Ideal points of and Incompressible Surfaces in Punctured Torus Bundles

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January 9, 2006

1 Introduction

Definition 1.1. Let \( \mathcal{T} \) be a tetrahedralisation of the 3-manifold \( M \), with \( N \) tetrahedra. Given a choice of one of the three dihedral angles in each tetrahedron, we define the tetrahedron variety of \( M \) with respect to the tetrahedralisation \( \mathcal{T} \), \( \Xi(M; \mathcal{T}) \) to be the affine variety in \( (\mathbb{C} \setminus \{0,1\})^N \) defined as the solutions of the gluing equations, where each dimension of the ambient space corresponds to a tetrahedron.

This is also known as Thurston’s parameter space, and is closely related to the deformation variety \( \mathcal{D}(M; \mathcal{T}) \), which retains the symmetry of the tetrahedralisation at the cost of using three times as many variables, by not making a choice of dihedral angle in each tetrahedron.

Definition 1.2. The character variety \( \chi(M) \) is the variety consisting of traces of representations of \( \pi_1(M) \) into \( PSL_2(\mathbb{C}) \).

Definition 1.3. An ideal point of the tetrahedron variety is a limit point \( p \) of \( \Xi(M; \mathcal{T}) \) at which one or more of the tetrahedra angles converges to 0, 1 or \( \infty \). We say that such a tetrahedron degenerates.

Definition 1.4. An ideal point of the character variety is a point at which one or more of the characters "blows up", i.e. the trace of an element of \( \pi_1(M) \) under the representation into \( PSL_2(\mathbb{C}) \) goes to \( \infty \) as we approach this point.

There is a strong correspondence between \( \Xi(M; \mathcal{T}) \) and \( \chi(M) \), although they are not isomorphic. Given a finite (not ideal) point of the tetrahedron variety, the holonomy of loops through the manifold is determined, and we obtain a point of the character variety.

As we approach an ideal point of the character variety, a loop through the manifold becomes infinitely long, so not all of the tetrahedra in our triangulation can stay finite. However, it can happen that some, or even all of the tetrahedra in the triangulation can degenerate, while the characters remain finite.
Yoshida [12] (in his Definition 3.1) has a slightly different definition of an ideal point of the tetrahedron variety, requiring that a certain slope on the boundary torus of the manifold is non trivial. Such an ideal point of the tetrahedron variety will necessarily correspond to an ideal point of the character variety, and rules out the kind of situation mentioned above.

**Definition 1.5.** A surface $S$ in a 3-manifold $M$ with $\partial S \subset \partial M$ is said to be **incompressible** if $S$ has no sphere components and if every loop in $S$ that bounds a disk in $M \setminus S$ also bounds a disk in $S$. A surface with boundary is said to be **$\partial$-incompressible** if every arc $\alpha$ in $S$ (with $\partial(\alpha) \subset \partial S$) which is homotopic to $\partial M$ is homotopic in $S$ to $\partial S$. We will ignore boundary parallel surfaces and in general refer to surfaces that are incompressible, $\partial$-incompressible and not boundary parallel as incompressible surfaces.

Culler and Shalen [3] give a method of constructing an incompressible surface in a 3-manifold from an ideal point of the character variety of that manifold. The question naturally arises of the degree to which a reverse construction might be possible. That is, given an incompressible surface, does it come from an ideal point?

**Definition 1.6.** If $M$ is a 3-manifold with a single torus boundary, orientable, irreducible and compact, we say that a boundary slope of an incompressible surface that comes from an ideal point is **detected**. If there is no closed surface that comes from the same ideal point then the slope is **strongly detected**, otherwise it is **weakly detected**. In our case, due to the classification of the incompressible surfaces of punctured torus bundles by Floyd and Hatcher [4] (and independently by Culler, Jaco and Rubinstein [8]), we know that there are no closed incompressible surfaces, so we have only strongly detected slopes. We also refer to an incompressible surface as being **strongly detected** if it corresponds to an ideal point of the character variety via either the twisted square construction of Yoshida [12] or the construction of Culler and Shalen [3].

Note that it is possible for different incompressible surfaces to have the same boundary slope, and so it doesn’t appear to be clear whether the two constructions result in the same surfaces, and hence if the same surfaces are strongly detected.

Previous results about strongly detected surfaces: Ohtsuki [9] shows that all boundary slopes of incompressible surfaces in 2-bridge knots are strongly detected, but that not every incompressible surface can be obtained from the construction. Schamuel and Zhang [10] gave the first examples of non-fiber (and non semi-fiber) boundary slopes that are not strongly detected, although they are weakly detected. Chesebro and Tillmann [2] give an infinite family of hyperbolic knots, each of which has at least one boundary slope of an incompressible surface (non-fiber and non semi-fiber) that is not strongly detected.

**Theorem 1.7.** All incompressible surfaces (other than the fiber) in punctured torus bundles over the circle with large enough fans, are strongly detected.
"Large enough" for us currently means with at least 3 tetrahedra in each
fan (not counting hinge tetrahedra). In fact this only seems to be a major issue
around a $L^*_p R^*_q$ join (these terms defined later), and there don’t seem to be any
obvious issues with the other cases, but further study is needed.

In 1982, Floyd and Hatcher [4], and Culler, Jaco and Rubinstein [8] classified
the incompressible surfaces in punctured torus bundles. We will work from the
an ideal point of the tetrahedra variety of non-zero (i.e. non-trivial) slope, based
on the work of Culler and Shalen.

Given an incompressible surface in a punctured torus bundle, obtained from
Yoshida’s construction, it is not immediately obvious which Floyd-Hatcher sur-
face it is isotopic to. In fact, as constructed, the Yoshida surface may need a
number of ambient 2-surgeries and deletions of sphere components before it is
incompressible (and boundary incompressible). However, it must of course be
reducible to one of the Floyd-Hatcher surfaces.

The plan of attack is to reverse this process: to start with a Floyd-Hatcher
surface, isotope and add sphere components until it is in the form of a Yoshida
surface, and then check that the corresponding (apparently) ideal point of the
tetrahedra variety is indeed an ideal point.

In fact it is generally easier to follow the isotopies from the more convoluted
Yoshida form to the simpler Floyd-Hatcher form, so our argument for that part
of the proof proceeds in that direction: For a surface given to us in Floyd-
Hatcher form, we give the corresponding surface in Yoshida form (in section 4)
and check that the Yoshida form surface simplifies to the given Floyd-Hatcher
form (in section 5).

Yoshida’s construction relates the rates at which various tetrahedra degener-
ate to the position of the surface with respect to the triangulation (the number
of twisted square pieces of the surface in each tetrahedron give the relative rates
of degeneration). We use this relation in reverse: we need to show that the
degeneration rates implied by the Yoshida form of our surface correspond to an
ideal point $p$ of the tetrahedra variety. That is, we need to obtain a solution to
some form of the gluing equations at the (hopefully) ideal point $p$ (and there
may be non-degenerate tetrahedra in the triangulation, as well as the ones that
are degenerate), and to show that $p$ is the limit of finite (no angle is $0, \infty$ or 1)
points of the tetrahedron variety.

The solution we will find for $p$ will be explicit, in the sense of giving actual
complex values for the angles in the non-degenerate tetrahedra, and the "direc-
tions" of degeneration for the degenerate tetrahedra.

The equations we find a solution to are not the gluing equations themselves,
since a number of variables are supposedly converging to 0 or \( \infty \) as we approach \( p \). Instead we make a number of changes of variable. First we change which dihedral angle within each degenerating tetrahedron is labelled, to standardise so that the labelled angle is the one converging to 0. Secondly we perform a kind of weighted "blow up" (of the algebraic geometry sort), replacing each variable which is now converging to 0 with a "direction" variable, multiplied by some power of a global "convergence variable", which we call \( \zeta \). So if \( Z \) is the complex angle in some tetrahedron, which is supposedly converging to 0 at rate \( k \), we replace \( Z \) in our equations with \( \zeta^k y \) (here \( y \) is a direction variable). Each gluing equation describes the complex angles around an edge of the tetrahedralisation, and for the gluing equations to be satisfied as our tetrahedra degenerate, we must have the sum of the rates at which angles around this edge converge to 0 to equal the sum of the rates at which angles converge to \( \infty \). This is the case for the rates we get from our Yoshida form of the surface. The effect this has on our equations is that a power of \( \zeta \) factors out from each equation corresponding to an edge at which some dihedral angles are converging to 0 or \( \infty \). Deleting this factor and rearranging the equations to form polynomial equations we reach a form of the equations we call \textbf{tilde} equations.

It is these equations we will find a solution for, when \( \zeta = 0 \), corresponding to being at \( p \). We bring in another idea here, concerning the angle variables (that is, the ones that do not degenerate as we approach \( p \)):

**Lemma.** \( a_k = \frac{1 - \cos k \beta}{1 - \cos \gamma} \) is a solution of \( a_{k-1}a_{k+1} - (1 - a_k)^2 = 0 \)

This recursion relation comes from the gluing equations at the 4-valent edges throughout one of the two types of fan of a torus bundle, and holds whether or not we are at a point at which some tetrahedra are degenerating. A very similar result, for a closely related recursion relation, holds for fans that face the other direction. It turns out that when tetrahedra are degenerating, the ends of a fan degenerate in such a way that we can add extra "fake" variables to either end of the fan, set their values to be 1, and then the recursion relations also hold at the ends of the fans.

Using these results we can assign values to variables corresponding to all of the non-degenerate tetrahedra. The values assigned will be real numbers, but not 0 or 1 (or \( \infty \)), so they correspond to non-degenerate tetrahedra. The remaining direction variables depend on each other in understandable ways: essentially, certain ratios between variables at the joins between fans are equal to each other throughout the torus bundle, and normalising by setting one particular direction variable to be 1, we can use these ratios to determine all other direction variables, with a choice of sign in some cases. None of the direction variables are assigned a value of 0.

We have a solution to the \textbf{tilde} equations, we now want to show that we also have solutions nearby that correspond to finite points of the tetrahedron variety. There are two parts to this:
• (i) We show that there are points of the variety defined by the tilde equations arbitrarily near to $p$.

• (ii) We show that points $q$ close enough to $p$ must have the variable $\zeta \neq 0$. Then by continuity, at $q$ all angle variables are still away from 0 or 1, and all direction variables are away from 0, and so multiplying them by the appropriate power of $\zeta$ (recall that the original complex angle, $Z = \zeta^k y$) we obtain a complex angle that is near, but not equal to 0. Hence all complex angles of tetrahedra for this point are non-degenerate and we have a finite point of the tetrahedron variety.

We can see (i) as a consequence of the fact that we have one more variable than we have equations, and provide an algebraic geometry proof of this. We can actually show something stronger, that the variety defined by the tilde equations is smooth at $p$. We do this by showing that the derivative matrix of the tilde equations with respect to the variables has full rank at $p$ and using (for instance) Theorem 5.1 of [6]. This uses our knowledge about the angle variable solutions again, and involves walking through the various sections of the torus bundle, row reducing the matrix as we go.

(ii) comes from the fact that (at least sufficiently near to $p$) there are only finitely many solutions with $\zeta = 0$. We show this by considering the steps we took to find the solution $p$, and show that we only ever had finitely many choices at each step (there are a finite number of choices for each fan of angle variables, so sufficiently close to $p$ we would have to choose the exact same parameters as for $p$ itself, then there are a finite number of choices of sign).

We obtain an ideal point of the tetrahedron variety, with non-trivial boundary slope, which therefore corresponds to an ideal point of the character variety.

2 The Canonical Tetrahedralisation of a Torus Bundle

We use the tetrahedralisation $\mathcal{H}$, sometimes called the Floyd-Hatcher or monodromy tetrahedralisation. It first (?) appears in Floyd-Hatcher [4], based on an idea from [11]. Guéritaud [5] gives an excellent exposition. Figure 1 shows a picture of the tetrahedralisation as seen from the torus boundary. $\mathcal{H}$ consists of a stack of tetrahedra, one on top of the next. Each (ideal) tetrahedron has four vertices at infinity, and we have truncated each tetrahedron at each of its four vertices to produce four triangles on the boundary torus. We can see the four triangles in the layers marked $t$ or $v$. The vertices of the resulting triangulation of the boundary torus are shown on the diagram with circles around them, labelled $\lambda_k$ or $\rho_k$. There are also special vertices,
Figure 1: Canonical tetrahedrization of a Torus Bundle.
labelled $\lambda_*$ and $\rho_*$ which have been stretched out on this diagram for clarity. We are to imagine collapsing these "long" vertices down to points. Doing this will also change all of the apparently 4 sided polygons in the diagram into triangles, as expected for the truncated ends of a tetrahedron. The edges of those triangles do not quite meet at the vertices in order to highlight which tetrahedron a boundary triangle comes from. A layer of triangles which is "connected" through the vertices are all the truncated boundary of the same tetrahedron.

The shape of an ideal tetrahedron in $\mathbb{H}^3$ is specified by one of its dihedral angles, together with a scaling factor across that angle. This information is encoded as a single complex number ("complex angle") assigned to one of the dihedral angles in the tetrahedron (See [11]). This shows up on the torus boundary as a complex angle at one of the three corners of each triangle. If we name one angle $z$, then moving clockwise around the triangle, the other two angles are $\frac{z-1}{z}$ and $\frac{1}{1-z}$. We use the convention of choosing the uppermost dihedral angle in the tetrahedron to refer to with a single variable, where by "uppermost" we mean in relation to the tetrahedralisation $\mathcal{H}$ of the torus bundle. It turns out that opposite edges of an ideal tetrahedron have the same complex angle, and so the bottommost dihedral angle is the same as the uppermost.

On the left side of the diagram we see the boundaries between tetrahedron "layers", labelled with either $L$ or $R$. These are the $L$ and $R$ from the decomposition of the monodromy $\phi$ into the generators. A tetrahedron that lies between an $L$ and an $R$ is called a hinge tetrahedron, and we use the variables $\ell$ and $v$ to describe the uppermost angle of those tetrahedra. All other tetrahedra are part of fans or tetrahedra, and we use the variables $x_i$ and $z_j$ to refer to the uppermost angles of those tetrahedra (when the "long" vertices are collapsed, the torus boundary picture of such a sequence of tetrahedra looks like a fan). It has been observed that fans of tetrahedra seem to act very much as a unit, and one of the themes of this paper is to make explicit some aspects of this notion.

3 Various Forms of Surfaces

As mentioned before we will be deforming surfaces that begin in Yoshida form into Floyd-Hatcher form in order to show that the ideal points the Yoshida form surfaces come from cover all of the incompressible surfaces we claim they do. In the following two subsections we will describe these two forms of surface, and convert them to our own format, with which we will perform the deformations. We first describe this format:

We alter our tetrahedralisation of $M$ slightly (but continue to refer to it as $\mathcal{H}$): We take a small neighbourhood of the 1-skeleton of the tetrahedralisation which is a product neighbourhood around each edge. Call this neighbourhood $\mathcal{N} = \bigcup \mathcal{N}_e$, where $\mathcal{N}_e$ is the cylindrical product neighbourhood of the edge $e$.
Then the manifold with boundary, $M$, is the union of the edge neighbourhoods $N_e$ with the union of the tetrahedra of $\mathcal{H}$, minus neighbourhoods of its 6 edges. We will refer to such a tetrahedron minus neighbourhoods of its edges as $T_z$, if the complex angle corresponding to that tetrahedron is $z$.

**Definition 3.1.** Our surfaces will be made from three kinds of piece:

- A twisted square that sits inside a $T_z$, with its four edges on four of the six "edge neighbourhood" boundaries.
- A triangle that sits inside a $T_z$, parallel and close to one of the 4 faces of the original tetrahedron, with its three edges on the edge neighbourhood boundaries that bound the face of the original tetrahedron.
- A long thin strip that sits inside an $N_e$ and respects its product structure.

The thin strips serve to glue together the twisted squares and triangles near to edges of the tetrahedralisation. All three types of surface in fact have boundary on the boundary torus of the punctured torus bundle, and so strictly speaking the twisted square is an octagon (it has an edge across the torus boundary at each "corner" of the twisted square) and the triangle is a hexagon (also has an edge at each "corner"). The strip has 4 edges: two long edges parallel to the $e$ of the $N_e$ in which the strip lies, and two short edges on the boundary torus. See Figures 5 and 6 for some pictures of these pieces of surface in a tetrahedron.

