Problem 21, Chapter 1

This problem asks for a description of the universal covering space of the space X obtained from the torus $S^1 \times S^1$ by attaching a Möbius band M using an identification of the boundary circle of M with the circle $S^1 \times \{0\}$. We will think of S^1 as the quotient group of the real numbers by the subgroup of integers. To see what this is in two stages, we first construct a two fold cover which is easier to understand. It will be defined as a quotient space of

$$W = A \coprod B \coprod C = S^1 \times S^1 \coprod S^1 \times I \coprod S^1 \times S^1$$

where $I = [0,1]$. W will be equipped with a free action by the group $G = \mathbb{Z}/2\mathbb{Z} = \{e, T\}$, so that the quotient is homeomorphic to the space X.

We first define a G action on W with the following conditions.

1. The action is defined on $S^1 \times I$ by

$$T([r], t) = ([r + \frac{1}{2}], 1 - t)$$

   It is easy to observe that M is the quotient of $S^1 \times I$ by this action.

2. The G action permutes A and C.

3. The action of T on A is given by $T([r],[s]) = ([r + \frac{1}{2}],[s])$, where the right side of the equality is understood to lie in B. The action of T on B is by the same formula, where again it is understood that the right hand side describes a point in A.

We next define the equivalence relation $\simeq$ on W via the following conditions.

1. $S^1 \times \{[0]\} \subseteq A$ is identified with $S^1 \times \{0\} \subseteq S^1 \times I$ via

$$([r],[0]) \simeq ([r], 0)$$

2. $S^1 \times \{[0]\} \subseteq B$ is identified with $S^1 \times \{1\} \subseteq S^1 \times I$ via

$$([r],[0]) \simeq ([r], 1)$$

It is clear that $\simeq$ is invariant under the G-action, and that the action is free. Moreover, the identifications are such that they preserve the product decompositions with the first S^1-factor, and that therefore W is homeomorphic to $S^1 \times W^0$, where W^0 is the identification space

$$A^0 \coprod B^0 \coprod C^0 = S^1 \coprod I \coprod S^1 / \simeq^0$$

where $\simeq^0$ glues the point $[0] \in A$ to the point $0 \in I$ and the point $[0] \in C$ to the point $1 \in I$. W^0 is clearly a graph, and its universal cover is the infinite tree T associated to a bouquet of two circles.

Since universal covering spaces of products are homeomorphic to products of universal covering spaces, we have that $W \cong \mathbb{R} \times T$.

It remains to show that the orbit space of the G-action on W is homeomorphic to X. We define a map

$$\pi : W \to X$$

as follows.

1. The torus A maps by the identity to the torus in the definition of X, and C maps to that torus by the map $([r],[s]) \to ([r+\frac{1}{2}],[s])$.
2. B maps to M by the identification map.

It is clear that the identifications defining W are respected by this map. Further, π factors through a map $\overline{\pi}$ defined on the orbit space W/G, since it is clear that $\pi(Tw) = \pi(w)$ for all $w \in W$. The map $\overline{\pi}$ is easily seen to be a bijection on points, and further both W/G and X are compact Hausdorff spaces, so it follows that $\overline{\pi}$ is a homeomorphism. Therefore, the universal covering space of X is homeomorphic to the universal covering space of W/G, which is identical to the universal covering space of W.