Derived Representation Theory and the Algebraic K-theory of Fields

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1 Introduction

Quillen's higher algebraic K-theory for fields F has been the object of intense study since their introduction in 1972 [26]. The main direction of research has been the construction of "descent spectral sequences" whose E_2-term involved the cohomology of the absolute Galois group G_F with coefficients in the algebraic K–theory of an algebraically closed. The form of such a spectral sequence was conjectured in [28] and [19], but it was soon realized that it could not be expected to converge to algebraic K exactly. However, it appeared likely that it could converge to algebraic K-groups in sufficiently high dimensions, i.e. in dimensions greater than the cohomological dimension of G_F. This observation was formalized into the *Quillen-Lichtenbaum conjecture*. This conjecture has attracted a great deal of attention. Over the years, a number of special cases [36], [16] have been treated, and partial progress has been made [9]. Voevodsky in [41] proved results which led to the verification of the conjecture at $p = 2$. More recent work of V. Voevodsky appears to have resolved this long standing conjecture at odd primes as well. Moreover, it also appears to resolve the Beilinson-Lichtenbaum conjecture about the form of a spectral sequence which converges exactly to the algebraic K-theory of F.

The existence of this spectral sequence does not, however, provide a homotopy theoretic model for the algebraic K-theory spectrum of the field F. It is the goal of this paper to propose such a homotopy theoretic model, and to verify it in some cases. We will construct a model which depends only on the complex representation theory (or twisted versions, when the roots of unity are not present in F) of G_F, without any other explicit arithmetic information about the field F, and is therefore contravariantly functorial in G_F. It is understood in algebraic topology that explicit space level models for spaces and spectra are generally preferable to strictly algebraic calculations of homotopy groups for the spaces. Knowledge of homotopy groups alone does not allow one to understand the behavior of various constructions and maps of spectra in the explicit way. In this case, there are specific reasons why one would find such a model desirable.

- It appears likely that one could begin to understand *finite* descent prob-

[1]Supported in part by NSF DMS 0104162

lems, since the behavior of restriction and induction maps for subgroups of finite index is relatively well understood.

- It appears that the Milnor K-groups should be identified with the homotopy groups of the so-called *derived completion* of the complex representation ring of G_F. This would relate an explicitly arithmetic invariant (Milnor K-theory) with an explicitly group-theoretic invariant coming out of the representation theory. The Bloch-Kato conjecture provides one such connection, relating Milnor K-theory with Galois cohomology. These ideas provide another one, arising out of the representation theory of the group rather than from cohomology.

- It is well understood that representation theory of Galois groups plays a key role in many problems in number theory. This construction provides another explicit link between representation theory and arithmetic in fields which it would be very interesting to explore.

The idea of the construction is as follows. We assume for simplicity that F contains an algebraically closed subfield. A more involved and more general version of the construction is discussed in the body of the paper. Let F be a field, with algebraic closure $\overline{F}$, and define the category of *descent data* for the extension $F \subset \overline{F}$ (denoted by $V^G(\overline{F})$) to have objects the finite dimensional $\overline{F}$-vector spaces V with G_F-action satisfying $\gamma(\overline{f}v) = \overline{f}^{\gamma}\gamma(v)$, with equivariant $\overline{F}$-linear isomorphisms. It is standard descent theory [15] that this category is equivalent to the category of finite dimensional F-vector spaces. On the other hand, let $Rep_F[G_F]$ denote the category of finite dimensional continuous representations of the profinite group G_F. There is a canonical homomorphism from $Rep_F[G_F]$ to $V^G(\overline{F})$ obtained by applying $\overline{F} \underset{F}{\otimes} -$ and extending the action via the Galois action of G_F on $\overline{F}$. We then have the composite

$$A_F : KRep_k[G_F] \longrightarrow KRep_F[G_F] \longrightarrow KV^G(\overline{F})$$

This map is of course very far from being an equivalence, since $\pi_0 KRep_k[G_F]$ is isomorphic to $R[G_F]$, a non-finitely generated abelian group, and $\pi_0 KF \cong \mathbb{Z}$. However, we observe that both spectra are *commutative S-algebras* in the sense of [11], and further that the map A_F is a homomorphism of commutative S-algebras. We describe in this paper a derived version of completion, which is applicable to any homomorphism of commutative S-algebras. For such a homomorphism $f : A \to B$, we denote this derived completion by $A^{\wedge}_f$. We also let $\mathbb{H}$ denote the mod-p Eilenberg spectrum. $\mathbb{H}$ is also a commutatve S-algebra, and we have an evident commutative diagram of commutative S-algebras

$$\begin{array}{ccc} KRep_k[G_F] & \xrightarrow{A_F} & KV^G \simeq KF \\ \downarrow{\scriptstyle \varepsilon_p} & & \downarrow{\scriptstyle \varepsilon_p} \\ \mathbb{H} & \xrightarrow{id} & \mathbb{H} \end{array}$$

