Lemma 1. $y_k = \frac{1 - \cos kb}{1 - \cos b}$ is a solution of $y_{k-1}y_{k+1} - (y_k - 1)^2 = 0$

Proof. Let $a = \frac{1 - \cos b}{1 - \cos b}$ (so $1 - \frac{1}{a} = \cos(b)$).

\[
y_{k-1}y_{k+1} - (y_k - 1)^2 = a \left( 1 - \cos ((k - 1)b) \right) a \left( 1 - \cos ((k + 1)b) \right) - (a(1 - \cos kb) - 1)^2
\]

\[
= a^2 \left( (1 - \cos (kb - b))(1 - \cos (kb + b)) - \left( 1 - \frac{1}{a} - \cos kb \right)^2 \right)
\]

\[
= a^2 \left( (1 - (\cos kb \cos b + \sin kb \sin b))(1 - (\cos kb \cos b - \sin kb \sin b)) - (\cos b - \cos kb)^2 \right)
\]

\[
= a^2 \left( (1 - 2 \cos kb \cos b + (\cos kb \cos b)^2) - (\sin kb \sin b)^2 - (\cos b - \cos kb)^2 \right)
\]

\[
= a^2 \left( 1 - 2 \cos kb \cos b + (\cos kb \cos b)^2 - (1 - \cos^2 kb)(1 - \cos^2 b) - (\cos b - \cos kb)^2 \right)
\]

\[
= a^2 \left( 1 - 2 \cos kb \cos b + (\cos^2 kb + \cos^2 b) - (\cos b - \cos kb)^2 \right)
\]

\[
= 0
\]

In our case, we want solutions with $y_1 = 1$ and $y_{m+1} = 1$. The first equation is automatically true for this form of solution, and the second may be satisfied by choosing $b = \frac{2\pi}{m}$. Other solutions can be found without too much trouble, but this one in particular gives us that $y_k > 1$ for all $1 < k < m + 1$, and thus $(\frac{1}{y_k})x_k < 1$.

Lemma 2. The matrix:

\[
\begin{bmatrix}
\frac{2}{x_2-1} & 1 & 0 \\
\frac{2}{x_3-1} & 1 & \ddots \\
\frac{2}{x_{m-1}-1} & \frac{2}{x_m-1} & 1
\end{bmatrix}
\]

has non-zero determinant with the $x_k$ chosen as above.
Proof. We prove the more general statement, that the matrix:

\[
\begin{bmatrix}
\gamma_2 & 1 & & & \\
1 & \gamma_3 & 1 & & \\
& & \ddots & \ddots & \\
0 & 1 & \gamma_{m-1} & 1 & \\
& & & 1 & \gamma_m
\end{bmatrix}
\]

has non zero determinant, when \( \gamma_k < -2 \) for all \( k \). Note that this is true of the above matrix if \( 0 < x_k < 1 \).

We use row operations to make the matrix upper triangular. After one such move, we have:

\[
\begin{bmatrix}
\gamma_2 & 1 & & & \\
0 & \gamma_3 - \frac{1}{\gamma_2} & 1 & & \\
& & \ddots & \ddots & \\
0 & 1 & \gamma_{m-1} & 1 & \\
& & & 1 & \gamma_m
\end{bmatrix}
\]

then:

\[
\begin{bmatrix}
\gamma_2 & 1 & & & \\
0 & \gamma_3 - \frac{1}{\gamma_2} & 1 & & \\
0 & \gamma_4 - \frac{1}{\gamma_3 - \frac{1}{\gamma_2}} & 1 & & \\
& & \ddots & \ddots & \\
0 & & & \ddots & 1 & \\
& & & & 1 & \gamma_{m-1} & 1 & \\
& & & & & 1 & \gamma_m
\end{bmatrix}
\]

and so on. The determinant is non-zero as long as none of the diagonal entries (these continued fractions) are zero. We use induction, assuming that the \( k \)th continued fraction is less than \(-1\). This is true for the first, \( \gamma_2 \), since by assumption it is less than \(-2\). Now if the \((k - 1)\)th continued fraction is less than \(-1\), then the \( k \)th is \( \gamma_k \) (less than \(-2\)) plus something less than 1, hence the \( k \)th continued fraction is also less than \(-1\). \( \square \)