### 3.1 Incompressible Surfaces in Floyd-Hatcher Form

Floyd-Hatcher [4] classify the connected, orientable, incompressible, $\partial$-incompressible surfaces in a torus bundle (excluding the boundary torus itself and the fiber) by edge paths $\gamma$ in the Farey graph diagram (see Figure 2) of $PSL_2(\mathbb{C})$ which are invariant by the monodromy $\phi$ and minimal, in the sense that no two successive edges of $\gamma$ lie in the same triangle. See Floyd-Hatcher [4], Theorem 1.1. The minimality condition implies that $\gamma$ is in fact constrained to lie on a strip (the "Farey strip"), and cannot divide a fan of the torus bundle in two. I.e. $\gamma$ can travel along either side of the strip, or cross from one side to the other at the border between fans.

The vertices of the Farey graph can be viewed as the rational numbers $\frac{a}{b}$, together with $\frac{1}{0}$. Two vertices $\frac{a}{b}$ and $\frac{c}{d}$ are joined by an edge if $ad - bc = \pm 1$. Putting aside the incompressible surface for a moment, we can see how to read off the tetrahedralisation of the punctured torus bundle from the monodromy $\phi$ using the Farey graph.

We begin with the punctured torus bundle seen as a cube $[0, 1] \times [0, 1] \times [0, 1]$, minus its vertical edges and with some identifications. We fix a reference basis for the torus taken from the cube edges. We identify the front face with the
back, and the left face with the right by translation to obtain \((T^2 \setminus \{0\}) \times [0,1]\).
Then identify the bottom with the top, after applying \(\phi\) (seen as a linear transformation preserving \(\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}\)) to the bottom face before gluing.

As we build \(\phi\) from \(Ls\) and \(R\)'s, we can build the punctured torus bundle as a stack of \((T^2 \setminus \{0\}) \times [0,1]\)'s, one for each \(L\) or \(R\). Let \(\phi_k\) be the \(k\)th generator (either \(L\) or \(R\)) in the decomposition of \(\phi\), where we count from the bottom of the stack, and \(\phi = \phi_1 \phi_2 \ldots \phi_N\) (acting on vectors to its right, as usual). Then the basis vectors for the punctured torus, \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\) at the bottom of the stack map up to the \(k\)th level (boundary between blocks) by \(\phi_1 \phi_2 \ldots \phi_k\). Thus we obtain a list of vectors (and so slopes: \(\frac{a}{b}\) corresponds to \(\begin{pmatrix} a \\ b \end{pmatrix}\)). At each level of the stack we have two vectors, the image of the basis vectors at the bottom of the stack under \(\phi_1 \phi_2 \ldots \phi_k\). The corresponding slopes are the points on the boundary of our Farey strip.

We can also see the tetrahedralisation \(\mathcal{H}\) of the torus bundle as a stack of tetrahedra. The six edges of each tetrahedron in the stack have one of four slopes: the bottom and top edges of the tetrahedron have their own slopes, for the other four "middle" edges, opposite edges will have the same slope. Each pair of neighbouring triangles of the Farey strip corresponds to a tetrahedron. The pair of triangles have 4 vertices, corresponding to these 4 slopes of the tetrahedron, the vertices touching both triangles correspond to the slopes of the
two pairs of opposite edges of the tetrahedron (the "middle" slopes) and the
vertices not touching both triangles correspond to the top and bottom edges of
the tetrahedron.

One can retrieve the triangulation of the boundary torus induced by the
tetrahedralisation of the punctured torus bundle from the Farey strip. Simply
take the Farey strip and reflect it across one of its two sides. We now have
a "double thickness" strip. Reflect this across one of its sides (equivalent to
taking a translate) to obtain a "quadruple thickness" strip. The strip we obtain
is combinatorially identical to the triangulation of the boundary torus. One can
see this by considering the relationship between the tetrahedra that make up
the tetrahedralisation, and the way they must connect to each other based on
the slopes that their edges have.

We can map the picture of the torus bundle as a stack of tetrahedra onto the
picture of the torus bundle as a stack of \((T^2 \setminus \{0\}) \times [0,1]\) blocks. We require
that each edge of a tetrahedron is contained within the first \(T^2 \setminus \{0\}\) at the
boundary between \((T^2 \setminus \{0\}) \times [0,1]\) blocks that has the correct slope, so we
have to allow "stretching out" of the vertices vertically. Even better, if we cut
out a cylindrical neighbourhood around the puncture \(\times [0,1]\) and truncate the
tetrahedra appropriately we can see them as in Figure 3. We show two copies
of a \((T^2 \setminus \{0\}) \times [0,1]\) block for clarity. This diagram actually contains three
\((T^2 \setminus \{0\}) \times [0,1]\) layers across which we make our \(L\) or \(R\) moves, resulting in
the four different slopes.

![Tetrahedron with slopes \(\frac{1}{6}, \frac{1}{7}, \frac{1}{1}\) and \(\frac{1}{2}\)](image)

Figure 3: Tetrahedron with slopes \(\frac{1}{6}, \frac{1}{7}, \frac{1}{1}\) and \(\frac{1}{2}\)

Back to the incompressible surfaces: Floyd-Hatcher index the non-fiber in-
compressible surfaces by edge paths \(\gamma\) in the Farey strip which are invariant by
\(\phi\) and minimal, in the sense that no two successive edges of \(\gamma\) lie in the same
triangle.

To construct a surface from such a path, we glue together a number of
saddles vertically through the stack. See Figure 4. The saddle is embedded in a cube which, after removing the vertical edges and gluing front with back and left with right, we view as \((T^2 \setminus \{0\}) \times [0, 1]\). We require one such saddle for each edge of the path \(\gamma\). Such an edge joins two vertices of the Farey graph, say \(\frac{a}{b}\) with \(\frac{a+1}{b+1}\). We transform this saddle by applying in each level the linear transformation:

\[
\begin{pmatrix}
  b_i & b_{i+1} \\
  a_i & a_{i+1}
\end{pmatrix}
\]

This has the effect of sending the bottom edges to slope \(\frac{a}{b}\) and the top edges to \(\frac{a+1}{b+1}\). We insert this block into our stack of \((T^2 \setminus \{0\}) \times [0, 1]\) blocks by putting the bottom of the saddle block at the level at which the slope \(\frac{a}{b}\) first appears, and the top of the saddle block at the level at which the slope \(\frac{a+1}{b+1}\) first appears.

We can now see where these surfaces lie with respect to the tetrahedralisation \(\mathcal{H}\). There are two cases, depending on if the edge \(e\) of \(\gamma\) crosses the strip or not:

- If the edge does cross the strip then there is a pair of neighbouring triangles of the Farey strip, with \(e\) as the shared edge. The pair of neighbouring triangles corresponds to a tetrahedron, and the surface has boundary on the four middle edges of the tetrahedron consisting of two pairs with the same slope within each pair. Thus this saddle section of the surface lies as a twisted square in that tetrahedron, seperating the top edge from the bottom. In Figure 3, the surface would have boundary on the \(\frac{0}{1}\) and \(\frac{1}{1}\) edges. The saddle connects through to other parts of the surface heading downwards through (the lower down) \(N_{\frac{1}{1}}\) and upwards through (the higher up) \(N_{\frac{1}{1}}\).

- The case of an edge \(e\) that does not cross the strip is a little more complex. As in the previous case we look for tetrahedra which have edges of the slopes which are the boundary of the saddle surface for \(e\). There is only one triangle of the Farey strip with \(e\) as a boundary, and so two pairs
of neighbouring triangles which touch $e$. If we look at the upper of the two pairs of neighbouring triangles, then $e$ joins the bottom slope of the tetrahedron with one of the two middle slopes. We can now see where this saddle is in Figure 3 if it joins the bottom slope, $\frac{1}{3}$ to $\frac{1}{2}$: the surface consists of the two lower faces of the tetrahedron, which we push inside the tetrahedron slightly. These two triangles connect to each other through the remaining middle slope edge ($N_4$ in Figure 3) to form the saddle, and connect downwards through the bottom edge ($N_4$) and upwards through the first middle slope edge ($N_4$). We could of course have looked at the position of this saddle on the lower of the two tetrahedra, in which case the saddle would be formed from the upper two faces.

The curve $\gamma$ cannot "split apart" a fan of tetrahedra due to its minimality requirement. Thus whenever we have an edge that does not cross the strip, we will have to continue along the side of the entire fan before having the choice to cross the strip instead. The surface section we get from going along the entire side of a fan consists of the boundary triangles between each pair of neighbouring tetrahedra in the fan, as well as the boundary triangles between the tetrahedra at the ends of the fan and the hinge tetrahedra next to them.

Note that the pieces we now have are types of piece allowed by Definition 3.1.

The last step in the construction of a Floyd-Hatcher incompressible surface is to check if the surface constructed so far is orientable or not. If it is not orientable then it is replaced with the boundary of a small tubular neighbourhood of the original. This has the effect of doubling the number of parallel surfaces in each tetrahedron. In all that follows, the fact that we may actually be manipulating two parallel copies of each piece of surface will not change any of the arguments, and so we will rarely refer to this issue.

### 3.2 Surfaces in Yoshida Form

Yoshida constructs a surface from information about the rates and ways in which tetrahedra in the triangulation are degenerating as we approach an ideal point. As a tetrahedron degenerates the three pairs of dihedral complex angles converge to 0, $\infty$ and 1. We put twisted squares in each tetrahedron that is degenerating such that it has boundary on the four $N_e$, for $e$ an edge whose dihedral angle is converging to 0 or $\infty$. The curve segment boundaries of a twisted square on the boundary torus are four arcs inside the four triangular truncated ends of the $T_z$ containing the twisted square. The twisted square joins 0 with $\infty$ edges inside the $T_z$, and so the arcs join 0 with $\infty$ vertices of the triangles. We orient these arcs within each triangle to point from the $\infty$ vertex to the 0 vertex. The relative rates of collapse to tell us how many parallel twisted squares to put in each tetrahedron. We connect two twisted squares to each other through an $N_e$ so as to connect an edge coming from a 0 dihedral
angle to one coming from a $\infty$ dihedral angle. Yoshida proves that this is always possible (i.e. that there are the same number of each around an edge). There may still be some choice in how the edges are connected to each other by these long strips through $N_e$, but this ambiguity is not important to the construction, or the proof that the surface obtained, although not necessarily incompressible itself, can be converted by a sequence of ambient 2-surgeries and removals of sphere components to an incompressible surface.

Again, the pieces used are allowed by Definition 3.1 (we only use twisted squares and long thin strips).

### 3.3 Torus Boundary Diagrams of the Surfaces

Working with surfaces inside of tetrahedra is difficult and time consuming. Fortunately however, all of the information encoded by a surface in the form given by Definition 3.1 can be read off from the pattern of the boundary of the pieces of surface on the boundary torus of the punctured torus bundle. We analyse this in Figures 5 and 6.
Figure 5: Twisted squares of type 1 and $\infty$ in a tetrahedron and the corresponding picture on the boundary torus.
Figure 6: Twisted square of type 0 and a triangle in a tetrahedron and the corresponding picture on the boundary torus.
In the figures, we see tetrahedra viewed from above, looking down the torus bundle. There are three ways to put a twisted square in a tetrahedron, named types 1, \( \infty \) and 0, with reference to the complex angle at the top and bottom edges (with respect to the torus bundle) of the tetrahedron. We also show a triangle piece, parallel to one of the upper faces of a tetrahedron. To the right we see the patterns formed on the boundary torus. The orientation on the boundary curves is Yoshida's orientation, which within a triangle on the boundary torus, points from the \( \infty \) corner to the 0 corner. This orientation may or may not agree with the orientation induced from the orientation on the triangle or twisted square. The labels "w" and "a" are to be read as "with the induced orientation" or "against the induced orientation". We show two parallel copies of each surface to demonstrate how the ordering of various copies translates to the boundary picture.

The curves on the boundary torus pictures will connect to curves on the boundary of neighbouring tetrahedra, passing through neighbourhoods of the vertex (corresponding to \( N_v \)). Within these neighbourhoods of the vertices we see the boundary edges of the long thin strips. In order to respect the product structure on each \( N_v \), we require that the way in which the curves connect to each other through a vertex is consistent with the way in which curves connect at the other end of the edge passing through the manifold. The vertex at the other end of an edge through the manifold can be found by moving two steps along the boundary torus picture to the right or left, and consistency requires that the picture near one vertex be the mirror image of the picture near the vertex at the other end of its edge. The axis of the reflection is roughly vertical in the torus boundary diagrams.

Given a surface in the form described by Definition 3.1, we can tell if a surface is orientable by looking at the boundary picture: The curve components on the boundary must be consistently oriented according to the induced orientation from a choice of orientation on each twisted square and triangle. A Yoshida form surface only contains twisted squares, and the curve components are already each oriented with Yoshida's orientation. Showing that such a surface is orientable amounts to showing that half of the curve segment orientations can be reversed, in the ways allowed looking at the diagram, and still having the curve components be oriented.

### 4 Translating Between the Two Forms of Surface

There are a total of ten types of block that we break a Floyd-Hatcher edge path \( \gamma \) into. The path can travel along the left edge of the strip, or the right, and can cross from one side of the strip to the other, in the following ways, which we denote by \( R^S_L, R^R_L, R^S_R, S^L_L, S^R_R, L^R_L, L^R_R, L^S_L, L^S_R \) and \( L^S_S \). (See Figure 7).

The notation is chosen based around the vertex in each diagram. The main symbol is \( L \) if \( \gamma \) passes through the vertex and the vertex is on the left side, \( R \) if
it is on the right side, and $S$ ("straight") if $\gamma$ does not pass through the vertex. The superscript and subscripts denote which other kinds of block can go above and below (respectively) this one. For example, $R_S^S$ can go above $L^R$, because the superscript on the lower matches the main symbol of the upper, and the subscript on the upper matches the main symbol on the lower. By gluing these "Lego blocks" together, we can make any valid $\gamma$.

Floyd and Hatcher give a translation from these parts of the curve on the Farey graph into surfaces within the torus bundle. At the end of section 3.1 we described those parts of surface: For edges of $\gamma$ that cross the strip we get a single type 1 twisted square inside the hinge tetrahedron given by the two triangles on the Farey strip that meet at the crossing edge. For edges that travel up either side of the strip we get triangle surface pieces on the boundary between each pair of neighbouring tetrahedra in the fan. The Yoshida forms of these blocks are considerably more convoluted. For one of the two crossing cases (i.e. the path $\gamma$ crosses from the right side of the strip to the left, or vice versa) nothing changes, for the other some extra surface parts need to be introduced in tetrahedra next to the hinge tetrahedra, due to an issue of orientation. For the blocks in which $\gamma$ travels up on side of the strip it seems that a great amount of extra "scrunching up" happens. The surface parts that in Floyd-Hatcher form are evenly spread throughout the fan are increasingly bunched up to one side of the fan. In the diagrams that follow we see in fact the number of twisted squares in each tetrahedra "ramping up" by two each time. This corresponds to
the tetrahedra degenerating faster and faster as we look along the fan. It isn't intuitively clear why this needs to happen.

We now give translations of each of the blocks into Yoshida form in the following torus boundary diagrams. To the right of each diagram we label the type and number of each twisted square in each tetrahedron. One can check that the picture near one vertex of these diagrams is the mirror image of the picture near the vertex at the other end of its edge (as required for the strip pieces to respect the product structure of $N_s$. The orientations on the curves are the Yoshida orientation, going from $\infty$ corners to 0 corners of each triangle, and so are necessarily identical for parallel curves going through a triangle.

There are often many parallel curve segments (coming from boundaries of the twisted squares) going through the same region on the boundary torus, and so we draw this as a single curve labelled with a number. There are the same number of curve segments going through each of the four triangular truncated ends of each tetrahedron, and the number of such segments entering a junction is equal to the number exiting, so one can quickly work out the number of parallel segments when an edge is not labelled.

In section 5 we show that these surfaces in Yoshida form are indeed equivalent to the Floyd-Hatcher forms. By "equivalent" we mean that after some isotopies and deletion of sphere components of the Yoshida form we obtain the Floyd-Hatcher form. Yoshida also allows ambient 2-surgeries, but we shall not need these in our construction.