The vertical maps are induced by forgetful functors which forget the vector space structure and only retain the dimension mod p. The completion construction is natural for such commutative squares, and we obtain a homomorphism $\alpha_F : KRep_k[G_F]^{\wedge}_{\varepsilon_p} \longrightarrow KF^{\wedge}_{\varepsilon_p}$. Another property of the completion construction shows that the spectrum $KF^{\wedge}_{\varepsilon_p}$ is naturally equivalent to the usual p-completion of the spectrum KF. We prove in this paper α_F is a weak equivalence in the case of finitely generated abelian absolute Galois groups. This is the desired explicit model for the homotopy type of the spectrum KF. We wish to make a few comments about this result.

- Although the result as stated can be formulated for any absolute Galois group, we do not expect it to hold for non-abelian groups. In order to formulate a conjecture which should hold generally, one has to recognize that both of the spectra $KRep_k[G_F]$ and KF are actually fixed points of equivariant spectra, and that the commutative square above can be induced as the fixed point square of a corresponding commutative square of equivariant spectra. The correct result should now be obtained by performing an equivariant completion, and restricting to fixed point sets. This is definitely a different construction from simply completing the fixed point sets, but it does preserve the crucial property that the completion on the right hand side is still the p-adic completion of KF.

- Actual commutative rings can be regarded as ring spectra by the Eilenberg-MacLane construction, and one can consequently construct derived completions of rings along homomorphisms between commutative rings. When the rings in question are Noetherian and the homomorphism is surjective, the result of derived completion is just ordinary completion. However, when the rings are not Noetherian, the result of completion is actually a ring spectrum, which may have higher homotopy groups corresponding to higher derived functors of completion. One consequence of the constructions in this paper is that one obtains a homomorphism from the higher homotopy groups of the derived completion of the representation ring (or perhaps Green functor) to the Milnor K-theory of the field F. Such a homomorphism can be regarded as another natural generalization to higher dimensions of Kummer theory.

- Although the derived completion construction depends explicitly only on the absolute Galois group G_F, it is not particularly geometric in character.

One may attempt to obtain a more geometric interpretation by studying K-theory spectra which take into account the possibility of *deformations* of representations of groups. The most geometric version of such constructions is applicable to infinite discrete groups, and is described in section 4.6. For an infinite discrete group Γ, we would write $K^{def}(\Gamma)$ for this construction. On obtains a map $K^{\hat{\Gamma}}(*) \rightarrow K^{def}(\Gamma)$, where $K^{\hat{\Gamma}}(*)$ denotes the equivariant K-theory of a point for the group $\hat{\Gamma}$, the profinite completion of the group Γ, defined as the direct limit of the equivariant K-theory spectra of the finite quotients of Γ. This map appears to be an equivalence after completion for abelian groups, encouraging the conjecture that the derived completion of the K-theory of representations of the profinite completion of Γ should be equivalent to the deformation K-theory of Γ for a large number of groups. Absolute Galois groups are of course not in general profinite completions of discrete groups, but one can attempt to construct a p-adic version of deformation K-theory $K^{def,p}(G)$ using $\overline{\mathbb{Q}_p}$ instead of $\mathbb{C}$, and to prove that for p-profinite groups G we have

$$KRep_k[G]^{\wedge}_{\varepsilon_p} \cong K^{def,p}(G)$$

If one proved the conjectures concerning KF described above, we would ultimately obtain

$$KF^{\wedge}_p \cong K^{def,p}(G_F)$$

This goal will be the subject of future work in this direction.

The ultimate hope is that the clarification of the relationship between arithmetically defined descriptions of algebraic K-theory, such as the motivic spectral sequence, with descriptions which involve the Galois group and its representation theory directly, will shed more light on arithmetic and algebraic geometric questions.

The author wishes to express his thanks to a number of mathematicians for useful conversations, including D. Bump, R. Cohen, W. Dwyer, A. Elmendorf, E. Friedlander, L. Hesselholt, M. Hopkins, W.C. Hsiang, J.F. Jardine, M. Levine, J. Li, I. Madsen, M.Mandell, J.P. May, H. Miller, J. Morava, F. Morel, K. Rubin, C. Schlichtkrull, V. Snaith, R. Thomason, R. Vakil, and C. Weibel.

2 Preliminaries

We assume the reader to be familiar with the category of spectra, as developed in [10] and [11] or [17]. In either of these references, the category of spectra is shown to possess a coherently commutative and associative monoidal structure, called smash product. Of course, on the level of homotopy categories, this monoidal structure is the usual smash product. The presence of such a monoidal structure makes it possible to define *ring spectra* as monoid objects in $\underline{Spectra}$. It is also possible to define the notion of a commutative ring spectrum, with appropriate higher homotopies encoding the commutativity as well as associativity and distributivity. In line with the terminology of [11], we will refer to these objects as S-algebras and commutative S-algebras, respectively. For the relationship with earlier notions of A_∞ and E_∞ ring spectra, see [11].

