; L) = 0 \) for all \( p + q > n \).

**Proof.** (Outline) The idea is to compare the Weitzenbock formula for the operator \( \bar{\partial}_L = \nabla^{0,1}_L \) with the one for \( \partial_L = \nabla^{1,0} \). (Careful: while \( \bar{\partial}_L \) is independent of the metric, \( \partial_L \) depends on the metric, and it is crucial that we choose the “right” metrics!)

Because \( L \) is positive, \( L \) has a hermitian metric so that \( i\text{Ric} > 0 \), and we can choose a Kahler form \( \omega = i\text{Ric} \). Weitzenbock formul\( ^{12} \) and its complex conjugate implies

\[ 2(\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) = \nabla^* \nabla + \mathcal{R} \quad 2(\partial \partial^* + \partial^* \partial) = \nabla^* \nabla + \mathcal{R}. \]

After some calculation (using crucially that \( \omega = i\text{Ric} \)) we get:

\[ 2(\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) = 2(\partial \partial^* + \partial^* \partial) + 2(p + q - n)\text{Id} \]

Now vanishing theorem applies for \( \bar{\partial} \) harmonic \((p, q)\)-forms with \( p + q > n \). Specifically, integrating both sides we get

\[ \int_M |\bar{\partial} \alpha|^2 + |\bar{\partial}^* \alpha| = \int_M |\partial \alpha|^2 + |\partial^* \alpha|^2 + (p + q - n)|\alpha|^2 \]

and so if \( \alpha \) is harmonic, \( \text{LHS} \) is 0, \( \text{RHS} \) is a sum of nonnegative terms thus they must identically vanish, so \( \alpha = 0. \)

\[ ^{12} \text{Kodaira used instead the Nakano identity} \ [\Lambda, \bar{\partial}_E] = -i(\nabla^{1,0}_{\text{ext}})^* \text{ for any holo v. bd} E \to M. \]
Example 18.3. (HW) Use this and Serre duality to show vanishing of $H^q(\mathbb{CP}^n, \mathcal{O}(k))$ in these ranges: (a) $0 < q < n$ (b) for $q = 0$ and $k < 0$ and (c) $q = n$ and $k > -n - 1$.

Recall: Serre duality $H^{p,q}(M, E) \cong H^{n-p,n-q}(M, E^*)^*$ so in particular for $p = 0$ it gives
\[ H^q(M, E) \cong H^{n-q}(M, K_M \otimes E^*)^*. \tag{18.1} \]
where $K_M = \Omega^n_M$ is the canonical line bundle. We also used Dolbeault Thm
\[ H^{p,q}(M, E) = H^q(M, \Omega^p_M \otimes E). \]

Theorem 18.4 (Kodaira-Serre vanishing). Assume $L$ is a positive holomorphic line bundle over a compact Kahler $M$. For any holo bundle $E \to M$ there exits $k_0$ such that
\[ H^q(M, E \otimes L^k) = 0 \]
for all $k \geq k_0$ and $q > 0$.

We don’t have time to prove this, but we discuss some consequences of it+HRR.

Example 18.5. Assume $C$ is a compact complex genus $g$ curve. Then RR implies that any degree $d = \deg(L) \geq g$ line bundle has a nontrivial holomorphic section, because
\[ h^0(C, L) - h^1(C, L) = \chi(C, L) = d + (1 - g) > 0. \tag{18.2} \]
In particular, the map $\text{Sym}^q(C) \to \text{Pic}^q(C)$, given by $\{x_i\} \mapsto \mathcal{O}(\sum x_i)$ is surjective. (SKIP)
Note that both are $g$ dimensional smooth complex manifolds. With more effort, one can show this map is generically injective (i.e. it fails to be injective along a codimension 1 subvariety). For genus 1, we get an isomorphism (even as complex groups) $C \cong \text{Pic}^0(C)$.

Moreover, any holomorphic bundle on $C$ must have a meromorphic section: tensor it with $\mathcal{O}(dx)$ for very large $d$ until RHS of (17.5) becomes positive.

Note: Here we only used RR. Its extension to higher dim also uses Kodaira vanishing.

Proposition 18.6. Any holomorphic line bundle on a projective $M$ (i.e. $M \subseteq \mathbb{CP}^N$) has a meromorphic section, so the map $\text{Div}(M)/\sim \to \text{Pic}(M)$ is a bijection.

Proof. The restriction of $\mathcal{O}(1)$ to $M$ is a positive line bundle with first chern class $h = [\omega_{FS}]$, so by Kodaira Vanishing $H^q(M, L \otimes \mathcal{O}(k)) = 0$ for $q > 0$ and $k$ large. On the other hand, by HRR the holomorphic Euler characteristic
\[ \chi(M, L \otimes \mathcal{O}(k)) = \int_M Td(M)e^{\chi_1(L) + kh} = k^n \int_M \frac{h^n}{n!} + \ldots \]
is a polynomial in $k$ that goes to infinity as $k \to \infty$ (the coefficient of $k^n$ is $\text{vol}(M) > 0$). In particular, for $k$ large, $H^0(M, L \otimes \mathcal{O}(k)) \neq 0$, i.e. $L \otimes \mathcal{O}(k)$ has a holomorphic section, so $L$ has a meromorphic section. \qed

Note: (HW) The argument extends to give $h^0(M, E \otimes L^k) = \chi(M, E \otimes L^k) > 0$ whenever $L$ is positive and $k \gg 0$. This in turn can be used to prove:

Theorem 18.7 (Grothendieck). Every holomorphic vector bundle $E \to \mathbb{CP}^1$ splits uniquely (up to reordering) as a direct sum $\bigoplus_{i=1}^r \mathcal{O}(a_i)$ of holomorphic line bundles with roots $a_i \in \mathbb{Z}$.

Proof. (Sketch) Use induction on the rank of $E$; true for a line bundle (including uniqueness). By Kodaira vanishing and HRR, $E$ has a nontrivial meromorphic section, i.e. a sub-line bundle $\mathcal{O}(a) \subseteq E$. In fact there is a maximum such $a$ by Serre duality and Kodaira vanishing: $H^0(\mathbb{CP}^1, E \otimes \mathcal{O}(-a)) = H^1(\mathbb{CP}^1, \Omega_{\mathbb{CP}^1} \otimes E^* \otimes \mathcal{O}(a))^* = 0$
for large \(a \gg 0\). Choose the maximal \(a\) such that \(\mathcal{O}(a) \subset E\) and assume for simplicity that \(a = 0\). (Note that if \(E \otimes \mathcal{O}(a)\) splits, so does \(E\).