As noted at the end of section 3.1, we may need to double up the number of surface pieces in each tetrahedron, depending on whether the complete surface, after all blocks are joined together, is orientable.

Note that these diagrams and the calculations coming from them assume there are greater than or equal to 3 tetrahedra in each fan (not counting the hinge tetrahedra). The arguments should go through for cases involving most smaller fans, although there are some interesting anomalies that deserve future work.
Figure 8: Torus Boundary pictures for Yoshida form surfaces in blocks $R^S_S$ and $R^S_L$
Figure 9: Torus Boundary pictures for Yoshida form surfaces in blocks $R^L_L$ and $R^L_S$
Figure 10: Torus Boundary pictures for Yoshida form surfaces in blocks $S_L^T$ and $S_R^R$
Figure 11: Torus Boundary pictures for Yoshida form surfaces in blocks $L^R_S$ and $L^R_R$
Figure 12: Torus Boundary pictures for Yoshida form surfaces in blocks $L^S_R$ and $L^S_S$
5 Converting Yoshida form surfaces to Floyd-Hatcher form surfaces

We will employ a number of moves to alter our surfaces. Some of these moves are illustrated in Figures 13 through 15.

Definition 5.1. The allowed moves are of the following types:

- Move 1: A twisted square in a tetrahedron may be pushed over to become two triangles parallel to the faces of the tetrahedron in one of two ways. For example, if the square is horizontal in the tetrahedron (has its boundary on the four middle slope edges) then it may either be pushed up to become the upper two triangle faces of the tetrahedron, or down to become the lower two faces. This move can of course be reversed, and two triangles, if connected to each other in the correct way, can be "flipped" through the center of a tetrahedron to become the other two triangle faces of the tetrahedron by moving first to the twisted square, then continuing to push.

- Move 2: A triangle parallel to a face of a tetrahedron may be pushed through the tetrahedron face into the neighbouring tetrahedron, to become a triangle within and parallel to the face of the second tetrahedron. This move may create or remove strips through the $N_e$ and does so in the obvious way.

- Move 3: Boundary bigon removal: if we have a bigon curve component on the boundary torus, where the two surface pieces the bigon is a boundary of are both triangles, necessarily each parallel to the shared face between neighbouring tetrahedra, then we may cap off the bigon, push it inside the tetrahedron, and deform away the cap and two triangles to leave only a strip, as detailed in Figure 15. This move is often preceded by a move or moves of type 1, to convert twisted squares into two triangles each, a pair of such triangles can then be removed using a move of type 3.

- Move 4: Local deformations of the positions of strips within a $N_e$ are often used for clarity.

- Move 5: Removal of sphere components: The sphere components we will see will be of the form of a cylinder of strips within a $N_e$, surrounding the edge $e$. They have two trivial circles on the boundary torus, which we cap off to form the sphere, which we then delete.

Notation: $R^*_L$ denotes either $R^I_L$ or $R^*_L$, when it doesn’t matter which. We use analogous notation for other cases also. We break the path $\gamma$ into its edges on the Farey strip. There are four cases to consider:

- $R^*_L L^R_*$ An edge of $\gamma$ crosses from the left to the right side of the strip.
- $L^*_R R^L_*$ An edge of $\gamma$ crosses from the right to the left side of the strip.
Figure 13: Examples of type 1 moves. Above is the picture for twisted squares of type 1 (horizontal with respect to the torus bundle), below is what happens for the other types of twisted squares.
Figure 14: An example of a type 2 move.

Figure 15: An example of a type 3 move. A good heuristic for seeing what this move does to the boundary picture is to imagine removing the bigon by pulling its two corners into the middle of its edge. As we do this, we also pull the corresponding corners at the other end of the edges through the manifold inwards, through to the opposite side of that boundary triangle edge.

- $L^*_S S^L_L L^*_S$ A sequence of edges of $\gamma$ travelling up the left side of the strip.
- $R^*_S S^R_R R^*_S$ A sequence of edges of $\gamma$ travelling up the right side of the strip.
In the first case, \( R_L^* L_R^* \), nothing need be done. The single twisted square in Yoshida form is already in the hinge tetrahedron, the correct place for the Floyd-Hatcher form. The second case is the first that needs any work. Figure 16 shows the sequence of moves.

We draw arrows to show how each move is being used. From the first picture to the second, we apply move 1 seven times. From the second to the third we show one use of move 3, then from the third to the fourth we do the rest of the move 3s. From the fourth picture to the fifth we remove a sphere component by move 5 and use move 2 twice to push two triangles into the hinge tetrahedron. We also use a move 4 to push the strip between those triangles up onto the hinge tetrahedron side of the edge in whose neighbourhood it lies. Finally, from the fifth picture to the sixth we use move 1 once more, to push two triangles inwards, to form a twisted square.

The third and fourth cases are considerably more complicated. We need only do one of them however, since (ignoring the Yoshida orientation arrows) reflecting the diagram for \( L^*_S S^L_r L^*_S \) across a horizontal line gives us the diagram for \( R^*_S S^L_r R^*_S \). The moves we employ do not care which way up the diagram is. We demonstrate the sequence of moves in the \( R^*_S S^L_r R^*_S \) case, starting with preliminary moves in Figure 17.

Some notes on the moves in these diagrams: In Figure 17, in changing diagram C into diagram D we first do move 1 on two strands in the hinge tetrahedron, then use move 3 to remove the bigon. In Figure 19, when changing diagram K into diagram L we use move 4 (local deformation of the strips around an edge through the manifold) on the edge we see the two ends of, just above the hinge tetrahedron.

Now we see that the situation in the first tetrahedron above the hinge in diagram G is the same as the situation in the second tetrahedron above the hinge in diagram L. In fact we can see the same relationship in diagrams F and K, E and J, and D and I. In any case, we can repeat the five moves that take us from G to L and work our way up the stack of tetrahedra, removing a series of concentric trivial loops on the right side of the pictures. There are \( m \) tetrahedra in the fan (between hinge tetrahedra) and every time we move up one tetrahedron we remove two loops. Looking at diagram A, we start with a total of \( m + 1 \) loops, so we will be able to continue this procedure until we run out of loops, half way up the fan. Exactly how this will happen depends on the parity of \( m \). If \( m + 1 \) is even (i.e. the fan contains an even number of tetrahedra), then we will reach a situation analogous to as in diagram I, but where the curve labelled \( m - 3 \) is missing, as that value is zero. If \( m + 1 \) is odd, we have a very slightly different procedure to remove the last loop. This is illustrated in Figure 20, in which diagram M shows the state in diagram K, but where "\( m - 3 \)" is 1. In diagram P we see the position after we have eliminated all trivial loops. Note that the 1's appear on the boundary between the middle two tetrahedra of the fan.
Figure 16: Moves to convert a $L^*_R R^*_L$ join from Yoshida form to Floyd-Hatcher form.
Figure 17: Converting a $R^*_S S^*_R R^*_S$ join from Yoshida form to Floyd-Hatcher form.
Figure 18: Converting a $R_S^e S^e R_S^e$ join from Yoshida form to Floyd-Hatcher form.
Figure 19: Converting a $R^*_S S^R_H R^*_S$ join from Yoshida form to Floyd-Hatcher form.
Figure 20: Converting a $RS_S R_S R_S$ join from Yoshida form to Floyd-Hatcher form when $m$ is even.
We now work backwards from the Floyd-Hatcher form to this stage. As discussed at the end of section 3.1 we get the triangles between each pair of neighbouring tetrahedra in the fan, plus the triangles between each tetrahedron on the end of the fan and the hinge tetrahedron next to it. We neglect which of the two tetrahedra a triangle between them belongs to and write only the number of triangular surface pieces are near the triangular boundary between two tetrahedra. In Figure 21 we see the Floyd-Hatcher surface in this form, a demonstration of the move we will apply to it, and the pictures we have reached thus far, for \( m \) odd and even.

The move is of type 1. First we apply the move to all of the tetrahedra in the fan. This only changes the numbers in the top and bottom layers, as in all other layers whenever a triangle is lost, another is gained. However, when we again apply the move everywhere we can (in all tetrahedra of the fan other than the two outermost) the same happens, and now the outermost 2 layers have the correct numbers. We continue applying the move to all layers we can, until eventually, depending on the parity, we reach one of the two results of our previous isotopies. Running this sequence of moves backwards completes the conversion of the surface in a \( R_3 S^1_H R_3^N \) join from Yoshida form to Floyd-Hatcher form.

We have shown that each section of the surface can be individually converted from Yoshida form to Floyd-Hatcher form, so the surface as a whole is equivalent to the corresponding surface in Floyd-Hatcher form. None of the preceding arguments are changed if we needed to double up each surface piece (if the complete surface would otherwise be non-orientable). By their result, Theorem 1.1 of [4], we can construct in this way all incompressible surfaces in the torus bundle, other than the fiber \( T^2 \setminus \{0\} \) and the peripheral torus.

We now need to show that the surfaces we have constructed correspond to ideal points.
Figure 21: Converting a $R^*_S S^*_R R^*_S$ join from Yoshida form to Floyd-Hatcher form.
6 Tilde Equations and Solutions at $p$

6.1 Preliminaries

These are the gluing equations for an $L^{m+1}R^{n+1}$ section (one can check these by looking at Figure 1):

\[
\begin{align*}
\lambda_n : \quad & \hat{z}_n \left( \frac{1}{1-\rho} \right)^2 \left( \frac{1}{1-\tau} \right)^2 \hat{z}_1 = 1 \\
\lambda_1 : \quad & \left( \frac{1}{1-\rho} \right)^2 \hat{z}_2 = 1 \\
\lambda_2 : \quad & z_1 \left( \frac{1}{1-\tau} \right)^2 z_3 = 1 \\
\lambda_3 : \quad & z_2 \left( \frac{1}{1-\tau} \right)^2 z_4 = 1 \\
& \quad \vdots \\
\lambda_{n-1} : \quad & z_{n-2} \left( \frac{1}{1-\tau} \right)^2 z_n = 1 \\
\lambda_n : \quad & z_{n-1} \left( \frac{1}{1-\tau} \right)^2 \hat{v} = 1 \\
\rho_1 : \quad & \hat{v} \left( \frac{x_1-1}{x_1} \right)^2 x_2 = 1 \\
\rho_2 : \quad & x_1 \left( \frac{x_2-1}{x_2} \right)^2 x_3 = 1 \\
\rho_3 : \quad & x_2 \left( \frac{x_3-1}{x_3} \right)^2 x_4 = 1 \\
& \quad \vdots \\
\rho_{m-1} : \quad & x_{m-2} \left( \frac{x_{m-1}-1}{x_{m-1}} \right)^2 x_m = 1 \\
\rho_m : \quad & x_{m-1} \left( \frac{x_m-1}{x_m} \right)^2 \hat{t} = 1 \\
\rho_\ast : \quad & x_m \left( \frac{1}{1-\tau} \right)^2 \left( \prod_{j=1}^{n} \left( \frac{z_j}{1-z_j} \right)^2 \right) \left( \frac{\hat{v}-1}{\hat{v}} \right)^2 \hat{v}_1 = 1
\end{align*}
\]

The whole torus bundle may contain many such sections, with different numbers of tetrahedra in each fan (so different values of $m$ and $n$). Since we are breaking down the problem into sections of the torus bundle, it is notationally convenient to not specify which $L^{m+1}R^{n+1}$ block a particular variable or gluing equation is from. We use notation such as $\hat{z}_n$ to denote a variable in the next section above the one currently in focus (and in this case the "n" above is denoted $\hat{n}$), or $\hat{v}$ for variables below, and generally use such symbol accents whenever they are needed for clarity.

We know from section 5 that the surfaces given in figures 8 through 12 are indeed equivalent to our original Floyd-Hatcher surfaces. We now consider the algebraic information obtained about the degeneration from the positions of the surfaces. Again running Yoshida’s construction backwards, we know that the orientation of twisted squares in a tetrahedron (if there are any) tells us how it is degenerating (which angle is supposedly converging to 0, which to $\infty$ and which to 1). The number of twisted squares is supposed to tell us the relative rates of degeneration. With these facts in mind, we make a number of changes of variables. We will follow the manipulations with the example $R^n_S$, then quickly do the same with the other 9 cases.

First we change variables (replacing lower case with upper case) so that in each tetrahedron that is degenerating, we use the angle that is converging to 0, rather than to $\infty$ or 1. So, for example, we replace $z_1$ (which is converging to 1 at rate $m+1$) with $Z_1 = \frac{z_1-1}{1-z_1}$ (and so $z_1 = \frac{1}{1-Z_1}$). We do not change variables corresponding to tetrahedra that are not degenerating. It turns out that for $R^n_S$, we will be interested in the gluing equations $\rho_\ast, \lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda_\ast$ (see the diagrams for $R^n_S$ and, for what happens with $\lambda_\ast$ below, $S^n_L$). After making these changes, those gluing equations look like this:
\[ \rho'_0 = \hat{X}_m(\frac{T-1}{T})^2Z_1^2(\prod_{j=2}^{n}(z_j - 1)^2)(\frac{z_2 - 1}{z_2})^2X_1 = 1 \]
\[ \lambda'_0 = T(\frac{z_2 - 1}{z_2})^2z_2 = 1 \]
\[ \lambda'_2 = \frac{1}{1-z_1}(\frac{1}{1-z_2})^2z_3 = 1 \]
\[ \lambda'_3 = z_2(\frac{1}{1-z_3})^2z_4 = 1 \]
\[ \vdots \]
\[ \lambda'_{n-1} : z_{n-2}(\frac{1}{1-z_{n-1}})^2z_n = 1 \]
\[ \lambda'_n : z_{n-1}(\frac{1}{1-z_n})^2v = 1 \]
\[ \lambda'_n : z_n(\frac{1}{1-z_n})^2(\prod_{j=1}^{n}(1 - X_j)^2)(\frac{1}{1-T})^2(\frac{1}{1-Z_n}) = 1 \]

A shortcut for seeing what the gluing equations will look like in this form is to notice the following: Because the orientation of the boundary curve within one triangle is such that the curve goes from the \( \infty \) vertex to the 0 vertex, we need only look at the directions of the arrows on parts of the curve touching that vertex. If a curve path enters a vertex of the torus bundle boundary torus from a given triangle (with variable \( z \), say), then that angle of the triangle must converge to 0, and so the relevant term in the gluing equation for that vertex must be \( Z \). If the curve leaves that vertex, then the angle must converge to \( \infty \), and the relevant term must be \( \frac{z}{z_2} \). Lastly, if the curve does not enter or exit the vertex, but goes between the other two corners of the triangle, then the angle must converge to 1, and the relevant term is \( \frac{1}{1-z} \).

Next, we convert the equations into polynomial equations, by multiplying up by denominators and moving all terms to the left side:

\[ \rho'' : \hat{X}_m(\hat{T} - 1)^2Z_1^2(\prod_{j=2}^{n}(z_j - 1)^2)(\frac{v - 1}{v})2X_1 - \hat{T}^2(\prod_{j=2}^{n}z_j^2)v^2 = 0 \]
\[ \lambda''_0 = \hat{T}(\hat{T} - 1)^2z_2 - Z_1^2 = 0 \]
\[ \lambda''_2 = z_3 - (1 - Z_1)(1 - z_2)^2 = 0 \]
\[ \lambda''_3 = z_2z_4 - (1 - z_3)^2 = 0 \]
\[ \vdots \]
\[ \lambda''_{n-1} : z_{n-2}z_n - (1 - z_{n-1})^2 = 0 \]
\[ \lambda''_n : z_{n-1}v - (1 - z_n)^2 = 0 \]
\[ \lambda''_n : z_n - (1 - v)^2(\prod_{j=1}^{n}(1 - X_j)^2)(\hat{T} - 1)^2(1 - Z_n) = 0 \]

Now we introduce a new variable, \( \zeta \), which will be the parameter which converges to 0, and to which all other rates of convergence are relative to. If the variable \( Z \) is (according to the number of twisted squares in the corresponding tetrahedron) supposed to converge to 0 at rate \( k \) (i.e. there are \( k \) parallel copies of the twisted square in that tetrahedron), then we set \( Z = \zeta^k \). In general we follow the same procedure for all variables, replacing the upper case letter with the lower case of the alphabetically previous letter. This is a kind of "blow up" (the algebraic geometry kind). The idea is to remove the singularity at our point by replacing variables with the "directions" (\( y \) above) near the point.
If we had to double up the surfaces due to non-orientability, then the only effect this has on the equations is to replace $\zeta$ with $\zeta^2$. This change will have no effect on the results of the following calculations.