The monoidal structure on $\underline{Spectra}$ also makes it possible to define the notion of a left or right R-module, where R is either an A_∞ or E_∞ ring spectrum. By analogy with the commutative algebra setting, it is also possible to define spectra $Hom_R(M, N)$ where M and N are left or right R-modules, and $M \underset{R}{\wedge} N$ when M is a right R-module and N is a left R-module. Of course, $\underset{R}{\wedge}$ is analogous to algebraic $\underset{R}{\otimes}$. The usual adjoint relationships familiar from the algebraic setting hol, and composition of maps is associative.

For certain classes of modules, it is possible to construct spectral sequences for computing the homotopy groups of Hom and smash product spectra. To understand this, we first observe that since the category of modules over an S-algebra R admits mapping cylinder and mapping cones, it should be thought of as analogous to the *derived category* of chain complexes of modules over a ring R. In the derived category of modules, a special role is played by by projective complexes. The analogous condition condition for modules is the condition of *cofibrancy*. Just as it is always possible to replace a any chain complex over a ring with a weakly equivalent projective one (a weak equivalence here means a chain map inducing an isomorphism on homology), so it is always possible to replace any R-module with a weakly equivalent cofibrant one, where a weak equivalence is an R-module map inducing an isomorphism on homotopy groups.

Proposition 2.1 *Let R be an S-algebra, and suppose that M and N are both left or right R-modules. Suppose further that M is cofibrant. Then there is a spectral sequence with*

$$E_2^{p,q} \cong Ext_{\pi_* R}^{-p,q}(\pi_*(M), \pi_*(N))$$

converging to $\pi_{p+q}(Hom(M, N))$. This spectral sequence will be known as the **Universal coefficient spectral sequence** *for this situation. Similarly, if M is a right R-module and N is a left R-module, and one of M and N is cofibrant, there is a spectral sequence with*

$$E_2^{p,q} \cong Tor_{p,q}^{\pi_*(R)}(\pi_*(M), \pi_*(N))$$

converging to $\pi_{p+q}(M \underset{R}{\wedge} N)$. In both cases, the q-index refers to the internal *grading coming from the graded structure on π_*, and the p-index refers tothe homological degree. This spectral sequence is referred to as the* **Künneth** *spectral sequence for this situation.*

Remark: Throughout this paper, all *Hom* and smash product spectra will be computed using only cofibrant modules. If a module is not cofibrant by construction, we will always replace it with a weakly equivalent cofibrant model. We will sometimes do this without comment.

An important method for constructing spectra from combinatorial data is via *infinite loop space machines* (see [22] or [31]), which are functors from the category of symmetric monoidal categories to spectra. The algebraic K-theory functor is a prime example of this construction, since it can obtained by applying an infinite loop space machine to the symmetric monoidal category of finitely generated projective modules over a ring. When the symmetric monoidal category has a coherently associative and distributive second monoidal structure, such as tensor product of modules, the spectrum constructed by an infinite loop space machine will have the structure of an S-algebra. If in addition the second monoidal structure is coherently commutative, the spectrum will be a commutative S-algebra. See [11] or [17] for these results. Since the tensor product of finitely generated projective modules over a commutative ring R is coherently commutative, we have

Proposition 2.2 *For any commutative ring A, the spectrum KA is equipped with a commutative S-algebra structure in a canonical way.*

We also observe, as is pointed out in [11], that given an S-algebra A and a simplicial group G, one may construct the group ring spectrum $A[G]$. For our purposes, we will make this construction in a particular way, which will be useful later. For any group G, let $C[G]$ denote the category whose objects are finite sets X equipped with a reference map $\varphi : X \to G$, and whose morphisms are isomorphisms of finite sets respecting the reference maps. $C[G]$ is a symmetric monoidal category with disjoint union as its notion of sum. It is also equipped with a multiplicative monoidal structure defined by $(X, \varphi_X) \times (Y, \varphi_Y) = (X \times Y, \varphi_{X \times Y})$, where $\varphi_{X \times Y}((x, y) = \varphi_X(x)\varphi_Y(y)$. The spectrum associated to this construction is itself an S-algebra, and it is a model for the group ring $S[G]$, where S denotes the sphere spectrum. For any S-algebra A, we will define the group ring $A[G]$ to be the S-algebra $A \underset{S}{\wedge} S[G]$. If G is an abelian group, and A is a commutative S-algebra, then $A[G]$ is a commutataive S-algebra in an obvious way. Moreover, if $G.$ is a simplicial group, then we obtain a simplicial S-algebra $A[G.]$, for which we may form the total S-algebra, which we will denote by $|A[G.]|$, and which is the right notion for the group ring of the given simplicial group. We have the following proposition about smash products over group rings.