The quotient \(F = E/\mathcal{O}\) is a holomorphic vector bundle, so by induction \(F = \oplus \mathcal{O}(a_i)\), where \(a_i \leq 0\) since \(a = 0\) was the maximum. Argue next that the sequence

\[
0 \to \mathcal{O} \to E \to F = \oplus \mathcal{O}(a_i) \to 0 \iff 0 \to \oplus \mathcal{O}(-a_i) = F^* \to E^* \to \mathcal{O} \to 0
\]

must holomorphically split, i.e. that the first map in the LES

\[
H^0(E^*) \to H^0(\mathcal{O}) \to H^1(F^*)
\]

is surjective. For that it suffices to show \(H^1(F^*) = 0\). But by Serre duality, and \(K_{\mathbb{P}^1} = \mathcal{O}(-2)\),

\[
H^1(F^*) = H^0(\mathcal{O}(-2 + a_i)) = 0
\]

because negative line bundles cannot have holomorphic sections. \(\square\)

**Example 18.8.** On a genus 1 curve \(C\), there is a rank 2 holomorphic vector bundle \(E\) which fits in the SES \(0 \to \mathcal{O} \to E \to \mathcal{O} \to 0\), but does NOT split as the direct sum of two holomorphic line bundles. The holomorphic structure on \(E\) can be defined starting from the trivial bundle \(\mathcal{O} \oplus \mathcal{O}\) and deforming it to a new \(\bar{\partial}\) operator

\[
\bar{\partial}_E = \left(\begin{array}{c} \partial \\ 0 \end{array}\right) \frac{\alpha}{\bar{\partial}}
\]

where \(\alpha\) is a \(\bar{\partial}\)-closed \((0,1)\)-form such that \(0 \neq [\alpha] \in H^{0,1}(C) = H^1(C, \mathcal{O}) = \mathbb{C}\). (HW) Flesh out this argument.

**Note:** One could also see this algebraically using the formula \(\text{Ext}^1(L_1, L_2) = H^1(C, L_1^* \otimes L_2)\).

We end with two more important consequences of Kodaira Vanishing:

**Theorem 18.9** (Kodaira Embedding). Assume \(M\) is a compact Kahler manifold, and \(L\) a positive line bundle (which exists e.g. when \([\omega] \in H^2(M, \mathbb{Q})\)).

Then \(M\) can be holomorphically embedded in \(\mathbb{C}P^N\) for large \(N\). In fact, the embedding is

\[
\varphi: M \to \mathbb{C}P^N \quad x \mapsto [s_0(x), \ldots, s_N(x)]
\]

where \(s_0, \ldots, s_N\) is a basis of \(H^0(M, L^k)\) for \(k \gg 0\).

**Note:** While all projective manifolds (i.e. cx smflds of \(\mathbb{C}P^N\)) are Kahler, not all Kahler manifolds are projective. Kodaira embedding implies that \(M\) is projective iff \(M\) has a Kahler form \(\omega\) with rational coefficients \([\omega] \in H^2(M, \mathbb{Q})\).

**Note:** Careful! Not all Kahler manifolds have a rational Kahler form. One example is the 2-complex dimensional torus \(T = \mathbb{C}^2/\Lambda\) whose lattice \(\Lambda\) is generated by \((1,0), (0,1), i(\sqrt{2}, \sqrt{3})\) and \(i(\sqrt{3}, \sqrt{7})\). One could show that \(T\) has no non constant global meromorphic functions.

**Note:** (HW) What is wrong with this "argument": Given a Kahler form, we can deform it to one with \(\mathbb{Q}\) coeff. Recall that a Kahler form is a positive, closed, real \((1,1)\) form. But being positive is an open condition, and we can also deform the cohomology class \(H^2_{dR}(M, \mathbb{R})\) to one with \(\mathbb{Q}\) coeff, so it should work.

**Note:** Kodaira embedding could be regarded as some replacement of Whitney embedding from real differential geometry. Recall that \(\mathbb{C}^N\) has no positive dimensional compact cx smflds, and all such smflds of \(\mathbb{C}P^N\) are Kahler with an integral Kahler class.
Proof. (sketch) The proof involves a repeated use of Kodaira vanishing in the context of LES sequences in cohomology associated to SES of sheaves. It also uses the blow up of $M$ at some points to trade some ideal sheaves (of holomorphic functions vanishing to certain order at those points) with line bundles associated to the exceptional divisors, so one can actually apply Kodaira Vanishing Thm.

For example, Kodaira vanishing and HRR implies that $H^0(M, L^k) \neq 0$ for $k \gg 0$.

To check (18.3) is well defined, one needs to show that for each $x \in M$, there is a section $\sigma \in H^0(M, L^k)$ such that $\sigma(x) \neq 0$. For that it suffices to show that the evaluation map/restriction

$$H^0(M, L^k) \xrightarrow{r} L_x^k \quad \sigma \mapsto \sigma(x)$$

is surjective, and that is where a SES of sheaves (and Kodaira vanishing) comes in. Note that the kernel of this map consists of holomorphic sections which vanish at $x$.

For example, when $M$ is 1-dimensional, start with the SES (cf (14.2))

$$0 \to \mathcal{O}_M(-x) \to \mathcal{O}_M \xrightarrow{r} \mathcal{O}_x \to 0$$

tensor it with $L^k$ to get $0 \to L^k \otimes \mathcal{O}(-x) \to L^k \to L_x^k \to 0$ and then take the LES in cohomology

$$H^0(M, L^k) \to L_x^k \to H^1(L^k \otimes \mathcal{O}(-x)) = 0 \quad \text{by Kodaira vanishing, when } k \gg 0.$$

Similarly to show that (18.3) is injective, it suffices to prove (dually) that for any $x \neq y \in M$, the restriction

$$H^0(M, L^k) \to L_x^k \oplus L_y^k$$

is surjective.

Showing that (18.3) is an immersion is very similar (except now looks at 1-jets, or better one first restricts to sections that vanish at $x$ and evaluates their "next" coefficient $d_x \sigma$).