**Definition 6.1.** **Angle variables** are those which have not been changed in the preceding steps, that are not supposed to be going to 0, $\infty$ or 1 as we approach our supposedly ideal point. **Direction variables** are the replacements for variables that are converging to 0 (e.g. $y$ in the above example).

For reference, $t, x_i, v$ and $z_j$ are angle variables and $s, w_i, u$ and $y_j$ are direction variables.

After these changes, the gluing equations become:

\[
\begin{align*}
\rho_k'' : & \quad \zeta^{2n_k} \hat{\nu}_{\hat{\nu}}(\zeta^{2n_k+2} \hat{s} - 1)^2(\zeta^{n_k+1} y_1)^2(\prod_{j=2}^{n_k} (z_j - 1)^2)(v - 1)^2 \zeta^2 w_1 - (\zeta^{2n_k+2} \hat{s})^2(\prod_{j=2}^{n_k} z_j^2) v^2 = 0 \\
\lambda_1' : & \quad \zeta^{2n_k+2} \hat{s} \left( \zeta^{n_k+1} y_1 - 1 \right) z_2 - y_1^2 = 0 \\
\lambda_2' : & \quad z_3 - (1 - \zeta^{n_k+1} y_1)(1 - z_2)^2 = 0 \\
\lambda_3' : & \quad z_2 z_4 - (1 - z_3)^2 = 0 \\
\vdots & \\
\lambda_{n-1}' : & \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0 \\
\lambda_n' : & \quad z_{n-1} v - (1 - z_n)^2 = 0 \\
\tilde{\lambda}_n : & \quad z_n - (1 - v)^2(\prod_{j=1}^{m} (1 - \zeta^{2j} w_j)^2)(1 - \zeta^{2m+2} \hat{s})^2(1 - \zeta^{m+1} y_1) = 0 \\
\end{align*}
\]

Note that in some of the equations, a power of $\zeta$ factors out. All of the above equations are of the form $A - B = 0$, and in fact, for any vertex of the torus boundary which the curve $\gamma$ passes through, the power of $\zeta$ that factors from $A$ is the number of edges of $\gamma$ entering the vertex, whereas that from $B$ is the number of edges exiting the vertex. These are of course equal. We factor out this power of $\zeta$ and delete it from our equations to obtain:

\[
\begin{align*}
\tilde{\rho}_k : & \quad \hat{\nu}_{\hat{\nu}}(\zeta^{2n_k+2} \hat{s} - 1)^2 y_1^2(\prod_{j=2}^{n_k} (z_j - 1)^2)(v - 1)^2 w_1 - \hat{s}^2(\prod_{j=2}^{n_k} z_j^2) v^2 = 0 \\
\tilde{\lambda}_1 : & \quad \hat{s}(\zeta^{n_k+1} y_1 - 1) z_2 - y_1^2 = 0 \\
\tilde{\lambda}_2 : & \quad z_3 - (1 - \zeta^{n_k+1} y_1)(1 - z_2)^2 = 0 \\
\tilde{\lambda}_3 : & \quad z_2 z_4 - (1 - z_3)^2 = 0 \\
\vdots & \\
\tilde{\lambda}_{n-1} : & \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0 \\
\tilde{\lambda}_n : & \quad z_{n-1} v - (1 - z_n)^2 = 0 \\
\tilde{\lambda}_n : & \quad z_n - (1 - v)^2(\prod_{j=1}^{m} (1 - \zeta^{2j} w_j)^2)(1 - \zeta^{2m+2} \hat{s})^2(1 - \zeta^{m+1} y_1) = 0 \\
\end{align*}
\]

We view these as equations in variables in $\mathbb{C}$. We will calculate similar equations for other blocks, and later normalise by fixing one of the direction variables to be equal to 1. These equations, together with the similar ones for
other blocks, and one equation of the form \(^n\) \((\text{direction variable}) - 1 = 0^n\) together define an affine variety we call \(\tilde{\mathcal{X}}(M, \mathcal{J})\). This variety depends of course on which point we are trying to show is in fact an ideal point of the original tetrahedra variety, \(\mathcal{X}(M, \mathcal{J})\), and which direction variable we will normalise with respect to. Given a point of \(\tilde{\mathcal{X}}(M, \mathcal{J})\) (a solution to the "tilde equations") there is an obvious procedure to try to convert this point back to a point of \(\mathcal{X}(M, \mathcal{J})\). Namely, set \(Z = \zeta^k y\) and so on. Of course, if \(\zeta\) or one of the direction variables were zero, or if \(\zeta^k y\) evaluates to 1, then we will not get a point of \(\tilde{\mathcal{X}}(M, \mathcal{J})\). We also require that no angle variable is 0, \(\infty\) or 1.

First we find a solution with \(\zeta = 0\). Call the point corresponding to this solution \(p\).

We will later show that there are points of \(\tilde{\mathcal{X}}(M, \mathcal{J})\) near \(p\), and in fact the stronger result that \(p\) is a regular point of \(\tilde{\mathcal{X}}(M, \mathcal{J})\). Further we will show that those nearby points correspond to finite solutions of \(\mathcal{X}(M, \mathcal{J})\), and thus that \(p\) is indeed an ideal point of \(\mathcal{X}(M, \mathcal{J})\). Since the slope on the boundary torus of the torus bundle will be non trivial in all of our cases, we will in fact have an ideal point of the character variety, as required.

Setting \(\zeta = 0\), the tilde equations become:

\[
\begin{align*}
\overline{w}_1 : & \quad \bar{w}_m \bar{y}_1^2 (\prod_{j=2}^n (z_j - 1)^2) (v - 1)^2 w_1 - s^2 (\prod_{j=2}^n z_j^2) v^2 = 0 \\
\overline{\lambda}_1 : & \quad \bar{s} z_2 - y_1^2 = 0 \\
\overline{\lambda}_2 : & \quad z_3 - (1 - z_2)^2 = 0 \\
\overline{\lambda}_3 : & \quad z_2 z_4 - (1 - z_3)^2 = 0 \\
\vdots & \quad \vdots \\
\overline{\lambda}_{n-1} : & \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0 \\
\overline{\lambda}_n : & \quad z_{n-1} v - (1 - z_n)^2 = 0 \\
\overline{\lambda}_n : & \quad z_n - (1 - v)^2 = 0
\end{align*}
\]

It should now be clear why we included \(\bar{\lambda}_n\) in our set of equations for \(R^S\).

The equations \(\overline{\lambda}_2\) through \(\overline{\lambda}_n\), then \(\overline{\lambda}_n\) for the angle variables, \(z_2\) through \(z_n\), then \(v\), form a clear pattern. If we imagine two extra variables, one at either end of the list of angle variables, which are set to have value 1, then all of these equations are of the form \(a_{k-1} a_{k+1} - (1 - a_k)^2 = 0\).

**Lemma 6.2.** \(a_k = \frac{1 - \cos \frac{k\beta}{2}}{1 + \cos \beta}\) is a solution of \(a_{k-1} a_{k+1} - (1 - a_k)^2 = 0\)
Proof. Let \( \alpha = \frac{1}{1 - \cos \beta} \) (so \( 1 - \frac{1}{\alpha} = \cos \beta \)).

\[
\begin{align*}
    a_{k-1}a_{k+1} - (1 - a_k)^2 &= \alpha \left( 1 - \cos \left( (k-1)\beta \right) \right) \alpha \left( 1 - \cos \left( (k+1)\beta \right) \right) - (1 - \alpha(1 - \cos k\beta))^2 \\
    &= \alpha^2 \left( (1 - \cos (k\beta - \beta))(1 - \cos (k\beta + \beta)) - \left( \frac{1}{\alpha} - 1 + \cos k\beta \right)^2 \right) \\
    &= \alpha^2 \left( (1 - \cos k\beta \cos \beta + \sin k\beta \sin \beta)(1 - \cos k\beta \cos \beta - \sin k\beta \sin \beta) \right) - (\cos k\beta - \cos \beta)^2 \\
    &= \alpha^2 \left( 1 - 2 \cos k\beta \cos \beta + (\cos k\beta \cos \beta)^2 - (\sin k\beta \sin \beta)^2 - (\cos k\beta - \cos \beta)^2 \right) \\
    &= \alpha^2 \left( 1 - 2 \cos k\beta \cos \beta + (\cos k\beta \cos \beta)^2 - (1 - \cos^2 k\beta)(1 - \cos^2 \beta) - (\cos k\beta - \cos \beta)^2 \right) \\
    &= \alpha^2 \left( 2 \cos k\beta \cos \beta + (\cos^2 k\beta + \cos^2 \beta) - (\cos k\beta - \cos \beta)^2 \right) \\
    &= 0
\end{align*}
\]

In our case, we want solutions with \( a_1 = 1 \) and \( a_{N+1} = 1 \) (the two "extra" variables). The first equation is automatically true for this form of solution, and the second may be satisfied by choosing \( \beta = \frac{2\pi}{N+2} \). There are other possible choices for \( \beta \) that give a solution, but this solution will turn out to be the correct one for later calculations. We note the following feature of such a solution for future reference:

Lemma 6.3. For the solution \( a_k \) as above with \( \beta = \frac{2\pi}{N+2} \), \( \Re \ni a_k > 1 \) for \( 2 \leq k \leq N \).

It is worth noting as an aside that this explicit solution, generalised slightly to \( a_k = \frac{1 - \cos (k\beta + \theta)}{1 - \cos \beta} \) is a solution to these blocks of gluing equations independent of our looking at a degenerate point. These blocks of equations are solved as a unit by specifying \( \beta, \theta \in \mathbb{C} \) (which determine what happens at either end of the fan of tetrahedra) for all points of the tetrahedron variety. It seems likely that this observation could be useful in studying torus bundles as collections of fans in contexts other than this. For instance, it seems likely that this formula should give complex versions of the "concave" sequences of angles Guéritaud [5] finds in fans of a torus bundle.

Independent of the \( a_k \) having this explicit solution, we also note the following fact:

Lemma 6.4. For variables \( a_k \) satisfying the sequence of equations in the above lemma,

\[
\frac{\prod_{k=2}^{N} a_k^2}{\prod_{k=2}^{N} (a_k - 1)^2} = a_{2N}
\]

Proof. This is clear, simply by substituting \( a_{k-1}a_{k+1} \) for \( (a_k - 1)^2 \) in the product.

We record also some similar results for another sequence of equations that will come up:

**Lemma 6.5.** $b_k = \frac{1-\cos \beta}{1-\cos_k \beta}$ is a solution of $b_{k-1}(b_k - 1)^2b_{k+1} - b_k^2 = 0$.

**Proof.** If we set $b_k = \frac{1}{a_k}$ then the equation becomes

$$a_k^2\left(\frac{1}{a_k} - 1\right)^2 - a_{k-1}a_{k+1} = 0$$

which is just

$$(1 - a_k)^2 - a_{k-1}a_{k+1} = 0$$

We do not worry about division by zero as all of the solutions we are interested in are positive. \(\square\)

**Lemma 6.6.** For $b_1 = 1$ and $b_{N+1} = 1$ we may choose $\beta = \frac{2\pi}{N+2}$, then $0 < b_k < 1$ for $2 \leq k \leq N$.

**Lemma 6.7.** For variables $b_k$ satisfying the sequence of equations in the above lemma,

$$\prod_{k=2}^{N} (1 - b_k)^2 = b_2b_N$$

**Proof.** This is clear, simply by substituting $\frac{b_k^2}{b_{k-1}b_{k+1}}$ for $(1 - b_k)^2$ in the product. \(\square\)

We also note the following for future use:

**Lemma 6.8.** The sequence of equations $a_{k-1}a_{k+1} - (1 - a_k)^2 = 0$ with $a_1 = 1$, $a_{N+1} = 1$ and $a_k \in \mathbb{C}$ have only finitely many solutions. The same is true for the $b_k$ equations.

**Proof.** One way to see this is as follows: First note that having chosen a value for $a_2$, and fixing $a_1 = 1$ but leaving $a_{N+1}$ free, all values for $a_k$, even if we extend the sequence in the obvious manner, for $k > N+1$, are fixed. In fact they are rational functions of $a_2$:

$$a_3 = (1 - a_2)^2$$
$$a_4 = \frac{(1 - a_3)^2}{a_2}$$
$$a_5 = \frac{(1 - a_4)^2}{a_3}$$

$$\vdots$$
$$a_k = \frac{(1 - a_{k-1})^2}{a_{k-2}}$$

$$\vdots$$

40
If we ever had to divide by zero in this sequence then we are not at a solution to the original equations. Thus solving the equations in the statement of this lemma is equivalent to finding solutions to $a_{N+1}(a_2) = 1$ (where we view $a_{N+1}$ as a rational function of $a_2$). Multiplying up by the denominator of this rational function we see that we have the number of possible solutions equal to the number of roots of a polynomial. The only way this can be infinite is if the polynomial is identically zero, or equivalently if $a_{N+1}(a_2)$ is identically 1. This is clearly untrue, as from Lemma 6.2, we have the existence of solutions to such sequences of equations with $a_{N+1} \neq 1$.

As we saw in Lemma 6.5, the equations for $b_k$ are essentially the same as those for $a_k$, and a similar argument goes through.

6.2 Block Calculations
6.2.1 $R^S$

\[ \tilde{\rho}_0 : \quad \tilde{w}_m(\zeta^{2n+2}\tilde{s} - 1)^2y_1^2(\prod_{j=2}^n(z_j - 1)^2)(v - 1)^2w_1 - \tilde{s}^2(\prod_{j=2}^n z_j^2)v^2 = 0 \]
\[ \tilde{\lambda}_1 : \quad 2\tilde{s}(\zeta^{m+1}y_1 - 1)^2z_2 - y_1^2 = 0 \]
\[ \tilde{\lambda}_2 : \quad z_2 - (1 - \zeta^{m+1}y_1)(1 - z_2)^2 = 0 \]
\[ \tilde{\lambda}_3 : \quad z_2z_4 - (1 - z_3)^2 = 0 \]
\[ \vdots \]
\[ \tilde{\lambda}_{n-1} : \quad z_{n-2}z_n - (1 - z_{n-1})^2 = 0 \]
\[ \tilde{\lambda}_n : \quad z_{n-1}v - (1 - z_n)^2 = 0 \]
\[ \tilde{\lambda}_s : \quad z_n - (1 - v)^2(\prod_{j=1}^m(1 - \zeta^2w_j)^2) - \tilde{s}2^{m+2}\tilde{s}(1 - \zeta^{m+1}y_1) = 0 \]

Setting $\zeta = 0$:

\[ \bar{\rho}_0 : \quad \bar{w}_m y_1^2(\prod_{j=2}^n(z_j - 1)^2)(v - 1)^2w_1 - \tilde{s}^2(\prod_{j=2}^n z_j^2)v^2 = 0 \]
\[ \bar{\lambda}_1 : \quad 2\zeta z_2 - y_1^2 = 0 \]
\[ \bar{\lambda}_2 : \quad z_2 - (1 - z_2)^2 = 0 \]
\[ \bar{\lambda}_3 : \quad z_2z_4 - (1 - z_3)^2 = 0 \]
\[ \vdots \]
\[ \bar{\lambda}_{n-1} : \quad z_{n-2}z_n - (1 - z_{n-1})^2 = 0 \]
\[ \bar{\lambda}_n : \quad z_{n-1}v - (1 - z_n)^2 = 0 \]
\[ \bar{\lambda}_s : \quad z_n - (1 - v)^2 = 0 \]

Dividing by $(\prod_{j=2}^n(z_j - 1)^2)(v - 1)^2$ and using Lemma 6.4, $\bar{\rho}_0$ becomes:

\[ \bar{w}_m y_1^2 w_1 - \tilde{s}^2z_2v = 0 \]

Using $\bar{\lambda}_1$, we get:

\[ \bar{w}_m y_1^2 w_1 - \tilde{s}y_1^2v = 0 \]
We are looking for solutions with the direction variables all non-zero, so we can
discard the factor of $y_i^2$ and rearrange to get:

$$\frac{\hat{s}}{\hat{w}_n} = \frac{w_1}{v}$$

In general the solutions we will find have no variables equal to zero, and we will
therefore not worry about dividing terms out.