Proposition 2.3 *Suppose that we are given a commutative S-algebra A and an abelian simplicial group $G.$, and a homomorphism $A \to B$ of commutative S-algebras. Suppose further that $i : G_. \to E_.$ is an inclusion of simplicial groups, and that $E.$ is contractible. Then the commutative S-algebra*

$$\underbrace{B \underset{|A[G.]|}{\wedge} B \underset{|A[G.]|}{\wedge} \cdots \underset{|A[G.]|}{\wedge} B}_{k \text{ factors}}$$

is equivalent to the commutative S-algebra

$$|\underbrace{B \underset{A}{\wedge} B \underset{A}{\wedge} \cdots \underset{A}{\wedge} B}_{k \text{ factors}}[E_.^k / G_.^{k-1}]|$$

where $G_.^{k-1}$ is included in $E_.^k$ via the homomorphism $(g_1, \ldots, g_{k-1}) \to (i(g_1), i(g_2) - i(g_1), i(g_3) - i(g_2), \ldots, i(g_{k-1}) - i(g_{k-2}), i(g_{k-1}))$. Note that the quotient group is a model for $BG_.^{k-1}$.

Proof: Immediate from the definitions. □

We also want to remind the reader of the construction by Bousfield-Kan of the l-completion of a space (simplicial set) at a prime l, $X_l^\wedge$. Bousfield and Kan construct a functorial cosimplicial space $T_l^{\cdot} X$, and define the l-completion of X to be $Tot(T_l^{\cdot})$. This construction gives rise to a functorial notion of completion.

We refer the reader to [26] for results concerning K-theory spectra, notably the localization, devissage, and reduction by resolution theorems. These theorems apply equally well to the completed versions $KA^\wedge$. We will also recall Suslin's theorem [35].

Theorem 2.4 *Let $k \to F$ be an inclusion of algebraically closed fields of characteristic $p \neq l$ (p may be 0). The the natural map $Kk \to KF$ induces an equivalence $Kk_l^\wedge \to KF_l^\wedge$. In fact, the proof shows that the map of pro-spectra $Kk^\wedge \to KF^\wedge$ is a weak equivalence in the sense that it induces an isomorphism of homotopy pro-groups.*

We also will free to use the standard results concerning higher algebraic K-theory, such as localization sequences, devissage, reduction by resolution, etc., as presented in [26].

We also recall the ideas of equivariant stable homotopy theory. See for example [14] or [7] for information about this theory. In summary, for a finite group G, there is a complete theory of G-equivariant spectra which includes suspension maps for one-point compactifications of all orthogonal representations of G. The proper analogue of homotopy groups takes its values in the abelian category of *Mackey functors*, which is a suitably defined category of diagrams over a category whose objects are finite G-sets, and whose morphisms include maps of G-sets, and also transfer maps attached to orbit projections. The category of Mackey functors admits a coherently commutative and associative tensor product, which is denoted by $\square$. Consequently, by analogy with the theory of rings, we define a *Green functor* to be a monoid object in the category of Mackey functors. The theory of ring functors has analogues for most of the standard theorems and constructions of ring theory. In particular, it is possible to define ideals, modules, completions, tensor products of modules over a Green functor, and modules $Hom_R(M, N)$ for any Green functor and modules M and N over R. The category of modules over a Green functor also has enough projectives, so homological algebra can be carried out in this category. It is also easy to verify that by passing to direct limits, it is possible to directly extend the ideas about Mackey and Green functors to profinite groups. See [3] for background material about Mackey and Green functors.

As mentioned above, the Mackey functor valued analogue of homotopy groups plays the same role for equivariant spectra that ordinary homotopy groups play for spectra. For example, it is shown in [18] that there exist Eilenberg-MacLane spectra attached to every Mackey functor. Moreover, a theory of ring and module spectra in the equivariant category has been developed by May and Mandell [20] and [24] in such a way that the Mackey-functor valued homotopy group becomes a Green functor in an evident way, and that the usual spectral sequences (Künneth and Universal coefficient) hold for modules in this category.

3 Completions

In this section we will discuss the properties of a completion construction in the category of ring spectra and modules, which is analogous to the completion construction in the category of rings. We will first motivate the construction.

In [4], Bousfield and Kan defined the completion of a space (actually simplicial set) at a prime l. The construction went as follows. Recall first that a *monad* in a category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$, equipped with natural transformations $\epsilon : Id \to T$ and $\mu : T^2 \to T$, satisfying various compatibility conditions. See [22] for details on these conditions.