Similar techniques are used to prove:

Theorem 18.10 (Lefschetz Hyperplane Thm). Assume $M$ is compact Kahler $n$ dimensional, $V \subset M$ a smooth hypersurface such that the line bundle $\mathcal{O}(V)$ is positive. Then the restriction

$$H^k(M, \mathbb{C}) \to H^k(V, \mathbb{C})$$

is bijective for $k \leq n - 2$, and injective for $k = n - 1$.

Note: In particular, if $V$ is a hypersurface in $\mathbb{CP}^n$, its cohomology (and Hodge decomposition) is determined except for the middle dimension $k = n - 1$, where it could be large.

Proof. (Outline) Use Hodge decomposition and then Kodaira vanishing to show that

$$H^{p,q}(M) \to H^{p,q}(V) \quad \text{i.e.} \quad H^q(M, \Omega^p_M) \to H^q(V, \Omega^p_V)$$

is an isomorphism for $p + q \leq n + 2$ and injective for $p = n - 1$. For that first factor the restriction $\Omega^q_M \to \Omega^q_V$ as the composition

$$\Omega^p_M \xrightarrow{r} \Omega^p_M|_V \xrightarrow{i^*} \Omega^p_V$$

and show that both maps

$$H^q(M, \Omega^q_M) \xrightarrow{r^*} H^q(M, \Omega^q_M|_V) \xrightarrow{i^*} H^q(V, \Omega^q_V)$$

are bijections and respectively injections in the desired range.

The map $r$ fits in the LES associated to the divisor SES (14.2):

$$0 \to \mathcal{O}(-V) \to \mathcal{O}_M \to \mathcal{O}_M|_V \to 0$$
after tensoring with $\Omega^p_M$. The previous term in the LES

$$0 = H^q(M, \Omega^p_M \otimes \mathcal{O}(-V)) \to H^q(M, \Omega^p_M) \xrightarrow{i^*} H^q(M, \Omega^p_M|_V) \to$$

vanishes by Kodaira vanishing if $p + q < n$.

The argument for the map $i$ is similar. Start with the dual of normal SES (13.2):

$$0 \to N^*_V \to \mathcal{T}^* M|_V \to \mathcal{T}^* V \to 0 \implies 0 \to \Omega^p_{V|-}\otimes N^*_V \to \Omega^p_M|_V \xrightarrow{i^*} \Omega^p_V \to 0$$

which induces

$$0 \to \Omega^p_{V|-}\otimes N^*_V \to \Omega^p_M|_V \xrightarrow{i^*} \Omega^p_V \to 0.$$

But by adjunction formula $N^*_V = \mathcal{O}(-V)|_V$, so the rest of the argument follows using the LES and Kodaira vanishing as above. □

**Note:** There is another proof due to Bott using Morse theory for (a small $C^2$ perturbation of) the function $\varphi(x) = \log |s|^2$ where $s$ is a holomorphic section of $L$ with divisor $V$. Recall that the curvature $\Theta = -\partial \bar{\partial} \log |s|^2 > 0$. This implies that at each critical point of $\varphi$, the Hessian has at least $n$ negative e-values, thus up to homotopy $M$ is obtained from $V$ by attaching cells of dimension at least $n$. So Bott’s argument proves the Lefschetz theorem at the homotopy level and therefore with $\mathbb{Z}$-coeff.

**Example 18.11.** A smooth curve $C \subseteq \mathbb{C}P^2$ may have nontrivial 1-dim cohomology.

**Example 18.12.** Lefshetz Thm determines the Hodge numbers of a Calabi-Yau hypersurfaces except in middle dimension. For a $K3$ surface, one can also determine the middle ones: $H^{2,0}(S) = H^0(S, K_S) = \mathbb{C}$ and so $h^{1,1} = 20$ (by HRR and the fact that the usual Euler characteristic is $24 = c_2(S)$).

**Note:** While every projective manifold is Kahler, not all Kahler manifolds are projective.

18.1. **Proof of Kodaira Embedding Theorem.** Here are more details on the proof of the Kodaira embedding theorem. Use Kodaira vanishing and HRR to show that, for large $k \gg 0$, the map (18.3) is

(a) well defined, i.e. $H^0(M, L^k) \neq 0$ and that not all sections vanish simultaneously, i.e. the evaluation/restriction map

$$r_x : H^0(M, L^k) \to L^k_x \quad s \mapsto s(x)$$

at each point $x \in M$ is surjective.

(b) injective, i.e. the evaluation/restriction map

$$r_{x,y} : H^0(M, L^k) \to L^k_x \oplus L^k_y$$

at any two distinct points $x, y \in M$ is surjective.

(c) immersion, i.e. dually the differential

$$d_x : \text{Ker } r_x = H^0(M, \mathcal{I}_x(L^k)) \to L^k_x \otimes \mathcal{T}^* M \quad s \mapsto d_xs$$

is surjective. Here the domain consists of holomorphic sections $s$ that vanish at $x$, therefore their "differential" $d_xs = (\nabla s)_x$ is well defined, independent of connection.
This involves a repeated use of Kodaira vanishing, and LES sequences in cohomology associated to SES of sheaves. For (a) and (b) use the SES
\[ 0 \to \mathcal{I}_x(L) \to \mathcal{O}(L) \to L_x \to 0 \quad 0 \to \mathcal{I}_{x,y}(L) \to \mathcal{O}(L) \to L_x \oplus L_y \to 0 \]
where \( \mathcal{I}_{x,y}(L) \) denotes the ideal sheaf of sections of \( L \) that vanish at \( x \) and \( y \). If \( \mathcal{I}_x^2(L) \) denotes the ideal sheaf of sections of \( L \) that vanish to second order at \( x \), then for (c) we use
\[ 0 \to \mathcal{I}_x^2(L) \to \mathcal{I}_x(L) \xrightarrow{d_x} L_x \otimes \mathcal{T}_x^* M \to 0. \]
Therefore we need to show that for large \( k \gg 0 \)
\[ H^1(M, \mathcal{I}_x(L)) = 0, \quad H^1(M, \mathcal{I}_{x,y}(L)) = 0, \quad H^1(M, \mathcal{I}_x^2(L)) = 0 \]
The best way to prove this is by blowing up \( M \) at one point \( x \) (respectively two distinct points \( x, y \) and finally twice at \( x \) "on top of each other"), and replace the ideal sheaves with line bundles associated to the exceptional divisor(s), and then apply Kodaira vanishing.

