**Recommended Problems # 5**

**Problem 1.** Show that if $M$ is compact Kahler then the Jacobian

$$\text{Pic}^0(M) \cong H^{0,1}(M)/H^1(M,\mathbb{Z})$$

is naturally a complex torus of dimension $b_1(M)$. 

Hint: Use the exponential sequence and argue that $H^1(M,\mathbb{Z}) \to H^1(M,\mathbb{C}) \overset{\pi^{0,1}}\to H^{0,1}(X)$ is injective with discrete image, aka a lattice.

**Problem 2.** Show that the curvature decreases in holomorphic subbundles and increases in quotients. Specifically, let $F$ be a holomorphic subbundle of $E$. Fix a hermitian metric on $E$ which induces one on $F$ and $E/F = Q$, and consider $\Theta_E$, $\Theta_F$ and $\Theta_Q$ the curvatures of the associated Chern connections. Show that

$$\Theta_F \leq \Theta_E|_F \quad \Theta_Q \geq \Theta_E|_Q$$

with equality iff the second fundamental form $A$ of $F$ in $E$ is zero.

Hint: Use formulas in PS #3.

**Problem 3.** Assume $E \to M$ is a holomorphic vector bundle which has finitely many global sections $\{s_1,\ldots,s_N\} \in H^0(M,E)$ spanning each fiber of $E$. Show that $\Theta_E \geq 0$.

Hint: the assumption is equivalent to $M \times \mathbb{C}^N \to E$ is surjective.

**Problem 4.** Assume $L$ is a holomorphic line bundle on a compact Kahler manifold $M$. Show that $L$ has a Chern connection with positive curvature iff $c_1(L) > 0$ (i.e. $c_1(L) \in H^2_{DR}(M)$ has a positive representative.)

Hint: Use Global $\partial\bar{\partial}$ lemma.

**Problem 5.** Assume $L \to M$ is a holomorphic line bundle on a compact Kahler manifold.

(a) show that $L$ has a Hermitian-Yang-Mills metric.

(b) show that for the HYM metric, the Chern form $c_\lambda$ is naturally a complex torus of dimension $b_1(M)$.

Hint: Start with some hermitian metric $h$ and use the global $\partial\bar{\partial}$ lemma to modify it to one $e^h$ for which the HYM equation has a solution.

**Problem 6.** Assume $E$, $F$ are holomorphic vector bundles on $M$ with a HYM metric. Show that the induced metrics on $E \otimes F$, Hom($E,F$) and $E^*$ are also HYM. Show that the sum $E \oplus F$ is HYM $\iff$ $E$ and $F$ have the same constant $\lambda_E = \lambda_F$ $\iff$ their slopes are equal $\mu(E) = \mu(F)$.

**Problem 7.** Fix $(M,g,\omega)$ a compact Kahler manifold. Assume $E \to M$ is a holomorphic vector bundle with a HYM metric i.e. a solution to

$$\text{tr}_g R^\nabla = \lambda \text{Id}_E.$$

Show that the constant $\lambda \in \mathbb{R}$ depends only on $c_1(E)$ and rank $E$, and in fact only on the slope

$$\mu(E) = \frac{\text{deg}(E)}{\text{rank}(E)} \quad \text{where} \quad \text{deg}(E) = \int_M c_1(E) \omega^{n-1}.$$

Hint: In fact, the HYM equation is equivalent to

$$iR^\nabla \wedge \frac{\omega^{n-1}}{(n-1)!} = \lambda \frac{\omega^n}{n!} \text{Id}_E \implies i\text{tr}R^\nabla \wedge \frac{\omega^{n-1}}{(n-1)!} = \lambda \text{rank} E \frac{\omega^n}{n!}.$$  

**Problem 8.** On a compact genus $g$ Riemann surface $\Sigma$ show that we get a correspondence

$\{\text{moduli space of degree 0 holo line bundles}\} \leftrightarrow \{\text{moduli space of flat } U(1)-\text{connections}\}$

\[
\begin{array}{c|c}
\text{exp sequence} & \text{holonomy} \\
\hline
\text{Pic}^0(\Sigma) = H^1(\Sigma,\mathcal{O})/H^1(\Sigma,\mathbb{Z}) = \mathbb{C}^g/\mathbb{Z}^{2g} & \text{Hom}(\pi_1(\Sigma), S^1) = (S^1)^{2g}
\end{array}
\]
Problem 9. Show that on a compact Kahler manifold $M$ the diagrams commute:

$$
\begin{array}{ccc}
A^k & \xrightarrow{d} & A^{k+1} \\
\downarrow^{\pi^{0,k}} & & \downarrow^{\pi^{0,k+1}} \\
A^{0,k} & \xrightarrow{\bar{\partial}} & A^{0,k+1}
\end{array}
\quad \Rightarrow 
\begin{array}{ccc}
H^k(M, \mathbb{C}) & \xrightarrow{\iota^*} & H^k(M, \mathcal{O}) \\
\downarrow^{deRham \cong} & & \downarrow^{Dolbeault \cong} \\
H^k_{dR}(M, \mathbb{C}) & \xrightarrow{\pi^{0,k}} & H^{0,k}(M)
\end{array}
$$

What happens if we project onto $(p, q)$ with $p > 0$?