Example 1: $\mathcal{C}$ is the category of based sets, and $T = G$, the *free group functor*, which assigns to any set the free group on that set, with the base point set equal to the identity.

Example 2: $\mathcal{C}$ is the category of based sets, and $T = F^{ab}$, the *free abelian group functor* which assigns to any set the free abelian group on that set, with the base point set equal to zero.

Example 3: $\mathcal{C}$ is the category of based sets, and $T = F_{\mathbb{F}_l}$, the *free* $\mathbb{F}_l$*-vector space functor.*

Example 4: Let $\mathcal{C}$ be the category of left R-modules, for a ring with unit R. Let A be an R-algebra. Let T be the functor $M \to A \underset{R}{\otimes} M$.

Example 5: Let R be an S-algebra, and let $\mathcal{C}$ be the category of left R-modules. Let A be an R-algebra. Then the functor $M \to A \underset{R}{\wedge} M$ is a monad on the category of left R-module .

In all cases, it is easy to see what ϵ and μ should be. Note that all functors can be extended to monads on the associated category of simplicial objects in the relevant category. Given any monad on a category $\mathcal{C}$ and object $c \in \mathcal{C}$, we can associate to it a cosimplicial object in $\mathcal{C}$, $T^{\cdot}(c)$. The object in codimension k is the $(k+1)$-fold iterate of the functor T, the coface maps are given by applying natural transformations of the form $T^l(\epsilon)$ to the object $T^{k-l+1}(c)$, and the codegeneracies are given by applying natural transformations of the form $T^l(\mu)$ to objects of the form $T^{k-l-1}(c)$ There is an evident natural transformation from the constant cosimplicial object with value c to $T^{\cdot}(c)$, induced by ϵ. See [6] for details of this construction.

Definition 3.1 Bousfield-Kan; [4]) *Let $X_.$ be a simplicial set, and let l be a prime. Then by the l-completion of $X_.$, $X_l^{\wedge}$, we will mean the total space of the cosimplicial simplicial set attached to the monad $F_{\mathbb{F}_l}$. l-competion becomes a functor from the category of simplicial sets to itself, and it is equpped with a natural transformation $X_. \to X_l^{\wedge}$. We will also write $X_.^{\wedge}$ for the pro-simplicial set obtained by applying Tot^k to this cosimplicial simplicial set for each k.*

We can extend this completion construction to spectra by first applying the construction to each space in the spectrum to obtain a prespectrum, and then applying the "spectrification" functor to obtain an Ω-spectrum. This creates a completion functor on the category of spectra. Let $\mathbb{H}$ denote the mod-l Eilenberg-MacLane spectrum. $\mathbb{H}$ is an algebra over the sphere spectrum S^0, and consequently we may consider the monad $X \to \mathbb{H} \underset{S^0}{\wedge} X$, as in example 5 above. There is a canonical equivalence between these two monads, since mod l spectrum homology of a spectrum X can be computed either by computing the homotopy groups of the functor $F_{\mathbb{F}_l}(X)$ or by computing the homotopy groups of the spectrum $\mathbb{H} \underset{S^0}{\wedge} X$. We may now define a second completion functor, which assigns to a spectrum X the total space of the cosimplicial object $S^{\cdot}(X)$, where S is the monad $X \to \mathbb{H} \underset{S^0}{\wedge} X$. These two completion constructions are equivalent, and suggests to us the following definition.

Definition 3.2 *Let R be an S-algebra, and let A be an R-algebra, with defining ring spectrum homomorphism $f\colon R \to A$. Let T_A denote the monad on the* *category of left R-module spectra defined by*

$$M \longrightarrow A \underset{R}{\wedge} M$$

We now define the completion of a left R-module M along the map f, $M_f^\wedge$, to be the total space of the cosimplicial spectrum $T_A^\cdot(M)$. This construction is clearly functorial in M, and is functorial for maps of S-algebra homomorphisms in the following sense. Suppose that we have a commutative square of S-algebra homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ R' & \xrightarrow{f'} & A' \end{array}$$

Then for any R-module M, we have a canonical homomorphism $M_f^\wedge \longrightarrow (R' \underset{R}{\wedge} M)^\wedge_{f'}$. In particular, there is a canonical map $\theta\colon R \to R_f^\wedge$. When both R and A are commutative S-algebras, and the homomorphism f is a homomorphism of commutative S-algebras, then the completion $R_f^\wedge$ is itself a commutative S-algebra, and the map θ is a homomorphism of commutative S-algebras.

The completion construction has a number of important properties. The following results will all be useful to us, and they will be proved in [8]

Proposition 3.3 *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a composite of homomorphisms of commutative S-algebras. Then there is a natural homomorphism $A_{gf}^\wedge \to B_g^\wedge$ of completions. Suppose further that the induced homomorphism of graded groups

$$Tor_*^{\pi_* A}(\pi_* C, \pi_* C) \longrightarrow Tor_*^{\pi_* B}(\pi_* C, \pi_* C)$$

is an isomorphism. Then the map $A_{gf}^\wedge \to B_g^\wedge$ described above is a weak equivalence of S-algebras.