For example, for (a), let \( \pi : \hat{M} \to M \) denote the blow up of \( M \) at \( x \) with exceptional divisor \( E = \pi^{-1}(x) \). We get a commutative diagram
\[
\begin{array}{ccc}
H^0(M, L^k) & \longrightarrow & H^0(\hat{M}, \pi^* L^k) \\
\downarrow \pi^* & & \downarrow r \\
H^0(\hat{M}, \pi^* E) & \longrightarrow & H^0(\hat{M}, \pi^* L^k)
\end{array}
\]
and it suffices to show the bottom restriction is surjective. But that follows because
\[ H^1(\hat{M}, \pi^* L^k \otimes \mathcal{O}(-E)) = 0, \quad \text{by Kodaira vanishing for } k \gg 0, \]
using the LES associated to the divisor SES
\[ 0 \to \mathcal{O}_{\hat{M}}(-E) \to \mathcal{O}_{\hat{M}} \to \mathcal{O}_E \to 0 \]
after tensoring it with \( \pi^* L^k \).

\[ \text{---} \text{---} \]
\[ ^{13} \text{see } \#9, 10 \text{ in PS2.} \]
We are interested in understanding the following questions:

- how many different complex structures can we put on a fixed smooth manifold?
- how many different holomorphic structures can we put on a fixed smooth vector bundle?

In general, the answer may be NONE (if we find topological obstructions e.g. if dimension or rank is odd), and sometimes it is "infinitely many" (up to isomorphism), so we want a much more qualitative answer. (e.g. if this was a linear problem, we would want to know at least the dimension of the space of solutions).

Say we have one such structure, can we describe its "neighborhood"? i.e. its local deformations (up to isomorphism) in the corresponding moduli space $M$:

- of holomorphic vector bundles $E \to X$ or
- of complex structures on $X$

up to isomorphism (so $M$ is a quotient space). Of course, we should give the moduli space $M$ a natural topology, i.e. make sense of what it means "to deform" and "small deformation". Perhaps we would even hope that the moduli space may be a smooth complex manifold. Unfortunately, the natural topology on moduli spaces is usually pretty bad (e.g. not smooth, and often not even Hausdorff!), except in some very special cases.

Still, we can hope at least to understand small local deformations of these structures, or at least the space of infinitesimal deformations aka $TM$, which can be identified with a Dolbeault cohomology group, thus finite dimensional (when $X$ is compact).

In general, this is only the "formal tangent space" i.e. $M$ may NOT be cut transversely, i.e. not all infinitesimal deformations can be integrated to actual deformations, contributing to singularities in the moduli space. There are obstructions to the integrability of the infinitesimal variations (conveniently enough, also in some Dolbeault cohomology group). When these obstructions vanish, the infinitesimal variations can be integrated and thus the moduli space is smooth.

19.1. Baby model. Consider the automorphism group $\text{Aut}(X)$ of a complex compact manifold $X$. Recall that infinitesimal automorphisms=holomorphic vector fields:

$$T_{id}\text{Aut}(X) = \{ \text{infinitesimal automorphisms} \} = H^0(X, TX),$$

is a finite dimensional complex Lie algebra. In this case, all infinitesimal automorphisms $v$ can be integrated to give rise to a path $\varphi_t$ of diffeomorphisms (actually automorphisms!) starting with $\varphi_0 = id$ and tangent to $v$ at $t = 0$, i.e.

$$\varphi_0 = id \quad \frac{d}{dt} \bigg|_{t=0} \varphi_t = v.$$ 

So there are no obstructions in this case. This can be used $^{[4]}$ to define a topology and in fact a complex Lie group structure on the automorphism group (cf Kobayashi’s classic book "Transformation Groups . . ").

**Note:** This discussion only recovers the identity component $\text{Aut}^0(M)$ of $\text{Aut}(M)$ i.e. automorphisms which are homotopic to $id$. There may also be automorphisms not homotopic to $id$. For example, one can show that a genus $g \geq 2$ curve has finitely many automorphisms, and that a genus 2 curve always has at least a $\mathbb{Z}_2$ automorphism group (it is hyperelliptic, i.e. a branch cover of $\mathbb{C}P^1$ and the deck transformation is an automorphism).

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$^{[4]}$ by "reverse engineering"
Note: Of course, the quotient
\[ \frac{\text{Aut}(M)}{\text{Aut}^0(M)} = \pi_0(\text{Aut}(M)) \]
is a discrete Lie group. Note that the infinitesimal diffeomorphisms \( T_{id} \text{Diff}(X) = \mathcal{A}^0(TX) \) are infinite dimensional, while the ones preserving the complex structure are finite dimensional, i.e. solutions \( v \in \mathcal{A}^0(TX) \) of the differential equation:
\[ \bar{\partial}v = 0 \quad \text{i.e. a class} \quad [v] \in H^0(X, TX). \]

(HW) Also describe this in terms of 0-Cech cocycles.

19.2. Warm-up case: moduli space of line bundles. Consider next \( \text{Pic}(X) \), the moduli space of holomorphic line bundles. We had many descriptions of it.

Restrict for simplicity to the Jacobian \( \text{Pic}^0(X) = \mathcal{M} \) the moduli space of topologically trivial holomorphic line bundles (up to isomorphism covering the identity on the base).

Recall that complex line bundles (up to smooth isomorphism) are classified by their first Chern class (by the exponential sequence). The Kodaira-Spencer Theorem 13.5 used the (holomorphic) exponential LES sequence
\[ H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\delta^*} H^2(X, \mathbb{Z}) \]
to show that
\[ \mathcal{M} = \text{Pic}^0(X) = H^1(M, \mathcal{O})/H^1(M, \mathbb{Z}). \]

Careful: in general this quotient may be terrible topologically (not Hausdorff); locally the quotient may look fine, but globally orbits may return to be dense! Typical example of ”bad quotient” is \( \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z}) = S^1/\alpha\mathbb{Z} \) where \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) (which has dense orbits). However,

**Proposition 19.1.** If \( X \) is compact Kahler then the Jacobian
\[ \text{Pic}^0(X) \cong H^{0,1}(X)/H^1(X, \mathbb{Z}) \]
is naturally a complex torus of dim \( b_1(X) \).