So, by this stage, we have chosen values for all the angle variables using
Lemma 6.2, $y_1$ depends on $\hat{s}$ and the already chosen $z_2$, up to sign, and we
have a relationship between ratios of variables at the top of this block and the
bottom. Ignoring the sign of $y_1$ for now, everything in this block is determined
once we have chosen the values for the angle variables and know the value of the
lower ratio. A similar situation will occur for the other types of block.

### 6.2.2 $R_L^S$

$R_L^S$ differs from $R_S^S$ only in what is happening to the bottom few tetrahedra, so
many of the equations follow through identically. We do not group $\lambda_j$ with this
block. Otherwise, with similar calculations to for $R_S^S$, we get:

$$\rho_s : \quad \hat{w}_n (\zeta^{2^{m+2}} - 1)^2 y_1^2 (\prod_{j=2}^n (z_j - 1)^2) u^2 x_1 - \hat{s}^2 (\prod_{j=2}^n z_j^2) = 0$$

$$\lambda_1 : \quad \hat{s} (\zeta^{m+1} y_1 - 1)^2 z_2 - y_1^2 = 0$$

$$\lambda_2 : \quad z_3 - (1 - \zeta^{m+1} y_1)(1 - z_2)^2 = 0$$

$$\lambda_3 : \quad z_2 z_4 - (1 - z_3)^2 = 0$$

$$\vdots$$

$$\lambda_{n-1} : \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0$$

$$\lambda_n : \quad z_{n-1} - (1 - z_n)^2 (1 - \zeta u) = 0$$

Setting $\zeta = 0$:

$$\rho_s : \quad \hat{w}_n y_1^2 (\prod_{j=2}^n (z_j - 1)^2) u^2 x_1 - \hat{s}^2 (\prod_{j=2}^n z_j^2) = 0$$

$$\lambda_1 : \quad \hat{s} z_2 - y_1^2 = 0$$

$$\lambda_2 : \quad z_3 - (1 - z_2)^2 = 0$$

$$\lambda_3 : \quad z_2 z_4 - (1 - z_3)^2 = 0$$

$$\vdots$$

$$\lambda_{n-1} : \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0$$

$$\lambda_n : \quad z_{n-1} - (1 - z_n)^2 = 0$$

Dividing this time by $(\prod_{j=2}^n z_j^2)(u - 1)^2$ and using Lemma 6.4, $\rho_s$ becomes:

$$\hat{w}_n y_1^2 u^2 x_1 \left(\frac{z_2 z_n}{z_{n+1}} - \hat{s}^2 = 0\right)$$
Using $\bar{\lambda}_1$, and cancelling a factor of $\hat{s}$, then rearranging gives:

$$\frac{u^2 x_1}{z_n} = \frac{\hat{s}}{\bar{w}_m}$$

6.2.3 $R^L_L$

$$\tilde{\rho}_z : \quad (\zeta^2 \bar{w}_m - 1)\hat{s}^2(\zeta^2 y_1 - 1)^2 y_2^2(\prod_{j=3}^n (z_j - 1)^2)u^2 x_1 - \bar{w}_m y_1^2(\prod_{j=3}^n z_j^2) = 0$$

$$\tilde{\lambda}_1 : \quad -\hat{s} - y_2 + \zeta(-2y_1 + \hat{s}y_2) + \zeta^2(\cdots) = 0$$

$$\tilde{\lambda}_2 : \quad y_1(\zeta y_2 - 1)^2 z_3 - y_2^2 = 0$$

$$\tilde{\lambda}_3 : \quad z_4 - (1 - \zeta y_2)(1 - z_3)^2 = 0$$

$$\tilde{\lambda}_4 : \quad z_3 z_5 - (1 - z_4)^2 = 0$$

$$\vdots$$

$$\tilde{\lambda}_{n-1} : \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0$$

$$\tilde{\lambda}_n : \quad z_{n-1} - (1 - z_n)^2(1 - \zeta u) = 0$$

A new feature comes up this time with $\tilde{\lambda}_1$ which deserves some explanation. The vertex $\lambda_1$ on the torus boundary has no curve parts entering or exiting, but is entirely surrounded by degenerating tetrahedra.

$$\lambda'_1 : \quad \frac{1}{1 - T(1 - Z_1)^2} = 1$$

$$\lambda''_1 : \quad 1 - (1 - T)(1 - Z_1)^2(1 - Z_2) = 0$$

$$\lambda''''_1 : \quad 1 - (1 - \zeta \hat{s})(1 - \zeta^2 y_1)^2(1 - \zeta y_2) = 0$$

$$\lambda_1 : \quad -\hat{s} - y_2 + \zeta(-2y_1 + sy_2) + \zeta^2(\cdots) = 0$$

The 1’s cancel, then we remove a factor of $\zeta$. We could, but will not need to calculate the higher order terms in this equation.

Setting $\zeta = 0$:

$$\bar{\rho}_z : \quad (1)\hat{s}^2 y_2^2(\prod_{j=3}^n (z_j - 1)^2)u^2 x_1 - \bar{w}_m y_1^2(\prod_{j=3}^n z_j^2) = 0$$

$$\bar{\lambda}_1 : \quad -\hat{s} - y_2 = 0$$

$$\bar{\lambda}_2 : \quad y_1 z_3 - y_2^2 = 0$$

$$\bar{\lambda}_3 : \quad z_4 - (1 - z_3)^2 = 0$$

$$\bar{\lambda}_4 : \quad z_3 z_5 - (1 - z_4)^2 = 0$$

$$\vdots$$

$$\bar{\lambda}_{n-1} : \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0$$

$$\bar{\lambda}_n : \quad z_{n-1} - (1 - z_n)^2 = 0$$

Manipulating $\bar{\rho}_z$ further, we divide by $(\prod_{j=3}^n (z_j - 1)^2)$ to get:

$$-\hat{s}^2 y_2^2 u^2 x_1 - \bar{w}_m y_1^2 z_3 z_n = 0$$

Using $\bar{\lambda}_2$ twice gives:

$$-\hat{s}^2 y_2^2 u^2 x_1 - \frac{\bar{w}_m y_1^2 z_3 z_n}{z_3} = 0$$

43
Cancelling and rearranging, then using $\overline{\lambda}_1$:

$$\frac{u^2 x_1}{z_n} = -\frac{\hat{w}_m y_2^2}{\hat{s}^2 z_3} = -\frac{\hat{w}_m}{z_3}$$

To fit in nicely with ratios from other blocks, we alter this a little more using equations $\overline{\rho}_{m-1}$ and $\overline{\rho}_m$ from the $L^*_R$ block above (recall the notation: $L^*_R$ refers to either $L^*_R$ or $L^*_R$, when it doesn’t matter which).

$$\overline{\rho}_{m-1} : \quad -\hat{x}_{m-2}\hat{w}_{m-1}^2 - \hat{w}_m = 0$$
$$\overline{\rho}_m : \quad -\hat{w}_{m-1} - \hat{s} = 0$$

These are calculated a few pages later. Using these,

$$\frac{u^2 x_1}{z_n} = -\frac{\hat{w}_m}{z_3} = \frac{\hat{x}_{m-2}\hat{w}_{m-1}^2}{z_3} = \frac{\hat{x}_{m-2}\hat{s}^2}{z_3}$$

### 6.2.4 $R^L_S$

$R^L_S$ is identical to the top half of $R^L_L$ and the bottom half of $R^S_S$, apart from $\overline{\rho}_*$. 

$$\overline{\rho}_* : \quad (\zeta^2\hat{w}_m - 1)\hat{s}^2(\zeta^2 y_1 - 1)^2 y_2^2(\prod_{j=3}^n(z_j - 1)^2)(v - 1)^2 w_1 - \hat{w}_m y_1^2(\prod_{j=3}^n z_j)^2 v^2 = 0$$

$$\overline{\lambda}_1 : \quad -s - y_2 + \zeta(-2y_1 + s y_2) + \zeta^2(\cdots) = 0$$

$$\overline{\lambda}_2 : \quad y_1(\zeta y_2 - 1)^2 z_3 - y_2^2 = 0$$

$$\overline{\lambda}_3 : \quad z_4 - (1 - \zeta y_2)(1 - z_3)^2 = 0$$

$$\overline{\lambda}_4 : \quad z_3 z_5 - (1 - z_4)^2 = 0$$

$$\vdots$$

$$\overline{\lambda}_{n-1} : \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0$$

$$\overline{\lambda}_n : \quad z_{n-1} v - (1 - z_n)^2 = 0$$

$$\overline{\lambda}_* : \quad z_n - (1 - v)^2(\prod_{j=1}^m(1 - \zeta^2 w_j)^2)(1 - \zeta^2 m + 2 \hat{s})^2(1 - \zeta^m + \hat{y}_1) = 0$$

Setting $\zeta = 0$:

$$\overline{\rho}_* : \quad (-1)\hat{s}^2 y_2^2(\prod_{j=3}^n(z_j - 1)^2)(v - 1)^2 w_1 - \hat{w}_m y_1^2(\prod_{j=3}^n z_j^2) v^2 = 0$$

$$\overline{\lambda}_1 : \quad -s - y_2 = 0$$

$$\overline{\lambda}_2 : \quad y_1 z_3 - y_2^2 = 0$$

$$\overline{\lambda}_3 : \quad z_4 - (1 - z_3)^2 = 0$$

$$\overline{\lambda}_4 : \quad z_3 z_5 - (1 - z_4)^2$$

$$\vdots$$

$$\overline{\lambda}_{n-1} : \quad z_{n-2} z_n - (1 - z_{n-1})^2 = 0$$

$$\overline{\lambda}_n : \quad z_{n-1} v - (1 - z_n)^2 = 0$$

$$\overline{\lambda}_* : \quad z_n - (1 - v)^2 = 0$$
As for $R_L^L$, we divide $\rho_\bullet$ by $(\prod_{j=3}^{n}(z_j - 1))(v - 1)^2$ to get
\[-s^2 y_2^2 w_1 - \hat{w}_m y_1^2 z_3 v = 0\]
Applying $\bar{\lambda}_2$ twice gives
\[-s^2 y_2^2 w_1 - \frac{\hat{w}_m y_1^2 v}{z_3} = 0\]
Then $\bar{\lambda}_1$ twice, cancelling and rearranging gives
\[\frac{w_1}{v} = -\frac{\hat{w}_m}{z_3}\]

We alter this again using equations $\bar{\rho}_{m-1}$ and $\bar{\rho}_m$ from the $L_R^L$ block above. They are identical to as listed for $R_L^L$. Then,
\[
\frac{w_1}{v} = \frac{\hat{w}_m}{z_3} = \frac{\hat{x}_{m-2} y_1^2 v}{z_3} = \frac{\hat{x}_{m-2} s^2}{z_3}
\]

6.2.5 $S_L^L$
\[
\begin{align*}
\bar{\lambda}_1 & : (\zeta^{2n+2} \hat{s} - 1)y_1^2 (\zeta^{2n-2} y_2 - 1) - \hat{s} y_2 = 0 \\
\bar{\lambda}_2 & : (\zeta^{2n} y_1 - 1)y_2^2 (\zeta^{2n-4} y_3 - 1) - y_1 y_3 = 0 \\
\bar{\lambda}_3 & : (\zeta^{2n-2} y_2 - 1)y_3^2 (\zeta^{2n-6} y_4 - 1) - y_2 y_4 = 0 \\
\vdots \\
\bar{\lambda}_{n-1} & : (\zeta^2 y_{n-2} - 1)y_{n-1}^2 (\zeta^2 y_{n-1} - 1) - y_{n-2} y_n = 0 \\
\bar{\lambda}_n & : (\zeta^4 y_{n-1} - 1)y_n v - y_{n-1} = 0 \\
\end{align*}
\]

Setting $\zeta = 0$:
\[
\begin{align*}
\bar{\lambda}_1 & : y_1^2 - \hat{s} y_2 = 0 \\
\bar{\lambda}_2 & : y_2^2 - y_1 y_3 = 0 \\
\bar{\lambda}_3 & : y_3^2 - y_2 y_4 = 0 \\
\vdots \\
\bar{\lambda}_{n-1} & : y_{n-1}^2 - y_{n-2} y_n = 0 \\
\bar{\lambda}_n & : (-1)y_n^2 v - y_{n-1} = 0 \\
\end{align*}
\]

We can rearrange these equations to get:
\[
\begin{align*}
\hat{s} & = y_1 \\
y_1 & = y_2 \\
y_2 & = y_3 \\
y_3 & = y_4 \\
\vdots & \\
y_{n-2} & = y_{n-1} \\
y_{n-1} & = -y_n v \\
\end{align*}
\]
6.2.6 \(S^R_R\)

We now move into the blocks dealing primarily with the \(x_i\).

\[
\begin{align*}
\bar{\rho}_1 : & \quad v(\zeta^2 w_1 - 1)^2 w_2 - w_1^2 = 0 \\
\bar{\rho}_2 : & \quad w_1(\zeta^4 w_2 - 1)^2 w_3 - w_2^2 = 0 \\
\bar{\rho}_3 : & \quad w_2(\zeta^6 w_3 - 1)^2 w_4 - w_3^2 = 0 \\
\vdots & \quad \vdots \\
\bar{\rho}_{m-1} : & \quad \bar{\rho}_{m-2}(\zeta^{2m-2} w_{m-1} - 1)^2 w_m - w_{m-1}^2 = 0 \\
\bar{\rho}_m : & \quad \bar{\rho}_{m-1}(\zeta^{2m} w_m - 1)^2 s - w_m^2 = 0
\end{align*}
\]

Setting \(\zeta = 0\):

\[
\begin{align*}
\bar{\rho}_1 & : \quad vv_2 - v_1^2 = 0 \\
\bar{\rho}_2 & : \quad w_1 v_3 - w_2^2 = 0 \\
\bar{\rho}_3 & : \quad w_2 v_4 - w_3^2 = 0 \\
\vdots & \quad \vdots \\
\bar{\rho}_{m-1} & : \quad \bar{\rho}_{m-2} w_m - w_{m-1}^2 = 0 \\
\bar{\rho}_m & : \quad \bar{\rho}_{m-1} w_m - w_m^2 = 0
\end{align*}
\]

These give:

\[
\begin{align*}
v_1 & = v_2 = v_3 = \ldots = v_m = s \\
\frac{w_1}{w_1} & = \frac{w_2}{w_2} = \frac{w_3}{w_3} = \ldots = \frac{w_m}{w_{m-1}} = \frac{s}{w_m}
\end{align*}
\]

6.2.7 \(L^R_S\)

\[
\begin{align*}
\tilde{\lambda}_s : & \quad \tilde{\zeta}_s(\zeta u - 1)^2(\zeta^{n+1} w_m - 1)^2 s^2(\zeta^{2n} y_1 - 1) - u^2(\prod_{i=1}^{m-1}(1 - x_i)^2)w_m^2 y_1 = 0 \\
\tilde{\rho}_1 & : \quad (x_1 - 1)^2 x_2 - (1 - \zeta u)x_1^2 = 0 \\
\tilde{\rho}_2 & : \quad x_1(x_2 - 1)^2 x_3 - x_2^2 = 0 \\
\tilde{\rho}_3 & : \quad x_2(x_3 - 1)^2 x_4 - x_3^2 = 0 \\
\vdots & \quad \vdots \\
\tilde{\rho}_{m-2} & : \quad x_{m-3}(x_{m-2} - 1)^2 x_{m-1} - x_{m-2}^2 = 0 \\
\tilde{\rho}_{m-1} & : \quad x_{m-2}(x_{m-1} - 1)^2 - x_{m-1}^2(1 - \zeta^{n+1} w_m) = 0 \\
\tilde{\rho}_m & : \quad x_{m-1} w_m^2(\zeta^{2n+1} s - 1) - s = 0
\end{align*}
\]