Proof: The hypothesis implies that the induced maps on the cosimplicial spectra defining the completions are equivalences in codimension 1, using the Künneth spectral sequence. An iteration of this argument shows that the map of cosimplicial spectra is a levelwise equivalence, which means that it will induce an equivalence on total spectra. □

Proposition 3.4 *Let k be an S-algebra, and let $A \to B$ be a homomorphism of commutative k-algebras. Let $\mathbb{H}$ denote the mod-l Eilenberg-MacLane spectrum, and suppose that we have a commutative square*

$$\begin{array}{ccc} k & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \varepsilon} \\ B & \xrightarrow{\varepsilon} & \mathbb{H} \end{array}$$

of homomorphisms of commutative S-algebras. Suppose further that the homomorphism $\mathbb{H} \underset{k}{\wedge} A \to \mathbb{H} \underset{k}{\wedge} B$ is a weak equivalence. Then the natural homomorphism $A^{\wedge}_{\varepsilon} \to B^{\wedge}_{\varepsilon}$ is a weak equivalence.

Proof: We observe that the natural map $\mathbb{H} \underset{A}{\wedge} \mathbb{H} \to \mathbb{H} \underset{B}{\wedge} \mathbb{H}$ is a weak equivalence, since we have a commutative diagram

$$\begin{array}{ccc} \mathbb{H} \underset{A}{\wedge} \mathbb{H} & \longrightarrow & \mathbb{H} \underset{B}{\wedge} \mathbb{H} \\ \downarrow & & \downarrow \\ \mathbb{H} \underset{k}{\wedge} \mathbb{H} \underset{\mathbb{H} \underset{k}{\wedge} A}{\wedge} \mathbb{H} & \longrightarrow & \mathbb{H} \underset{k}{\wedge} \mathbb{H} \underset{\mathbb{H} \underset{k}{\wedge} B}{\wedge} \mathbb{H} \end{array}$$

where the vertical arrows are canonical equivalences, and where the lower horizontal arrow is a weak equivalence since we have the assumption that $\mathbb{H} \underset{k}{\wedge} A \to \mathbb{H} \underset{k}{\wedge} B$ is a weak equivalence. This result works equally well for multiple tensor products, and we conclude that the cosimplicial spectra defining the completions $A^{\wedge}_{\varepsilon}$ and $B^{\wedge}_{\varepsilon}$ are levelwise equivalent. $\square$

Proposition 3.5 *Let $f : A \to B$ and $g : B \to C$ be homomorphisms of S-algebras, and let M be an B-module. Suppose that A,B, and C are (-1)-connected, and that the ring homomorphism $\pi_0 B \to \pi_0 C$ is an isomorphism. Then the canonical homomorphism $M^{\wedge}_{gf} \to M^{\wedge}_{f}$ is a homotopy equivalence.*

The following corollary will be useful.

Corollary 3.6 *Let A be a commutative S-algebra, equipped with a homomorphism ϵ of commutative S-algebras to $\mathbb{H}_l$, the mod-l Eilenberg-MacLane spectrum. Suppose further that A is (-1)-connected, and that $\pi_0 A \cong \mathbb{Z}$. Then $A^{\wedge}_{\epsilon}$ is weakly equivalent to the l-adic completion of A as described above.*

Proof: We apply the preceding proposition to the situation

$$S \xrightarrow{\eta} A \xrightarrow{\epsilon} \mathbb{H}_l$$

Where η is the canonical homomorphism of S-algebras. We find that $A^\wedge \epsilon \cong A^\wedge_{\epsilon\eta}$, and the latter is just the l-adic completion. □

Next, we note that the completion construction may be applied to rings, rather than S-algebra, by viewing a ring as an S-algebra via the Eilenberg-MacLane spectrum construction. For any map $f : R \to A$, we have the induced map of Eilenberg-MacLane spectra $H(f) : H(R) \to H(A)$, and we refer to $H(R)^\wedge_{H(f)}$ as the *derived completion* of the homomorphism f. Its homotopy groups are interesting invariants of the ring homomorphism f. Of course, $\pi_0 R$ and $\pi_0 A$ are rings. Note further that if M is an R-module, then $\pi_q M$ is a $\pi_0 R$-module. The following result concerns the **algebraic to geometric** spectral sequence for computing the homotopy groups of derived completions. It generalizes the lim^1 exact sequence for computing the homotopy groups of the l-completion of a space X.