**Proof.** Argue that \( H^1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathbb{C}) \xrightarrow{\alpha_{0,1}} H^{0,1}(X) \) is injective with discrete image, aka a lattice, thus the quotient is indeed a torus. \( \Box \)

**Example 19.2.** If \( C \) is a compact cx curve, \( \text{Pic}^0(C) = H^{0,1}(C)/H^1(C, \mathbb{Z}) \) is a \( g \)-dimtorus.

What is the geometric interpretation of \( H^1(X, \mathcal{O}) \) and \( H^1(X, \mathbb{Z}) \)? Clearly
\[ H^1(X, \mathcal{O}) = \{ \text{1st order variations in the holomorphic structure} \} \]
but why??

Recall that a holomorphic structure on \( L \) is equivalent to a \( \bar{\partial} \)-operator on sections of \( L \), i.e. a first order operator \( D : \mathcal{A}^0(X, L) \rightarrow \mathcal{A}^{0,1}(X, L) \) satisfying Leibnitz rule and which squares to 0. Assume \( L \) is topologically trivial, fix an identification \( L \cong X \times \mathbb{C} \), and let \( \bar{\partial}_0 \) be the trivial \( \bar{\partial} \)-operator.

But we are interested in the moduli space, which is quotient up to isomorphism, so we need to divide by the action of the gauge group \( \mathcal{G} = \text{Maps}(X, \mathbb{C}^*) \)
\[ \mathcal{M} = \{ \text{\( \bar{\partial} \)-operators on a topologically trivial line bd}\}/\mathcal{G} \]
It is easier to first consider the quotient by the identity component \( \mathcal{G}^0 \) and then take the quotient by the leftover group
\[ \mathcal{G}/\mathcal{G}^0 = \pi_0(\mathcal{G}) = [X, S^1] = H^1(X, \mathbb{Z}). \]
But the difference between two $\partial$-operators is a 1-form $\alpha$, so

$$\overline{\partial}_\alpha = \overline{\partial}_0 + \alpha$$

where the integrality condition $\overline{\partial}_\alpha^2 = 0$ becomes $\overline{\partial}_0 \alpha = 0$. So

$$\{ \overline{\partial} \text{-operators on a top trivial line bd} \} \leftrightarrow \{ \alpha \in \mathcal{A}^{0,1}(X) \mid \overline{\partial}_0 \alpha = 0 \}.$$ A gauge transformation $g \in \mathcal{G}$ acts on $\alpha \in \mathcal{A}^{0,1}(X)$ by translation:

$$\alpha \mapsto \alpha + g^{-1} \overline{\partial} g = \alpha + \overline{\partial} \log g$$

The (Lie algebra) exponential map $T_{id} \mathcal{G} \exp \rightarrow \mathcal{G}$ is

$$\text{Maps}(X, \mathbb{C}) \rightarrow \text{Maps}^0(X, \mathbb{C}^*) \quad f \mapsto e^f$$

Moreover, if $\alpha = \overline{\partial} f$ where $f : X \rightarrow \mathbb{C}$ is a smooth map, then $g = e^f$ pulls back $\overline{\partial}_0$ to $\overline{\partial}_\alpha$. Therefore we get an identification

$$\{ \overline{\partial} \text{-operators on a top trivial line bd}\}/\mathcal{G} \leftrightarrow H^{0,1}(X).$$

To summarize, infinitesimal deformations of the holomorphic structure on a line bundle are in 1-1 correspondence with classes $[\alpha] \in H^{0,1}(X) = H^1(X, \mathcal{O})$. So the moduli space of line bundles is locally modeled on $H^{0,1}(X)$.

**Note:** For line bundles, the moduli space is cut out by a linear equation $\overline{\partial}_0 \alpha = 0$, so infinitesimal deformations $= \text{holomorphic structures up to isomorphisms isotopic to the identity.}$

The space of infinitesimal automorphisms of $L$ is identified with

$$\{ \text{infinitesimal automorphisms} \} = H^0(X, \mathcal{E}nd(L)) = H^0(X, \mathcal{O}) = \mathbb{C}.$$ where $\mathcal{E}nd(L)$ denotes the sheaf of germs of holomorphic endomorphisms of $L$.

For higher rank bundles $E$, the deformation $\alpha$ is an element $\alpha \in \mathcal{A}^{0,1}(X, \mathcal{E}nd(E))$ but now we get a quadratic equation

$$\overline{\partial}_0 \alpha - \alpha \land \alpha = 0$$

The space of infinitesimal deformations is still $v \in H^1(X, \mathcal{E}nd(E))$, but now there are first-order obstructions $v \land v \in H^2(X, \mathcal{E}nd(L))$. The infinitesimal automorphisms are $H^0(X, \mathcal{E}nd(E))$. Of course, global constants are always automorphisms. If $H^0(X, \mathcal{E}nd(E))$ is larger than that, we get singularities in moduli space, and the bundle would not be stable (see discussion of stability at the end of Lecture 16).

**Note:** Stability (and the borderline case of semi stability) is needed so we get a Hausdorff quotient. Typical example is the $\mathbb{C}^*$ action on $\mathbb{C}^n$ where 0 is an unstable orbit (the other orbits accumulate to it), and needs to be tossed out to get a nice quotient $\mathbb{C}P^n$.

**Note:** The moduli space of (stable) holomorphic bundles $E$ can also be described the HYM equations to pick out the "best representative" out of all Chern connections on $E$. Recall that the space of Chern connections is also modeled on $\mathcal{A}^{0,1}(X; \mathcal{E}nd(E))$, but instead of taking the quotient by the gauge group, we look for the connection which minimizes HYM functional, i.e. a solution of

$$\text{tr}_g R^\nabla = \lambda \text{Id}_E$$

For line bundles, this is easy

**Proposition 19.3.** Any holomorphic line bundle has a unique HYM connection.

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15 see Kobayashi’s classic book "Differential geom of complex v. bds"
Proof. Start with a hermitian structure $h$, whose Chern connection has curvature $R^\nabla$, a $(1,1)$-form. Then $i \text{tr}_g R^\nabla$ is a real function $\varphi$ which by Hodge theory can be (uniquely) decomposed as $\varphi = \lambda - \partial^* \partial f$ where $\lambda$ is a constant=harmonic and $f = G_\varphi$. Using the Kahler identities, one can check that $e^f h$ is a HYM metric with constant $\lambda$. □

This discussion extends to higher rank bundles, except that again HYM is no longer a linear problem, and solutions to HYM may not always exist (existence of solutions is related to the stability of the bundle, cf Theorem 16.9). See PS#5 for more properties.
Lecture 20. Deformations of complex structures

Assume $X$ is compact complex connected manifold. On the same smooth manifold, there may be many different complex structures (up to biholomorphisms). Consider the moduli space of complex structure on $X$:

\[ \mathcal{M}_{cx}(X) = \{ \text{complex structures on } X \} / \text{Diff}(X) \]  

(20.1)

where $\varphi \in \text{Diff}(X)$ acts on a complex structure $J$ by pullback. We might hope that this moduli space is a nice complex manifold, but unfortunately that is almost never the case (only in some rare cases).