Setting \(\zeta = 0\):

\[
\begin{align*}
\tilde{\lambda}_s & : \quad \tilde{\zeta}_s s^2(-1) - u^2(\prod_{i=1}^{m-1}(1 - x_i)^2)w_m^2 y_1 = 0 \\
\tilde{\rho}_1 & : \quad (x_1 - 1)^2 x_2 - x_1^2 = 0 \\
\tilde{\rho}_2 & : \quad x_1(x_2 - 1)^2 x_3 - x_2^2 = 0 \\
\tilde{\rho}_3 & : \quad x_2(x_3 - 1)^2 x_4 - x_3^2 = 0 \\
\vdots & \quad \vdots \\
\tilde{\rho}_{m-2} & : \quad x_{m-3}(x_{m-2} - 1)^2 x_{m-1} - x_{m-2}^2 = 0 \\
\tilde{\rho}_{m-1} & : \quad x_{m-2}(x_{m-1} - 1)^2 - x_{m-1}^2 = 0 \\
\tilde{\rho}_m & : \quad x_{m-1} w_m^2(-1) - s = 0
\end{align*}
\]

Using Lemma 6.7, \(\tilde{\lambda}_s\) becomes:

\[
\tilde{\zeta}_s s^2 - u^2 x_1 x_{m-1} w_m^2 y_1 = 0
\]
By \( \mathbf{m} \) this is:
\[
\dot{z}_n s^2 + u^2 x_1 s y_1 = 0
\]
Cancelling the \( s \) and rearranging gives:
\[
\frac{s}{y_1} = \frac{u^2 x_1}{\dot{z}_n}
\]

6.2.8 \( L^R_H \)

\[ \begin{align*}
\lambda_n : & \quad \dot{z}_n (\zeta u - 1)^2 (\zeta w_{m-1} - 1)^2 w_m^2 (\zeta s - 1)^2 y_1 - u^2 (\prod_{i=1}^{m-2} (1 - x_i)^2) w_{m-1}^2 s^2 = 0 \\
\rho_1 : & \quad (x_1 - 1)^2 x_2 - (1 - \zeta u) x_2^2 = 0 \\
\rho_2 : & \quad x_1 (x_2 - 1)^2 x_3 - x_2^2 = 0 \\
\rho_3 : & \quad x_2 (x_3 - 1)^2 x_4 - x_3^2 = 0 \\
& \quad \vdots \\
\rho_{m-3} : & \quad x_m (x_{m-1} - 1)^2 x_{m-2} - x_{m-3}^2 = 0 \\
\rho_{m-2} : & \quad x_{m-3} (x_{m-2} - 1)^2 - x_{m-2}^2 (1 - \zeta w_{m-1}) = 0 \\
\rho_{m-1} : & \quad x_{m-2} w_{m-1}^2 (\zeta^2 w_{m-1} - 1) - w_m = 0 \\
\rho_m : & \quad w_{m-1} + s + \zeta (2 w_m - s w_{m-1}) + \zeta^2 (\cdots) = 0 
\end{align*} \]

Setting \( \zeta = 0 \):

\[ \begin{align*}
\lambda_n : & \quad \dot{z}_n w_m^2 y_1 - u^2 (\prod_{i=1}^{m-2} (1 - x_i)^2) w_{m-1}^2 s^2 = 0 \\
\rho_1 : & \quad (x_1 - 1)^2 x_2 - x_2^2 = 0 \\
\rho_2 : & \quad x_1 (x_2 - 1)^2 x_3 - x_2^2 = 0 \\
\rho_3 : & \quad x_2 (x_3 - 1)^2 x_4 - x_3^2 = 0 \\
& \quad \vdots \\
\rho_{m-3} : & \quad x_m (x_{m-1} - 1)^2 x_{m-2} - x_{m-3}^2 = 0 \\
\rho_{m-2} : & \quad x_{m-3} (x_{m-2} - 1)^2 - x_{m-2}^2 = 0 \\
\rho_{m-1} : & \quad x_{m-2} w_{m-1}^2 (-1) - w_m = 0 \\
\rho_m : & \quad w_{m-1} + s = 0 
\end{align*} \]

\( \lambda_n \) becomes:
\[
\dot{z}_n w_m^2 y_1 - u^2 x_1 x_{m-2} w_{m-1}^2 s^2 = 0
\]
and by \( \rho_{m-1} \) twice:
\[
\dot{z}_n x_{m-2} w_{m-1}^2 y_1 + u^2 x_1 x_{m-2} w_{m-1}^2 s^2 = 0
\]
by \( \rho_m \):
\[
\dot{z}_n x_{m-2} s^4 y_1 + u^2 x_1 x_{m-2} s^4 = 0
\]
Cancelling and rearranging:
\[
x_{m-2} y_1 = \frac{u^2 x_1}{\dot{z}_n}
\]

47
Taking a couple of equations from the $R_s^L$ block below (note that $\dot{s}$ from below is $s$ here):

$$\begin{align*}
\bar{\lambda}_1 & : -s - y_2 = 0 \\
\bar{\lambda}_2 & : y_1 z_3 - y_2^2 = 0 \\
\frac{w^2 x_1}{z_n} & = x_{m-2} y_1 = \frac{x_{m-2} y_2^2}{z_3} = \frac{x_{m-2} s^2}{z_3}
\end{align*}$$

6.2.9 $L^S_R$

Once again, one of the neighbours of an $S^*_s$ block takes its "long" vertex equation, in this case $\bar{\rho}_s$.

$$\begin{align*}
\bar{\lambda}_s : & \quad (\zeta^2 \dot{y}_n - 1)(\zeta w_{m-1} - 1)^2 w_m^2 (\zeta s - 1)^2 y_1 - \dot{y}_n (1 - v)^2 (\prod_{i=1}^{m-2} (1 - x_i)^2) w_{m-1} s^2 = 0 \\
\bar{\rho}_s : & \quad (v - 1)^2 x_1 - v^2 (1 - \zeta^{n+1} \dot{w}_n)(1 - \zeta^{2(n+2)})(\prod_{j=1}^{n} (1 - \zeta^{2(n-j+1)}) \dot{y}_j)^2 = 0 \\
\bar{\rho}_1 : & \quad v(x_1 - 1)^2 x_2 - x_1^2 = 0 \\
\bar{\rho}_2 : & \quad x_1(x_2 - 1)^2 x_3 - x_2^2 = 0 \\
\bar{\rho}_3 : & \quad x_2(x_3 - 1)^2 x_4 - x_3^2 = 0 \\
\vdots\end{align*}$$

$$\begin{align*}
\bar{\rho}_{m-3} : & \quad x_{m-4}(x_{m-3} - 1)^2 x_{m-2} - x_{m-3}^2 = 0 \\
\bar{\rho}_{m-2} : & \quad x_{m-3}(x_{m-2} - 1)^2 x_{m-2} - x_{m-1}(1 - \zeta w_{m-1}) = 0 \\
\bar{\rho}_{m-1} : & \quad x_{m-2} w_{m-1}^2 (\zeta^2 w_{m-1} - 1) - w_{m} = 0 \\
\bar{\rho}_m : & \quad w_{m-1} + s + \zeta(2w_m - sw_{m-1}) + \zeta^2(\cdots) = 0
\end{align*}$$

Setting $\zeta = 0$:

$$\begin{align*}
\bar{\lambda}_s : & \quad (-1) w_m^2 y_1 - \dot{y}_n (1 - v)^2 (\prod_{i=1}^{m-2} (1 - x_i)^2) w_{m-1} s^2 = 0 \\
\bar{\rho}_s : & \quad (v - 1)^2 x_1 - v^2 = 0 \\
\bar{\rho}_1 : & \quad v(x_1 - 1)^2 x_2 - x_1^2 = 0 \\
\bar{\rho}_2 : & \quad x_1(x_2 - 1)^2 x_3 - x_2^2 = 0 \\
\bar{\rho}_3 : & \quad x_2(x_3 - 1)^2 x_4 - x_3^2 = 0 \\
\vdots\end{align*}$$

$$\begin{align*}
\bar{\rho}_{m-3} : & \quad x_{m-4}(x_{m-3} - 1)^2 x_{m-2} - x_{m-3}^2 = 0 \\
\bar{\rho}_{m-2} : & \quad x_{m-3}(x_{m-2} - 1)^2 x_{m-2} - x_{m-2}^2 = 0 \\
\bar{\rho}_{m-1} : & \quad x_{m-2} w_{m-1}^2 (1 - w_{m} - w_{m-1}) = 0 \\
\bar{\rho}_m : & \quad w_{m-1} + s = 0
\end{align*}$$

By Lemma 6.7, $\bar{\lambda}_s$ becomes:

$$-w_m^2 y_1 - \dot{y}_n v x_{m-2} w_{m-1} s^2 = 0$$

and by $\bar{\rho}_{m-1}$ twice:

$$-w_m^4 x_{m-2}^2 y_1 - \dot{y}_n v x_{m-2} w_{m-1} s^2 = 0$$

by $\bar{\rho}_m$:

$$-s^4 x_{m-2}^2 y_1 - \dot{y}_n v x_{m-2} s^4 = 0$$

48
Cancelling we obtain:
\[-x_{m-2}y_1 - \hat{y}_n v = 0\]

However, again we alter this to fit with the other blocks using the two equations from the $R_L^\delta$ block below:

\[
\begin{align*}
\bar{\lambda}_1 & : \quad -\hat{s} - y_2 = 0 \\
\bar{\lambda}_2 & : \quad y_1 z_3 - y_2^2 = 0 \\
-\hat{y}_n v &= y_1 x_{m-2} = \frac{y_2^2 x_{m-2}}{z_3} = \frac{s^2 x_{m-2}}{z_3}
\end{align*}
\]

6.2.10 $L_S^\delta$

\[
\begin{align*}
\bar{\lambda}_s : \quad & (\zeta^2 \hat{y}_n - 1)(\zeta w_{m-1} - 1)^2 s^2(\zeta^2 y_1 - 1) - \hat{y}_n (1 - v)^2 (\prod_{i=1}^{m-1} (1 - x_i)^2) w_m^2 y_1 = 0 \\
\bar{\rho}_s : \quad & (v - 1)^2 x_1 - v^2 (1 - \zeta^2 + \hat{u}_m) (1 - \zeta^2) (\prod_{i=1}^{n-1} (1 - \zeta^{2n-j+1}) y_j)^2 = 0 \\
\bar{\rho}_1 : \quad & v(x_1 - 1)^2 x_2 - x_1^2 = 0 \\
\bar{\rho}_2 : \quad & x_1 (x_2 - 1)^2 x_3 - x_2^3 = 0 \\
\bar{\rho}_3 : \quad & x_2 (x_3 - 1)^2 x_4 - x_3^3 = 0 \\
& \vdots \\
\bar{\rho}_{m-2} : \quad & x_{m-3} (x_{m-2} - 1)^2 x_{m-1} - x_{m-2}^2 = 0 \\
\bar{\rho}_{m-1} : \quad & x_{m-2} (x_{m-1} - 1)^2 - x_{m-1}^2 (1 - \zeta^{n+1} w_m) = 0 \\
\bar{\rho}_m : \quad & x_{m-1} w_m^2 (\zeta^{2n+1} s - 1) - s = 0
\end{align*}
\]

Setting $\zeta = 0$:

\[
\begin{align*}
\bar{\lambda}_s : \quad & (-1)s^2(-1) - \hat{y}_n (1 - v)^2 (\prod_{i=1}^{m-1} (1 - x_i)^2) w_m^2 y_1 = 0 \\
\bar{\rho}_s : \quad & (v - 1)^2 x_1 - v^2 = 0 \\
\bar{\rho}_1 : \quad & v(x_1 - 1)^2 x_2 - x_1^2 = 0 \\
\bar{\rho}_2 : \quad & x_1 (x_2 - 1)^2 x_3 - x_2^3 = 0 \\
\bar{\rho}_3 : \quad & x_2 (x_3 - 1)^2 x_4 - x_3^3 = 0 \\
& \vdots \\
\bar{\rho}_{m-2} : \quad & x_{m-3} (x_{m-2} - 1)^2 x_{m-1} - x_{m-2}^2 = 0 \\
\bar{\rho}_{m-1} : \quad & x_{m-2} (x_{m-1} - 1)^2 - x_{m-1}^2 = 0 \\
\bar{\rho}_m : \quad & x_{m-1} w_m^2 (-1) - s = 0
\end{align*}
\]

By Lemma 6.7, $\bar{\lambda}_s$ becomes:

\[s^2 - \hat{y}_n v x_m - 1 w_m^2 y_1 = 0\]

Using $\bar{\rho}_m$:

\[s^2 + \hat{y}_n v s y_1 = 0\]

Cancelling $s$ and rearranging:

\[\frac{s}{y_1} = -\hat{y}_n v\]
6.3 Putting it all together

Summarising, we get the following equations:

\[ \begin{align*}
R_S^S & : \frac{s}{\omega_m^2} & R_L^S & : \frac{s}{\omega_m^2} & \frac{s}{\omega_m^2} \frac{\bar{x}^2}{\bar{z}_n^2} & \frac{s}{\omega_m^2} \\
R_L^L & : \frac{s}{\omega_m^2} & R_S^L & : \frac{s}{\omega_m^2} \\
S_R^R & : \frac{s}{\omega_m^2} & L_S^R & : \frac{s}{\omega_m^2} & L_R^R & : \frac{s}{\omega_m^2} \\
S_L^L & : \frac{s}{\omega_m^2} & \frac{s}{\omega_m^2} & \frac{s}{\omega_m^2} & \frac{s}{\omega_m^2} \\
\end{align*} \]

Whenever two blocks are possible to glue together, the ratios at the ends of those blocks agree. They typographically differ in hats of course, since a given variable is above one block but below the other, but it is not hard to check that they refer to the same variables.

We can now give an explicit algorithm for finding a solution to the tilde equations at \( p \) (when \( \zeta = 0 \)).

Step I: Using Lemmas 6.2 and 6.5, find values for all of the angle variables, choosing the solution with \( n\beta = \frac{\pi}{N+2} \) as in Lemmas 6.3 and 6.6.

Step II: Set one of the direction variables to be equal to 1 (normalising the rates). There are three cases to consider, and which case we are in will determine which variable we set to be 1.

Case 1: Our blocks describe a path entirely on the left side of the Farey strip, made of alternating \( L_S^S \) and \( S_L^L \) blocks.

Case 2: Our blocks describe a path entirely on the right side of the Farey strip, made of alternating \( R_S^S \) and \( S_R^R \) blocks.

Case 3: There is at least one place at which the path crosses from one side to the other.

In Case 1, we choose a \( y_n \) variable to set equal to 1, in Case 2 we choose a \( s \) variable, and in Case 3, we choose an \( s \) variable that sits in between \( L_R^R \) and \( R_L^L \) blocks (there must be such a pair of blocks if there is any crossing of the path).

Case 1: Consider the string of equations we get at the end of the \( S_L^L \) section:

\[ \frac{\bar{s}}{\bar{y}_1} = \frac{\bar{y}_2}{\bar{y}_2} = \frac{\bar{y}_3}{\bar{y}_3} = \cdots = \frac{y_{n-2}}{y_{n-2}} = \frac{y_{n-1}}{y_{n-1}} = -y_n \]

Working from the right leftwards (which is upwards on our torus boundary pictures): Knowing \( v \) (it is an angle variable, chosen in
Step I) and \( y_n \) we get a value for \( y_{n-1} \), then \( y_{n-2} \), and so on all the way to a value for \( \hat{s} \). Now that we know \( \frac{s}{y_n} \), the \( L_S^2 \) block above us tells us the value of \( -\hat{y}_n \hat{v} \), and we repeat again, moving upwards through this block. We continue upwards until we wrap around and cover all blocks. We cannot reach a contradiction of course, because we already know that all of these ratios of variables are equal throughout. Note that all of the values we assign are non-zero, since they are ratios of non-zero numbers.

Case 2: This works similarly to Case 1, except that we move from the left rightwards (which is downwards on our torus boundary pictures).