Proposition 3.7 (Algebraic to geometric spectral sequence) *Let $f : R \to A$ be a homomorphism of commutative S-algebras, and let M be an R-module. Then we may consider the rings $\pi_0 R$ and $\pi_0 A$ as well as the ring homomorphism $\pi_0 f$. There is a spectral sequence with $E_2^{p,q}$-term*

$$\pi_p(\pi_* R^\wedge_{\pi_0 f})_q$$

(Note that $\pi_p(\pi_* R^\wedge_{\pi_* f})$ is a graded group, using the internal grading of the rings $\pi_* R$ and $\pi_* A$).

We finally discuss the completion construction on group rings. We suppose that we have a commutative S-algebra A, equipped with a homomorphism of commutative S-algebras $\varepsilon : A \to H(\mathbb{F}_p)$. For any abelian simplicial group $G.$, we consider the composite

$$\varepsilon_G : |A[G.]| \to A \xrightarrow{\varepsilon} H(\mathbb{F}_p)$$

Proposition 3.8 *Suppose that we have a homomorphism $\varphi : G. \to H.$ of abelian simplicial groups, so that $B\varphi$ induces an isomorphism on mod-p homology groups. Then the evident homomorhphism of commutative S-algebras $|A[G.]|^\wedge_{\varepsilon_G} \to |A[H.]|^\wedge_{\varepsilon_H}$ is a weak equivalence of S-algebras.*

Proof: It follows immediately from 2.3 and the Atiyah-Hirzebruch spectral sequence that the map of cosimplicial spectra induced by φ is a levelwise equivalence, since in this case the graded algebras

$$\pi_*(B \underset{A}{\wedge} B \underset{A}{\wedge} \cdots \underset{A}{\wedge} B)$$

are $\mathbb{F}_p$-vector spaces. It is standard that levelwise equivalences of cosimplicial spectra induce equivalences of total spectra. □

4 Representations of Galois groups and descent in K-theory

This section contains the main results of this paper.

4.1 Categories of descent data

Let F be any field, equipped with a group action by a profinite group G. We assume that the action of G is continuous, in the sense that the stabilizer of any element of F is an open and closed subgroup of finite index in G.

Definition 4.1.1 *By a* linear descent datum *for the pair (G, F), we will mean a finite dimensional F-vector space V, together with a continuous action of G on V so that $g(fv) = f^g g(v)$ for all $g \in G$, $f \in F$, and $v \in V$. We define two categories of linear descent data, $V^G(F)$ and $V(G, F)$. The objects of $V^G(F)$ are all linear descent data for the pair (G, F), and the morphisms are all equivariant F-linear morphisms. The objects of $V(G, F)$ are also all linear descent data for (G, F), but the morphisms are all F-linear morphisms (without any equivariance requirements). The group G acts on the category $V(G, F)$, by conjugation of maps (so the action is trivial on objects), and the fixed point subcategory is clearly $V^G(F)$. Note that both categories are symmetric monoidal categories under direct sum.*

We note that $\underset{F}{\otimes}$ provides a coherently associative and commutative monoidal structure on $V(G, F)$ and $V^G(F)$.

Definition 4.1.2 *We define the spectra $K^G(F)$ and $K(G, F)$ to be the spectra obtained by applying an infinite loop space machine ([22] or [31]) to the symmetric monoidal categories of isomorphisms of $V^G(F)$ and $V(G, F)$, respectively. $K(G, F)$ is a spectrum with G-action, with fixed point spectrum $K^G(F)$. The tensor product described above makes each of these spectra into commutative S-algebras using the results of [23].*

There are various functors relating these categories (and therefore their K-theory spectra). We have the fixed point functor

$$(-)^G : V^G(F) \longrightarrow V^{\{e\}}(F^G) \cong Vect(F^G)$$

defined on objects by $V \to V^G$. We also have the induction functor

$$F \underset{F^G}{\otimes} - : VectF \cong V^{\{e\}}(F) \longrightarrow V^G(F)$$

given on objects by $V \to F \underset{F^G}{\otimes} V$. The following is a standard result in descent theory. See [15] for details.

Proposition 4.1.3 *Suppose that the action of G on F is faithful, so that $F^G \hookrightarrow F$ is a Galois extension with Galois group G. Then both $(-)^G$ and $F \underset{F^G}{\otimes} -$ are equivalences of categories.*

We also have

Proposition 4.1.4 *The category $V(G, F)$ is canonically equivalent to the category $Vect(F)$.*

Proof: The definition of the morphisms has no dependence on the group action, and the result follows immediately. □

Remark: Note that when the group action is trivial, i.e. G acts by the identity, then the category $V^G(F)$ is just the category of finite dimensional continuous F-linear representations of G.