Example 20.1. Consider elliptic curves $E = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} \oplus \tau \mathbb{Z}$. They are all diffeomorphic to a torus, but will not all be biholomorphic. Given an elliptic curve, we can see at least a 1-dimensional space of deformations of the complex structure on $E$, given by small deformations in the period $\tau$. We shall see that the moduli space is indeed a 1-dimensional complex orbifold, a manifold except for 2 singular points, corresponding to the two lattices (square+ equilateral triangle) that have extra automorphisms.

Start first with a cover of the moduli space, the Teichmüller space of $X$:

\[ \text{Teich}(X) = \{ \text{complex structures on } X \} / \text{Diff}^0 \]

where $\text{Diff}^0$ denotes the identity component of Diff. The quotient

\[ \Gamma = \text{Diff}/\text{Diff}^0 = \pi_0(\text{Diff}) \]

is called the mapping class group, and we can worry last about the quotient by $\Gamma$.

Our plan is to understand

(a) local deformations and infinitesimal deformations (first order variations in the structure)
(b) obstructions and first order obstructions to integrability of the infinitesimal deformations

Definition 20.2. Assume $T$ is a connected complex manifold. A holomorphic family of compact complex manifolds parameterized by $T$ is a (paracompact) complex manifold $X$ with a proper, surjective, holomorphic submersion $\pi : X \to T$.

Note: Ehresmann Thm (in diff top) implies that $X$ is a locally trivial smooth fibration. In particular, the fibers $X_t$ are diffeomorphic to each other. We usually take $T$ a small ball $B$ centered at 0 (even a germ), and regard the fibers over $t \neq 0$ as deformations of the central fiber $X = X_0$. Equivalently (after fixing a smooth trivialization) we can regard the family $X_t$ for $t \in B$ small as a family $J_t$ of complex structures on the central fiber $X$.

Careful:

(i) the fibers are diffeomorphic, but may NOT be biholomorphic to each other (e.g. an elliptic fibration.)
(ii) the fibers $X_t$ over the punctured ball $B^*$ may all be biholomorphic, while the limit $X_0$ is diffeo, but may NOT be biholomorphic to the rest! (e.g. a family of Hirzebruch surfaces.)
(iii) the fibers may all be biholomorphic to each other, but the family may not be a product! (e.g. projectivize a holomorphic bundle); however, Fischer-Grauert Thm implies that such family is locally holomorphically trivial.

Theorem 20.3 (Kodaira Spencer). Assume $X$ is a compact complex manifold. Then there is a natural correspondence between

\[ \{ \text{infinitesimal deformations of the complex structure} \} \leftrightarrow H^1(X, TX). \]
The infinitesimal obstructions are elements of $H^2(X, \mathcal{T}X)$, and the first order obstruction is 

$$H^1(X, \mathcal{T}X) \rightarrow H^2(X, \mathcal{T}X), \quad v \mapsto [v, v].$$

**Warning:** Because of singularities, in general one needs to allow for more general (non-smooth) families of deformations. e.g. in alg geometry work over "schemes" or even better "stacks". I completely ignore this issue here.

**Note:** One could try to look for the "universal" deformation space (i.e. any other one is pulled back from the universal one – by a unique "classifying map"), e.g. by making the space of deformations into a category and looking for the final object. The presence of automorphisms obstructs the existence of the universal family, so one can at most hope for a "versal" family (the pullback map is unique up to 1st order)

**Note:** Other useful results:
- Kuranishi proved that any complex manifold admits a versal deformation.
- if $H^2(X, T) = 0$ then $X$ has a smooth versal family of deformations.
- if $H^0(X, T) = 0$ then any versal deformation is universal (thus unique).
- Kodaira showed that small deformation of compact Kahler manifolds are Kahler.

**Example 20.4.** Any genus $g$ curve has $H^2(C, \mathcal{T}C) = 0$ by dimensional reasons. If $g \geq 2$ then also $H^0(C, \mathcal{T}C) = 0$ since $\deg \mathcal{T}C = 2 - 2g < 0$. In fact one can show $C$ has finitely many automorphisms (such a curve is called stable).

For $g \geq 2$ the moduli space of complex structures on a genus $g$ curve

$$\mathcal{M}_g = \{ \text{complex structures on } C \} / \text{Diff}(C)$$

is a $3g - 3$ dimensional orbifold locally modeled on $H^1(C, \mathcal{T}C)$, technically up to $\text{Aut}(C)$, but the generic genus $g \geq 3$ has no nontrivial automorphisms. The dimension is calculated using Riemann-Roch

$$\chi(C, \mathcal{T}C) = 0 - h^1(C, \mathcal{T}C) = (2 - 2g) + 1 - g.$$  

**Note:** Any genus 2 curve has at least a $\mathbb{Z}_2$ automorphism: it is hypereliptic, i.e. can be written as degree 2 branch cover of $\mathbb{CP}^1$, and the deck transf is an automorphism.

**Note:** (skip??) In genus 0, the automorphism group is 3 dim (Moebius transformations), and there is a unique complex structure (up to isom). In genus 1 the automorphism group is 1 dim. To deal with these special cases, add 3 respectively one marked points to kill the infinitesimal automorphism group, i.e. make the punctured surface stable/have negative Euler characteristic. More generally, can consider $\mathcal{M}_{g,n}$ the moduli space of complex structures on a genus $g$ curve with $n$ marked points (i.e. ordered distinct points). Here we restrict to $2g - 2 + n > 0 \iff$ curve is stable $\iff$ punctured Euler characteristic is negative $\iff$ punctured curve admits a hyperbolic metric, $\iff$ there are no infinitesimal automorphisms fixing the punctures.