Case 3: Given a value for \( s \), all other \( s \) and \( u \) variables are determined (up to sign) by following the equations around. \( \frac{z_{m-2}^2}{z_1} \) from one part of the torus bundle is equal to the same in another. The values of \( x_{m-2} \) and \( z_1 \) may be different if the \( m \) and \( n \) are different, but in any case, they were determined in Step I, so now all instances of \( s^3 \) are determined. Similarly \( \frac{z_1}{z_{m-1}} \) gives us \( u \) variables throughout the torus bundle, up to sign. Any part of our path that follows the left edge is of the form \( L_S^3 S_L^3 L_S^2 S_L^2 \cdots S_L^2 L_R^1 \), and can be dealt with in the same way as Case 1 (we know the value of \( v \), and of \( -y_n v \), so we know \( y_n \), and then the argument from Case I applies). A part that follows the right edge can similarly be dealt with as in Case 2.

Step III: there are 6 kinds of join, and we look at each in turn to determine remaining variables:

- From a \( L^*_R R^L_* \) join: \( w_{m-1} = -s = y_2 \), \( w_m \) is given by \( \sqrt{m-1} \) and \( y_1 \) by \( \sqrt{2} \).
- From a \( S^R_H R^L_* \) join: \( y_1 \) is determined up to sign by \( \tau_1 \).
- From a \( L^*_S S^L_* \) join: \( w_m \) is determined up to sign by \( \tau_m \).
- From a \( R^L_* L^R_* \) join: nothing left to determine.
- From a \( S^L_* L^R_* \) join: nothing left to determine.
- From a \( R^L_* S^R_H \) join: nothing left to determine.

We have now determined the values of all variables, up to sign in some cases. Whenever we had a choice of sign, either option gives a solution to the equations for \( \zeta = 0 \). For reference, those choices are: \( s \) from a \( L^*_R R^L_* \) join, \( u \) from a \( R^L_* L^R_* \) join, \( y_1 \) from a \( S^R_H R^L_* \) join and \( w_m \) from a \( L^*_S S^L_* \) join.

We have the existence of a point on the variety defined by the tilde equations (and the equation setting one direction variable equal to 1), \( \tilde{X}(M, T) \), with \( \zeta = 0 \). We now want to show that there are other points of the variety nearby, and moreover that we have nearby points that correspond to finite points of the
original variety $\mathcal{X}(M,J)$. The following discussion and result prove the first part:

Suppose we have $N$ tetrahedra in our torus bundle. The torus bundle is made up of some number of $L^{m+1}R^{n+1}$ sections, and $N$ is the sum of all of those $m+1$s and $n+1$s. We begin with $N$ equations (the gluing equations), and the $N$ variables (the original complex angles). One of the gluing equations is dependent on the other $N - 1$. This is a standard fact for triangulations (tetrahedralisations) of 3-manifolds with a single boundary component (starting with the original gluing equations, multiply $N - 1$ of them together, and use the identities between the 3 angles in each tetrahedron to obtain the $N$th). Thus we can remove one gluing equation and now have $N - 1$ equations in $N$ variables. Next we convert all these to tilde equations, and add a variable, $\zeta$. We then set one of the direction variables equal to one, effectively adding the equation "$\text{(direction variable)} - 1 = 0"$. This brings us to $N$ equations in $N + 1$ variables.

**Proposition 6.9.** If $p \in \mathbb{C}^{N+1}$ satisfies polynomial equations $f_1, f_2, \ldots, f_N \in \mathbb{C}[x_1, x_2, \ldots x_{N+1}]$ then there exist other solutions to these equations arbitrarily close to $p$.

The heuristic reason for this is that starting from $\mathbb{C}^{N+1}$, every polynomial we add to our set of equations cuts down the dimension of the set of solutions by at most one (unless it results in an inconsistent set of equations). Since we only make $N$ cuts, and started with $N + 1$ dimensions, we will have at least one left by the end. The existence of $p$ demonstrates the consistency of the equations. Here is a more formal proof:

**Proof.** Let $p$ be an irreducible component of $(f_1, f_2, \ldots, f_N)$, the ideal generated by the polynomials, $\mathbb{C}[x_1, x_2, \ldots x_{N+1}]$ has transcendence degree $N + 1$, so by (for example) Chapter IX, Exercise 18, p412 of [7], every maximal chain of prime ideals

$$\mathbb{C}[x_1, x_2, \ldots x_{N+1}] \supset p_1 \supset p_2 \supset \cdots \supset p_M \supset \{0\}$$

with $\mathbb{C}[x_1, x_2, \ldots x_{N+1}] \neq p_1, p_1 \neq p_1, p_M \neq \{0\}$ must have $M = N + 1$. Thus any such chain that passes through $p$ must have the same property, and so

$$N + 1 = \dim \frac{\mathbb{C}[x_1, x_2, \ldots x_{N+1}]}{p} + \text{ht}(p)$$

Corollary 11.16 of [1] states that every minimal ideal $p$ belonging to (i.e. a prime factor of) $(f_1, f_2, \ldots, f_N)$ has ht $(p) \leq N$, and so

$$\dim \frac{\mathbb{C}[x_1, x_2, \ldots x_{N+1}]}{p} \geq 1$$

Let $m \supset p$ be a maximal ideal. A form of Hilbert’s Nullstellensatz, Corollary 5.24 of [1], gives us that since $\dim \frac{\mathbb{C}[x_1, x_2, \ldots x_{N+1}]}{m}$ is a finitely generated $\mathbb{C}$-algebra,
and \( m \) is maximal, then \( \dim \mathbb{C}[x_1, x_2, \ldots, x_{N+1}] \) is a field and hence a finite algebraic extension of \( \mathbb{C} \), so it is \( \mathbb{C} \). Hence \( m = (x_1 - a_1, x_2 - a_2, \ldots, x_{N+1} - a_{N+1}) \) so every point of \( \mathbb{C}[x_1, x_2, \ldots, x_{N+1}] \) is an actual point in \( \mathbb{C}^{N+1} \) and thus a solution to our polynomial equations. 

However, we are actually able to prove a stronger result.

**Proposition 6.10.** \( p \) is a regular point of \( \tilde{\mathcal{T}}(M, \mathcal{T}) \).

We prove this in section 7.

### 6.4 \( p \) is an Ideal Point

We have shown that a solution \( p \) to the tilde equations exists, with \( \zeta = 0 \). We also know that the variety defined by the tilde equations, \( \tilde{\mathcal{T}}(M, \mathcal{T}) \) contains points arbitrarily near to \( p \). Moreover, we know something about what such a nearby point looks like:

**Proposition 6.11.** There exist points of \( \tilde{\mathcal{T}}(M, \mathcal{T}) \) arbitrarily near \( p \) which are finite (when we convert them back to points of \( \mathcal{T}(M, \mathcal{T}) \), no angle is 0, \( \infty \) or 1).

**Proof.** We want to show first that for points near enough to \( p \), \( \zeta \neq 0 \). In order to show this, we consider the steps we took to find \( p \) (i.e. a solution with \( \zeta = 0 \)). We first set \( \zeta = 0 \), then chose among finitely many solutions for each sequence of angle variables (see Lemma 6.8). If we look for points with \( \zeta = 0 \) and near enough to \( p \), then the choices of angle variables must be the same as for \( p \), since there are only finitely many such choices, and any two choices will have some distance between them. Then the only choice we had for direction variables were some signs. Choosing a different sign again puts us at some distance from \( p \) and therefore when we look near enough to \( p \), the only solution to the equations with \( \zeta = 0 \) is \( p \) itself. Therefore we must have points nearby for which \( \zeta \neq 0 \).

We should also consider if at any point an assumption we made about not dividing by zero when finding a solution to \( p \) could be false now that we are interested in any solution with \( \zeta = 0 \). If however such a solution does exist, it is not near to \( p \), since our solution \( p \) has no variable near 0, and whenever we divided, it was always by products of the variables. Thus we can ignore these possible solutions when trying to find solutions near \( p \).

Suppose \( q \) is a point of \( \tilde{\mathcal{T}}(M, \mathcal{T}) \) near \( p \) for which \( \zeta \) is near to but not equal to 0. Then by continuity, we can ensure that for \( q \) all angle variables are bounded away from 0, \( \infty \) or 1, since they are so for \( p \). When we change variables back to the original angles of the original gluing equations, all direction variables become \( \zeta^k y \) for some \( k > 0 \) and \( y \) a direction variable. By continuity, \( y \) is near whatever value it had at \( p \), that is, bounded away from 0 and \( \infty \). For \( q \) sufficiently close to (but not equal to) \( 0 \) then, \( 0 < |\zeta^k y| < 1 \), and this angle is also finite. 

53
Theorem 6.12. \( p \) is an ideal point of the character variety.

Proof. By Proposition 6.11, \( p \) is an ideal point of the tetrahedron variety. By Lemma 4.2 of Yoshiida [12], The slope of \( p \) is the same as the slope of the homology class formed by the boundary curve segments of our surface, with the Yoshiida orientation (within a triangle on the boundary torus, the arrow points from the \( \infty \) corner to the 0corner). The Yoshiida orientation is shown in Figures 8 through 12. The arrows always enter the bottom of each block and exit the top, so they cannot describe a trivial element in the homology of the boundary torus. Hence the slope of \( p \) is non zero (i.e. not trivial).

Lastly we need the fact that as we approach an ideal point of the tetrahedron variety, the length of some path on the boundary torus diverges to infinity. The ingredients to show this can be found in sections 2 and 3 of Yoshiida [12], although the result itself is not explicitly stated.

Assuming this fact, we conclude that \( p \) is an ideal point of the character variety. \( \square \)

7 Derivative Matrices at \( p \)

7.1 Preliminaries

We show that \( \tilde{\xi}(M, \mathcal{T}) \) is smooth at \( p \) by calculating the derivative matrix, restricting to our solution at \( \zeta = 0 \), and showing that the matrix is of full rank, essentially by performing row operations on it. Then by a standard result from algebraic geometry, e.g. Theorem 5.1 of [6], the variety is smooth at that point.

As discussed in section 6.3, we have \( N \) equations in \( N + 1 \) variables having removed one redundant gluing equation, or \( N + 1 \) equations in \( N + 1 \) variables if we leave it in. The equation setting one direction variable equal to 1 gives a row of the derivative matrix which is entirely zero, apart from at the column corresponding to the direction variable we set equal to 1, at which the entry is 1. We use this row first, to eliminate all terms in the column corresponding to the direction variable. Effectively, we can delete the row corresponding to the extra equation and the column corresponding to the direction variable, taking us to \( N \) equations in \( N \) variables.

The plan of attack is to ignore one column entirely (remove it from the matrix, taking us to \( N \) rows and \( N - 1 \) columns) and perform row operations on the others to reduce the matrix to the \( N - 1 \) by \( N - 1 \) identity, plus a row of zeros corresponding to the redundant gluing equation. It turns out that we will always be able to choose the \( \zeta \) column to ignore.

The form of the problem we will tackle is this: We start with the matrix with \( N \) rows corresponding to the \( N \) tilde equations, and \( N \) variables (we are
ignoring the $\zeta$ column). We kill the column corresponding to the direction variable we set equal to one, then start reducing. We will not need to use one of the rows (this is the fact that $N - 1$ of the gluing equations determine the $N$th), so can delete it, and should end up with an $N - 1$ by $N - 1$ matrix that can be row reduced to the identity.

This being possible is equivalent to the determinant of the matrix being non-zero. Column operations do not change the non-zero-ness or otherwise of a determinant, so we allow them also.

As before, we break the problem up by blocks. There are many calculations needed, and we will tend to only include calculations that are necessary to the proof. Since we will be ignoring the $\zeta$ column, we will not be calculating any derivatives with respect to $\zeta$. First, a couple of useful lemmas regarding the sequences of angle variables:

**Lemma 7.1.** The submatrix formed from columns corresponding to a sequence of angle variables $a_2$ through $a_N$ and equations $\eta_k : \frac{a_k - 1}{a_{k+1}} - (1 - a_k)^2 = 0$ (where $k$ ranges from 2 to $N$ and we assume $a_1 = 1 = a_{N+1}$ as before) has non zero determinant.

**Proof.** The derivative matrix is this:

$$
\begin{bmatrix}
2(1 - a_2) & 1 & 0 & \cdots & 0 \\
a_4 & 2(1 - a_3) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 2(1 - a_{N-2}) & a_{N-3} & 0 \\
0 & \cdots & 0 & a_{N-2} & 2(1 - a_N)
\end{bmatrix}
$$

Divide the row corresponding to $\eta_k$ by $(1 - a_k)^2$ and use $\eta_k$ to get:

$$
\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 \\
1 & a_2 - a_3 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 2(1 - a_{N-2}) & a_{N-3} & 0 \\
0 & \cdots & 0 & a_{N-2} & 2(1 - a_N)
\end{bmatrix}
$$

Multiply the column corresponding to $a_k$ by $a_k$ to get:

$$
\begin{bmatrix}
\frac{2a_k}{1-a_2} & 1 & 0 & \cdots & 0 \\
1 & \frac{2a_k}{1-a_3} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \frac{2a_k}{1-a_{N-2}} & 1 & 0 \\
0 & \cdots & 0 & \frac{2a_k}{1-a_N} & 1
\end{bmatrix}
$$

55
To show this has non-zero determinant, we prove a more general statement, that the matrix:

\[
\begin{bmatrix}
\delta_2 & 1 \\
1 & \delta_3 & 1 & 0 \\
& & \ddots & \ddots \\
& & & 1 & \delta_{m-1} \\
& & & & 1 & \delta_m \\
\end{bmatrix}
\]

has non-zero determinant when \( \delta_k < -2 \) for all \( k \). Note that this is true of the above matrix if \( a_k > 1 \), and it is by Lemma 6.3. We use row operations to make the matrix upper triangular. After one such move, we have:

\[
\begin{bmatrix}
\delta_2 & 1 \\
0 & \delta_3 - \frac{1}{(\delta_2)} & 1 & 0 \\
& & \ddots & \ddots \\
& & & 1 & \delta_{m-1} \\
& & & & 1 & \delta_m \\
\end{bmatrix}
\]

then:

\[
\begin{bmatrix}
\delta_2 & 1 \\
0 & \delta_3 - \frac{1}{(\delta_3)} \\
& & \ddots & \ddots \\
& & & 1 & \delta_{m-1} \\
& & & & 1 & \delta_m \\
\end{bmatrix}
\]

and so on. The determinant is non-zero as long as none of the diagonal entries (these continued fractions) are zero. We use induction, assuming that the \( k \)th continued fraction is less than \(-2\). This is true for the first, \( \delta_2 \), since by assumption it is less than \(-2\). Now if the \((k-1)\)th continued fraction is less than \(-1\), then the \( k \)th is \( \delta_k \) (less than \(-2\)) plus something less than 1, hence the \( k \)th continued fraction is also less than \(-1\).

We also record a similar statement for the other angle variables:

**Lemma 7.2.** The submatrix formed from columns corresponding to a sequence of angle variables \( b \) through \( b_N \) and equations \( \nu_k : b_{k-1}(b_k - 1)^2b_{k+1} - b_k^2 = 0 \) (where \( k \) ranges from 2 to \( N \) and we assume \( b_0 = 1 = b_{N+1} \) as before) has non zero determinant.

**Proof.** This works very similarly to the proof for the \( a_k \), and uses Lemma 6.6. □

56
In reducing the derivative matrix for the whole torus bundle, we will start by "killing off" all of the rows and columns containing the submatrices in these lemmas. Thus, we will not need to calculate any derivatives with respect to angle variables, or derivatives of equations that only contain angle variables. Note however, that the first and last equations referenced in the lemmas necessarily involve $\zeta$ and at least one direction variable. See for example $\lambda_2$ of $R^S$. Whenever a direction variable is involved in one of these equations, it always appears multiplied by a factor of $\zeta$. Thus a derivative of this equation with respect to the direction variable will always be zero once we set $\zeta = 0$.

With these comments in mind, we list the rest of the derivative matrix (restricted to $\zeta = 0$), by block. Note that we use Lemmas 6.4 and 6.7 without comment, dividing the entire row of the matrix by $\prod_{k=2}(a_k - 1)^2$ if necessary.

### 7.2 Block Calculations

#### 7.2.1 $R^S$

The relevant variables are $\hat{\omega}_n, \hat{s}, y_1$ and $w_1$ (all others are angle variables and have already been dealt with).

