Points nearby

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**Proposition 1.** If \( p \in \mathbb{C}^N \) satisfies polynomial equations \( f_1, f_2, \ldots, f_{N-1} \in \mathbb{C}[x_1, x_2, \ldots, x_N] \) then there exist other solutions to these equations arbitrarily close to \( p \).

The heuristic reason for this is that starting from \( \mathbb{C}^N \), every polynomial we add to our set of equations cuts down the dimension of the set of solutions by at most one (unless it results in an inconsistent set of equations). Since we only make \( N-1 \) cuts, and started with \( N \) dimensions, we will have at least one left by the end. \( p \) demonstrates the existence of solutions.

Here is a more formal proof:

**Proof.** Let \( p \) be an irreducible component of \((f_1, f_2, \ldots, f_{N-1})\), the ideal generated by the polynomials. \( \mathbb{C}[x_1, x_2, \ldots, x_N] \) has transcendence degree \( N \), so by (for example) Ex 18 p412 of Lang, every maximal chain of prime ideals

\[
\mathbb{C}[x_1, x_2, \ldots, x_N] \supset p_1 \supset p_2 \supset \cdots \supset p_M \supset \{0\}
\]

with \( \mathbb{C}[x_1, x_2, \ldots, x_N] \neq p_1, p_i \neq p_{i+1}, p_M \neq \{0\} \) must have \( M = N \). Thus any such chain that passes through \( p \) must have the same property, and so

\[
N = \dim \frac{\mathbb{C}[x_1, x_2, \ldots, x_N]}{p} + \text{ht}(p)
\]

Corollary 11.16 of Atiyah MacDonald states that every minimal ideal \( p \) belonging to (i.e. a prime factor of) \((f_1, f_2, \ldots, f_{N-1})\) has \( \text{ht}(p) \leq N - 1 \), and so

\[
\dim \frac{\mathbb{C}[x_1, x_2, \ldots, x_N]}{p} \geq 1
\]

Let \( m \supset p \) be a maximal ideal. A form of Hilbert’s Nullstellensatz, Corollary 5.24 of Atiyah MacDonald, gives us that since \( \dim \frac{\mathbb{C}[x_1, x_2, \ldots, x_N]}{m} \) is a finitely generated \( \mathbb{C} \)-algebra, and \( m \) is maximal, then \( \dim \frac{\mathbb{C}[x_1, x_2, \ldots, x_N]}{m} \) is a field and hence a finite algebraic extension of \( \mathbb{C} \), so it is \( \mathbb{C} \). Hence \( m = (x_1 - a_1, x_2 - a_2, \ldots, x_N - a_N) \) so every point of \( \frac{\mathbb{C}[x_1, x_2, \ldots, x_N]}{m} \) is an actual point in \( \mathbb{C}^N \) and thus a solution to our polynomial equations.

\[\square\]