One can similarly calculate $\dim \mathcal{M}_{g,n} = 3g - 3 + n$.

**Note:** The universal family is $\pi : \mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ the map that forgets the marked point, or more generally $\pi : \mathcal{M}_{g,n+1} \rightarrow \mathcal{M}_{g,n}$. If the (punctured) curve $C$ has no automorphisms, then the fiber of $\pi$ is a copy of $C$.

**Example 20.5.** Start with two generic cubics $Q_1, Q_2$ in $\mathbb{CP}^2$. These are tori by adjunction formula, and intersect (transversally) in $9=3H \cdot 3H$ points. Get a pencil of cubics

$$Q_1(z) + \lambda Q_2(z) = 0 \quad ** \text{picture} **$$

\[^{16}\text{careful, may not be true for large deformations}\]
all passing through the 9 points (thus no other common intersection – holomorphic submanifolds intersect in positive multiplicity UNLESS contained into each other).

Blow up the 9 points to get an elliptic fibration $X \to \mathbb{P}^1$. For topological reasons, there must be singular fibers ($X$ is diffeo to the $\mathbb{CP}^2 \# 9\mathbb{CP}^2$, which has Euler characteristic 12).

The generic pencil of cubics has only simple nodal singularities ("fishtail" fibers). Show that it MUST have singular fibers. Less generic examples will have worse singularities. For example, two fishtails can be traded for a cuspidal singularity. One can even get a fiber with an $E_8$ singularity (start with a smooth cubic $Q_1$ an a triple line $Q_2$ tangent to it).

Note: Here "generic" means on an open, dense set, but in fact one can show a stronger result: the subspace of cubics which are singular (i.e. either not immersed or with double point) is a finite union of complex codimension at least one submanifolds.

Note: (Skip) If $C$ is a curve and $x_1, \ldots, x_n$ are the marked points, tensoring the divisor SES \([14.2]\) with $\mathcal{T}C$ gives the SES

$$0 \to \mathcal{T}C \otimes \mathcal{O}(-\sum x_i) \to \mathcal{T}C \to \oplus \mathcal{T}_{x_i}C \to 0$$

If we denote by $\mathcal{T}(C, x) = \mathcal{T}C \otimes \mathcal{O}(-\sum x_i)$, regarded as the holomorphic tangent space to the marked curve, then the SES induces a LES in cohomology

$$0 \to H^0(\mathcal{T}(C, x)) \to H^0(\mathcal{T}C) \to \oplus \mathcal{T}_{x_i}C \to H^1(\mathcal{T}(C, x)) \to H^1(\mathcal{T}C) \to 0$$

where the rest of the terms vanish by dimensional reasons. The $H^0$ terms are the infinitesimal automorphisms of the marked curve $(C, x)$ (i.e. those that preserve the marked points) and respectively of $C$. The $H^1$ terms are the infinitesimal deformations; the infinitesimal obstructions $H^2 = 0$. The middle term $\oplus \mathcal{T}_{x_i}C$ can be interpreted as the infinitesimal variation in the marked points. When $C$ is stable, the both $H^0$ terms vanish, so as complex vector spaces

$$H^1(\mathcal{T}(C, x)) = H^1(\mathcal{T}C) \oplus \oplus \mathcal{T}_{x_i}C$$

This is consistent with the expectation that a variation in a marked curve consists of a variations in the curve plus a variation in the marked points.

Example 20.6. If $E \to M$ is a rank $r$ holomorphic vector bundle, its fiberwise projectivization $\mathbb{P}(E) \to M$ is a (locally holomorphically trivial) $\mathbb{CP}^{r-1}$ bundle. Moreover, $\mathbb{P}(E \otimes L) \cong \mathbb{P}(E)$ for any line bundle $L$.

Example 20.7. Hirzebruch surfaces are $H_n = \mathbb{P}(\mathcal{O}(n) \oplus \mathcal{O}) \to \mathbb{P}^1$, holomorphic $\mathbb{P}^1 = \mathbb{P}(\mathbb{C} \oplus \mathbb{C})$ bundles over $\mathbb{P}^1$ with a holomorphic zero section $D_0$ and infinity section $D_\infty$. Then

$$H_n \cong H_m, \iff n \equiv m \mod 2 \quad \text{while} \quad H_n \cong H_m^\text{biholo}, \iff n = \pm m.$$
(HW) Prove that topologically there are only two smooth $\mathbb{P}^1$ bundles over $\mathbb{P}^1$, the trivial one $H_0$ and the twisted one $H_1$, the blow up of $\mathbb{C}P^2$ at a point. *picture **. Show that $H_n \cong H_{-n}$ using the fact that $\mathbb{P}(L \oplus O) = \mathbb{P}(L \oplus (O \oplus L^*)) = \mathbb{P}((O \oplus L^*))$. To show $H_n \ncong H_m$ for $n \neq \pm m$, use the fact that the infinity section is the smooth curve with the most negative self intersection.

**Example 20.8.** One can cook up a 1-parameter family of Hirzebruch surfaces $H_0 = \mathbb{P}^1 \times \mathbb{P}^1$ degenerating to $H_2$, using a family of cubic equations in $\mathbb{P}^1 \times \mathbb{P}^2$, see example 6.2.1(iv) in Huybrechts. This is a manifestation of the fact that $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is NOT compact.

20.1. *The Kodaira-Spencer map.* The infinitesimal deformation of a family $\pi : \mathcal{X} \to B$ at $0 \in B$ is a linear map, called the Kodaira-Spencer map

$$\kappa : \mathcal{T}_0 B \to H^1(X_0, TX_0)$$

that can be defined in several different ways.

One definition uses Cech cocycles. By the implicit function theorem, we can find local holomorphic coordinates on the total space $\mathcal{X}$ in which the map $\pi$ is a linear projection. Cover the central fiber $X = X_0$ with such "product" coordinate charts $U_\alpha$ on the total space. Then each (holomorphic) tangent vector $v \in \mathcal{T}_0 B$ can be locally lifted (by the IFThm) to a local holomorphic section $v_\alpha$ of the tangent bundle $\mathcal{T}\mathcal{X}|_{U_\alpha}$ to the family (which projects to $v$). Therefore on the overlap $U_\alpha \cap U_\beta$, $v_\alpha - v_\beta$ has to be tangent to the fibers. The elements $\varphi_{\alpha \beta} = v_\alpha - v_\beta$ clearly satisfy the cocycle condition, thus giving rise to a 1-Cech cocycle $[\varphi_{\alpha \beta}] \in H^1(X_0, TX_0)$. One can show this cycle is well defined, independent of choices.