<table>
<thead>
<tr>
<th>$\rho_s$</th>
<th>$\hat{\omega}_n$</th>
<th>$\hat{s}$</th>
<th>$y_1$</th>
<th>$w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$y_1^2w_1$</td>
<td>$2s z_2 v$</td>
<td>$2w_n y_1 w_1$</td>
<td>$w_n y_1^2$</td>
</tr>
</tbody>
</table>

We put a divider between direction variables either side of the sequence of angle variables.

Divide the $\rho_s$ row by $\hat{\omega}_n y_1^2 w_1$, which is equal to $-\hat{s}^2 z_2 v$ (see the previous calculations for $R^S$), and the $\lambda_1$ row by $\hat{s} z_2$ which is equal to $y_1^2$, then use $\lambda_1$:

<table>
<thead>
<tr>
<th>$\rho_s$</th>
<th>$\hat{\omega}_n$</th>
<th>$\hat{s}$</th>
<th>$y_1$</th>
<th>$w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$\frac{\hat{\omega}_n}{\hat{\omega}_n}$</td>
<td>$\frac{\hat{s}}{\hat{s}}$</td>
<td>$\frac{y_1}{y_1}$</td>
<td>$\frac{w_1}{w_1}$</td>
</tr>
</tbody>
</table>

This trick will also be possible in the other blocks (we prove it in Lemma 7.3), and so we finish off by multiplying the column corresponding to a direction variable by that variable:

<table>
<thead>
<tr>
<th>$\rho_s$</th>
<th>$\hat{\omega}_n$</th>
<th>$\hat{s}$</th>
<th>$y_1$</th>
<th>$w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$1$</td>
<td>$-2$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

**Lemma 7.3.** We can manipulate each row of the derivative matrix so that each entry in a direction column is an integer divided by the variable that column corresponds to.

**Proof.** Suppose we are differentiating by a variable $y$. There are two types of tilde equation to consider. Most of them (we deal with the exceptions below)
are of the form $A - B = 0$, where $y$ appears either as some monomial factor in $A$ or $B$, or as some factor of the form $(1 - \zeta y)$. In the latter case, when we set $\zeta = 0$ after differentiating, any terms we get from this will vanish. Assume the monomial power of $y$ is $y^a$. Then if $y$ appears as a monomial in $A$ we get $\frac{\partial}{\partial y} (A - B) = a \frac{\partial A}{\partial y}$. If $y$ appears as a monomial in $B$ we get $\frac{\partial}{\partial y} (A - B) = -a \frac{\partial B}{\partial y}$. We set $\zeta = 0$, and then divide by $A$, which is equal to $B$ to get $\frac{\partial A}{\partial y}$ or $-\frac{\partial B}{\partial y}$, depending on which of $A$ and $B$ our monomial appeared in.

The second kind of tilde equation to consider is of the form $a + b + \zeta (\cdots) = 0$ (for example $\lambda_1$ of $R^L_L$). Here $a$ and $b$ are direction variables or negatives of them. In this case, the appropriate derivatives yield 1 or $-1$. After setting $\zeta = 0$, we divide the row by $a$, which is equal to $-b$, to get the result.

We note that for the more common kind of tilde equation the integer we see is the power of the monomial if it appears in $A$ or its negative if it appears in $B$. For the second kind of tilde equation we always get a row containing a single 1 and a single $-1$.

Given this lemma, we are justified in multiplying the columns corresponding to direction variables by those variables: the same manipulations will work in all cases. Without further ado, we therefore list the remaining nine cases.

### 7.2.2 $R^S_L$

<table>
<thead>
<tr>
<th>$\hat{w}_m$</th>
<th>$\hat{s}$</th>
<th>$y_1$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_s$</td>
<td>1</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0</td>
<td>1</td>
<td>-2</td>
</tr>
</tbody>
</table>

### 7.2.3 $R^L_L$

<table>
<thead>
<tr>
<th>$\hat{w}_m$</th>
<th>$\hat{s}$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1$</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
</tr>
</tbody>
</table>

### 7.2.4 $R^S_S$

This works out almost identically to $R^L_L$.

<table>
<thead>
<tr>
<th>$\hat{w}_m$</th>
<th>$\hat{s}$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_s$</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
</tr>
</tbody>
</table>
7.2.5 $S_L^L$

Lemma 7.3 applies also in this larger case:

<table>
<thead>
<tr>
<th>$s$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$...$</th>
<th>$y_{n-2}$</th>
<th>$y_{n-1}$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$-1$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$2$</td>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_{n-2}$</td>
<td>$-1$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_{n-1}$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$2$</td>
<td>$-1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_n$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$2$</td>
<td>$-1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7.2.6 $S_R^R$

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>$...$</th>
<th>$w_{m-2}$</th>
<th>$w_{m-1}$</th>
<th>$w_m$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\rho}_1$</td>
<td>$-2$</td>
<td>$1$</td>
<td>$0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\rho}_2$</td>
<td>$1$</td>
<td>$-2$</td>
<td>$1$</td>
<td>$0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\rho}_3$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-2$</td>
<td>$1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\rho}_{n-2}$</td>
<td>$1$</td>
<td>$-2$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\rho}_{n-1}$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-2$</td>
<td>$1$</td>
<td>$0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\rho}_n$</td>
<td>$0$</td>
<td>$1$</td>
<td>$-2$</td>
<td>$1$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7.2.7 $L_S^R$

<table>
<thead>
<tr>
<th>$u$</th>
<th>$w_m$</th>
<th>$s$</th>
<th>$y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_u$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\rho_m$</td>
<td>$0$</td>
<td>$2$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

7.2.8 $L_R^R$

<table>
<thead>
<tr>
<th>$u$</th>
<th>$w_{m-1}$</th>
<th>$w_m$</th>
<th>$s$</th>
<th>$y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_u$</td>
<td>$-2$</td>
<td>$-2$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$\rho_{m-1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\rho_m$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

7.2.9 $L_R^S$

<table>
<thead>
<tr>
<th>$\hat{y}_n$</th>
<th>$w_{m-1}$</th>
<th>$w_m$</th>
<th>$s$</th>
<th>$y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_u$</td>
<td>$-1$</td>
<td>$-2$</td>
<td>$2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$\rho_{m-1}$</td>
<td>$0$</td>
<td>$2$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\rho_m$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

59
7.2.10 \( L^S_L \)

<table>
<thead>
<tr>
<th>( \hat{y}_n )</th>
<th>( w_m )</th>
<th>( s )</th>
<th>( y_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>-1</td>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>( \rho_m )</td>
<td>0</td>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

Notice that for two blocks which connect in the same way in one direction (e.g. \( L^R_S \) and \( L^R_R \) connect in the same way upwards), the parts of the derivative matrix corresponding to that direction (i.e. the terms to the left of the dividing line if upwards, or to the right if downwards) are identical. Notice also that for \( L^*_R \) we can use the \( \hat{\rho} \) rows to kill all terms in the \( \lambda \) row in columns \( w_{m-1}, w_m \) and \( s \). We will assume this move in the following, as well as the analogous move for \( R^L_L \).

7.3 Putting it all together II

We are now ready to show that the derivative matrix is of full rank. As mentioned at the start of section 7.1, we will start with the matrix with \( N \) rows corresponding to the \( N \) tilde equations, and \( N \) variables (not including \( \zeta \)). We need to choose a direction variable to set equal to 1, kill that column then start reducing. First, using Lemmas 7.1 and 7.2 we kill off all rows and columns belonging to the submatrices referenced in those lemmas. As when finding solutions to the equations at \( p \), at the next step we break the problem into three cases:

Case 1: Our blocks describe a path entirely on the left side, made of alternating \( L^S_S \) and \( S^L_L \) blocks.

Case 2: Our blocks describe a path entirely on the right side, made of alternating \( R^S_S \) and \( S^R_R \) blocks.

Case 3: There is at least one place at which the path crosses from one side to the other.

7.3.1 Case 1

Our blocks describe a path entirely on the left side, made of alternating \( L^S_S \) and \( S^L_L \) blocks. We kill one of the \( y_n \) columns. We need to show that we can move across each transition between blocks, row reducing everything as we go. In this case there are only two types of transition to consider, \( L^S_S S^L_L \) and \( S^L_L L^S_S \). We
deal with both of them, and the whole of $S_L^L$ at once:

$$
\begin{array}{cccccccc}
\lambda_0 & -2 & 2 & -1 & 0 & 0 \\
\tilde{\rho}_0 & 2 & -1 & 0 & 0 & 0 \\
\lambda_1 & 0 & -1 & 2 & -1 & 0 \\
\tilde{\lambda}_2 & 0 & 0 & -1 & 2 & -1 & 0 \\
\lambda_3 & 0 & 0 & 0 & -1 & 2 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\tilde{\lambda}_{n-2} & 0 & -1 & 2 & -1 & 0 \\
\tilde{\lambda}_{n-1} & 0 & 0 & -1 & 2 \\
\lambda_n & 0 & 0 & 0 & -1 & 1 \\
\end{array}
$$

We kill this $y_n$ column, then use $\tilde{\lambda}_n$ to clear the $y_{n-1}$ column, and so on, moving upwards. At the top, we have cleared all terms in $\tilde{\lambda}_n$ apart from the $\tilde{y}_n$ term from the next transition up. We use this to continue this process on to the next transition, and continue again until we eventually come around to the start again. The $\tilde{\lambda}_n$ from the transition we started at is the redundant row.

### 7.3.2 Case 2

Our blocks describe a path entirely on the right side, made of alternating $R_S^S$ and $S_L^L$ blocks. This works very similarly to as in Case 1, except we move downwards, starting by killing off a $w_1$ column.

$$
\begin{array}{cccccccccc}
\hat{\rho}_0 & 1 & 0 & 0 \\
\hat{\rho}_1 & -2 & 1 & 0 \\
\hat{\rho}_2 & 1 & -2 & 1 & 0 \\
\hat{\rho}_3 & 0 & 1 & -2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hat{\rho}_{n-2} & 0 & 1 & -2 & 1 & 0 & 0 \\
\hat{\rho}_{n-1} & 1 & -2 & 1 & 0 & 0 \\
\tilde{\rho}_n & 0 & 1 & -2 & 1 & 0 \\
\tilde{\lambda}_1 & 0 & 0 & 1 & -2 \\
\tilde{\rho}_s & 0 & 1 & -2 & 2 \\
\end{array}
$$

### 7.3.3 Case 3

As for when we were showing the existence of a solution at $p_i$ we will kill off a column corresponding to an $s$ variable that sits in between $L_i^N$ and $R_s^L$ blocks.
The join between two such blocks looks like this:

\[
\begin{array}{c|cccc}
\lambda_s & \tilde{w}_{m-1} & w_m & s & y_1 & y_2 \\
\tilde{\rho}_{m-1} & 0 & 0 & 0 & 1 & 0 \\
\tilde{\rho}_m & 2 & -1 & 0 & 0 & 0 \\
\tilde{\lambda}_1 & 1 & 0 & -1 & 0 & 0 \\
\tilde{\lambda}_2 & 0 & 0 & 1 & 0 & -1 \\
\tilde{\rho}_s & 0 & 0 & 0 & 1 & -2 \\
\end{array}
\]

We kill the \( s \) column. We now have a "free" non zero term in the \( y_2 \) column (in the \( \tilde{\lambda}_1 \) row), which we use to clear that column. Then we use the \( \tilde{\lambda}_2 \) row to clear the \( y_1 \) column. The \( \tilde{\rho}_m \) row we use to clear the \( w_{m-1} \) column, which allows us to use the \( \tilde{\rho}_{m-1} \) row to clear the \( w_m \) column. Having done this, we have all of the direction variable columns shown here cleared or otherwise accounted for, and we have yet to use \( \lambda_s \) or \( \tilde{\rho}_s \). The plan of action is to move either upwards (or downwards) from this starting point, first using \( \lambda_s \) (or \( \tilde{\rho}_s \)). We will go through the various blocks, reducing the matrix as we go. Eventually we will come back around to our starting point, and will not need to use the last equation, \( \tilde{\rho}_s \) (or \( \lambda_s \)). We choose to move upwards.

We again need to show that we can move across each transition between blocks, row reducing everything as we go. There are again (as for when we were finding a solution at \( p \)) the 6 transitions to consider, although we will group things slightly differently this time:

\[
L^*_R R^*_s \quad \text{This is the same as from where we started:}
\]

\[
\begin{array}{c|cccc}
\lambda_s & \tilde{w}_{m-1} & w_m & s & y_1 & y_2 \\
\tilde{\rho}_{m-1} & 0 & 0 & 0 & 1 & 0 \\
\tilde{\rho}_m & 2 & -1 & 0 & 0 & 0 \\
\tilde{\lambda}_1 & 1 & 0 & -1 & 0 & 0 \\
\tilde{\lambda}_2 & 0 & 0 & 1 & 0 & -1 \\
\tilde{\rho}_s & 0 & 0 & 0 & 1 & -2 \\
\end{array}
\]

As we are moving upwards, we have that the other end of \( \tilde{\rho}_s \) has already been cleared, and so the \(-1\) in the \( w_m \) column is free. We use this to clear that column, which lets us use \( \tilde{\rho}_{m-1} \) to clear the \( w_{m-1} \) column. Then \( \tilde{\rho}_m \) to clear \( s \), \( \tilde{\lambda}_1 \) to clear \( y_2 \) then \( \tilde{\lambda}_2 \) to clear \( y_1 \). \( \tilde{\lambda}_s \) is now clear on this end, so we are free to use it on the next transition upwards.
We have rearranged the rows slightly from their positions when we were dealing with Case 2 in order to better show the order of operations. \( \tilde{\rho}_* \) is "free" (meaning that it has no other non-zero terms in any columns not in the part shown), since the other end has already been cleared. This is also true for all other rows here, apart from \( \tilde{\rho}_* \). We leave \( \tilde{\rho}_* \) aside for now, and show that the remaining submatrix has determinant non-zero. After some row operations, we get:

\[
\begin{array}{cccccccccc}
\tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \cdots & \tilde{\rho}_{m-2} & \tilde{\rho}_{m-1} & \tilde{\rho}_m & s & \tilde{y}_1 \\
-\frac{2}{3} & 1 & 0 & 0 & & & & & \\
0 & -\frac{2}{3} & 1 & 0 & & & & & \\
0 & 0 & -\frac{2}{3} & 1 & & & & & \\
\vdots & \vdots & \vdots & \ddots & & & & & \\
0 & 0 & 0 & -\frac{n+1}{n} & 1 & 0 & 0 & & \\
0 & 0 & -\frac{n+1}{n} & 1 & 0 & 0 & & & \\
0 & -\frac{n+2}{n+1} & 2 & & & & & & \\
0 & -\frac{n+2}{n+2} & & & & & & & \\
\end{array}
\]

so the determinant is non-zero. Thus this submatrix can be row reduced, clearing all of the variables shown, and freeing up \( \tilde{\rho}_* \) so we can carry on moving upwards.
Another pair of transitions:

\[
\begin{array}{cccccccc}
\lambda_s & -2 & 2 & -1 & 0 & 0 & \cdots & y_{n-2} & y_{n-1} & y_n \\
\rho_m & 2 & -1 & 0 & 0 & 0 & & & & \\
\tilde{\lambda}_1 & 0 & -1 & 2 & -1 & 0 & & & & \\
\tilde{\lambda}_2 & 0 & 0 & -1 & 2 & -1 & 0 & & & \\
\tilde{\lambda}_3 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & & \\
\vdots & & & & & & & & & \\
\tilde{\lambda}_{n-2} & & & & & & & & \ddots & \\
\tilde{\lambda}_{n-1} & 0 & & & \vdots & & & & & \\
\tilde{\lambda}_n & 0 & & & & & & & & \\
\end{array}
\]

This one is easier, and works just as in Case 1. \( \tilde{\lambda}_s \) is free, so we use it to clear the \( y_n \) column, and continue upwards. Finally we use \( \rho_m \) to clear \( \tilde{w}_m \), and \( \tilde{\lambda}_s \) is free.

\[
R^L_1 L^R_*
\begin{array}{c|c}
\rho_s & 2 \\
\tilde{\lambda}_s & -2 \\
\end{array}
\]

\( \tilde{\lambda}_s \) clears the \( u \) column, and \( \rho_s \) is free.

We are able to make all transitions, clear all columns as we go, and always have the next row free. When we finally get back to the start again, that last free row is the redundant one, and we are done.

References