An equivalent definition uses variations in an almost complex structure and their integrability condition (the Maurer-Cartan equation). In this description, which may seem longer, we also see the obstructions to integrability appearing naturally.

Consider the space of almost complex structures on $X$:

$$\mathcal{J}(X) = \{ J \in \text{End}(TX) \mid J^2 = -\text{id} \}$$

which can be given the structure of an infinite dimensional manifold, and its subset $\mathcal{J}_{\text{int}}(X)$ of integrable ones (which may not be cut transversally). Still, we want to understand at least

(i) the (formal) tangent space $T_J \mathcal{J}_{\text{int}}(X) \subset T_J \mathcal{J}(X)$ of infinitesimal variations,

(ii) the induced action of $\text{Diff}^0$ on $\mathcal{J}_{\text{int}}(X)$.

(iii) the (formal) tangent space to the quotient $\mathcal{J}_{\text{int}}(X) / \text{Diff}^0 = \text{Teich}(X)$.

(iv) the obstructions, especially the 1st order obstruction.

Recall: an almost complex structure $J_t$ is determined uniquely by a splitting of the complexified tangent space $T_c X = T^{1,0}_t \oplus T^{0,1}_t$ into eigen-bundles, and $J_t$ is integrable iff

$$[T^{0,1}_t, T^{0,1}_t] \subseteq T^{0,1}_t.$$  

(20.2)

A small variation $J_t$ of $J_0 = J$ can be therefore encoded as the graph of a map

$$\varphi(t) : T^{0,1}_t \to T^{1,0}_t$$

where $T^{0,1}_t = \text{Graph} \varphi(t)$

and $\varphi(0) = 0$. The integrability condition (20.2) is equivalent to the Maurer-Cartan equation

$$\overline{\partial} \varphi(t) + [\varphi(t), \varphi(t)] = 0$$  

(20.3)

on the variation $\varphi(t) \in \mathcal{A}^{0,1}(TX_0)$.
Note: Assume \( \varphi(t) \) had a (formal) power series expansion \( \varphi(t) = t\varphi_1 + t^2\varphi_2 + \ldots \). Then \([20.3]\) is equivalent to the system

\[
\overline{\partial}\varphi_1 = 0, \quad \overline{\partial}\varphi_2 + [\varphi_1, \varphi_1] = 0, \quad \ldots \quad \overline{\partial}\varphi_k + \sum [\varphi_1, \varphi_{k-1}] = 0.
\]

The first coefficient \( \varphi_1 = \frac{d}{dt}|_{t=0} \varphi(t) \) is the first order deformation, giving rise to a class

\[
v = [\varphi_1] = \left[ \frac{d}{dt}|_{t=0} \varphi(t) \right] \in H^1(X, TX)
\]

which is precisely the Kodaira-Spencer class associated to this 1-parameter family of deformations. The first order obstruction is

\[
[v, v] \in H^2(X, TX),
\]

the obstruction to solving the next equation \( \overline{\partial}\varphi_2 + [\varphi_1, \varphi_1] = 0 \). (i.e. the second equation can be solved iff \( [v, v] = 0 \)). The other equations give higher order obstructions.

Note: The usual Lie bracket on holomorphic vector fields combines with the wedge product on forms to induce a Lie bracket in cohomology

\[
\{\cdot, \cdot\} : H^p(X, TX) \times H^q(X, TX) \to H^{p+q}(X, TX).
\]

In fact, we get a Lie bracket on \( A^{0,*}(TX) \)

\[
\{\cdot, \cdot\} : A^{0,p}(TX) \times A^{0,q}(TX) \to A^{0,p+q}(TX).
\]

defined in local holomorphic coordinates by

\[
(dz_I \otimes v_I, dz_J \otimes v_J) = dz_I \wedge dz_J[v_I, v_J].
\]

One can check that \( \overline{\partial}[\alpha, \beta] = [\overline{\partial} \alpha, \beta] \pm [\alpha, \overline{\partial} \beta] \), thus the Lie bracket descends to cohomology.

**Proposition 20.9.** Assume \((X, J)\) is a complex compact manifold. Then there is a natural bijection between the space of infinitesimal deformations of \( X \) and \( H^1(X, TX) \), given by the Kodaira-Spencer map

\[
\kappa : T_J \text{Teich}(X) \to H^1(X, TX).
\]

**Proof.** We saw that infinitesimal deformations of the complex structure correspond to

\[
T_J \mathcal{J}_{int}(X) \leftrightarrow \{ \varphi_1 \in A^{0,1}(X, TX) \mid \overline{\partial}\varphi_1 = 0 \}.
\]

To complete the proof we need to also take into account the action of \( \text{Diff}^0(\mathcal{J}(X)) \), and the induced action on infinitesimal variations. A 1-parameter family \( F_t \in \text{Diff}^0(X) \) induces a 1-parameter family \( J_t = F_t^* J \) with

\[
T_t^{0,1} = dF_t(T^{0,1}), \quad \text{so} \quad \varphi(t) = (dF_t|_{T^{0,1}})^{1,0} = \sum \frac{\partial(F_t)}{\partial z_k} dz^k \otimes \frac{\partial}{\partial z_l},
\]

(i.e. the upper left block \( \frac{\partial F}{\partial z} \) in the Jacobian matrix of \( F \) in holomorphic coordinates, under the decomposition \( T_C X = T^{1,0} + T^{0,1} \)). So the infinitesimal variation in \( T^{0,1} \) (i.e. in \( J \)) induced by \( F_t \) is equal to

\[
\varphi_1 = \frac{d}{dt}|_{t=0} \varphi(t) = \overline{\partial}w^{1,0} \quad \text{where} \quad w = \frac{dF_t}{dt}|_{t=0} \in T_{id} \text{Diff}(X).
\]

This implies the desired conclusion about the formal tangent space to the quotient. \( \square \)

**Note:** Kodaira-Spencer map appears as a coboundary map \( \delta^* \) in a LES induced by a SES

\[
0 \to T \text{Vert} \to TX \to \pi^* TB \to 0
\]

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