Definition 4.1.5 *In the case when the G-action is trivial, we will also write $Rep_F[G]$ for $V^G(F)$.*

Proposition 4.1.6 *The functor*

$$Rep_{F^G}[G] \cong V^G(F^G) \xrightarrow{F \underset{F^G}{\otimes} -} V^G(F) \cong V^{\{e\}}(F^G) \cong Vect(F^G)$$

respects the tensor product structure, and $K(F^G)$ becomes an algebra over the S-algebra $KRep_{F^G}[G]$.

4.2 An example

Let k denote an algebraically closed field of characteristic 0, let $k[[x]]$ denote its ring of formal power series, and let $F = k((x))$ denote the field of fractions of $k[[x]]$. In this section, we wish to show that the representation theory of its absolute Galois group can be used to create a model for the spectrum $Kk((x))$.

The field F contains the subring $A = k[[x]]$ of actual power series, and is obtained from it by inverting x. We begin by analyzing the spectrum KF. It follows from the localization sequence (see [26]) that we have a fibration sequence of spectra

$$Kk \longrightarrow KA \longrightarrow KF$$

Further, the ring A is a *Henselian local ring* with residue class field k. It follows from a theorem of O. Gabber (see [12]) that the map of spectra $KA \to Kk$ induces an isomorphism on homotopy groups with finite coefficients, and consequently an equivalence on l-adic completions $KA_l^\wedge \to Kk_l^\wedge$. The fiber sequence above now becomes, up to homotopy equivalence, a fiber sequence

$$Kk_l^\wedge \longrightarrow Kk_l^\wedge \longrightarrow KF_l^\wedge$$

All three spectra in this sequence become module spectra over the commutative S-algebra $Kk_l^\wedge$, and the sequence consists of maps which are $Kk_l^\wedge$-module maps. Since we have $\pi_0 Kk_l^\wedge \cong \pi_0 KF_l^\wedge \cong \mathbb{Z}_l$, we find that the inclusion map $\pi_0 Kk_l^\wedge \to \pi_0 KA_l^\wedge$ is the zero map. Since the module $\pi_* Kk_l^\wedge$ is cyclic over the ring $\pi_* Kk_l^\wedge$, this means that the inclusion induces the zero map on all homotopy groups. The conclusion is that as a $\pi_* Kk_l^\wedge$-module,

$$\pi_* KF_l^\wedge \cong \pi_* Kk_l^\wedge \oplus \pi_* Kk_l^\wedge[1]$$

where the second summand is topologicaly generated by the unit x viewed as an element of $K_1(F)$. In fact, the algebra structure is also determined, since the square of the one dimensional generator is zero. The conclusion is that

$$\pi_* KF_l^\wedge \cong \Lambda_{\mathbb{Z}_l[x]}(\xi)$$

where Λ denotes the Grassmann algebra functor, where the polynomial generator x is in dimension 2, and where the exterior generator ξ is in dimension 1.

The following result is proved in [43]. It is derived from the fact that the algebraic closure of $k((x))$ is the field of *Puiseux series*, i.e. the union of the fields $k((x^{\frac{1}{n}}))$.

Proposition 4.2.1 *The absolute Galois group G of $k((x))$ is the group $\hat{\mathbb{Z}}$, the profinite completion of the group of integers.*

Let F denote the field $k((x))$, and let E denote $\overline{k((x))}$. We have observed in 4.1.6 that the spectrum KF, which is equivalent to $K^G(E)$, becomes an algebra spectrum over the S-algebra $KRep_F[G] \cong K^G F$. We wish to explore the nature of this algebra structure, and to use derived completion to demonstrate that the algebraic K-theory of F can be constructed directly from the representation theory of G over an algebraically closed field, e.g. $\mathbb{C}$.

We have the map of S-algebras $KRep_F[G] \to KV^G(E) \cong KF$, induced by the functor

$$id_E \underset{F}{\otimes} - : V^G(F) \longrightarrow V^G(E)$$

We may compose this map with the canonical map $KRep_k[G] \to KRep_F[G]$ to obtain a map of S-algebras

$$\hat{\alpha} : KRep_k[G] \to K^G(E)$$

We note that as it stands this map doesn't seem to carry much structure.

Proposition 4.2.2 *$\pi_* KRep_k[G] \cong R[G] \otimes ku_*$, where $R[G]$ denotes the complex representation ring. (The complex representation ring of a profinite group is defined to be the direct limit of the representation rings of its finite quotients.)*

Proof: Since k is algebraically closed of characteristic zero, the representation theory of G over k is identical to that over $\mathbb{C}$. This shows that $\pi_0 KRep_k[G] \cong R[G]$. In the category $Rep_k[G]$, every object has a unique decomposition into irreducibles, each of which has $\mathbb{C}$ as its endomorphism ring. The result follows directly. $\square$

We also know from the above discussion that $K_*F \cong K_*k \oplus K_{*-1}k$. It is now easy to check that the map $KRep_k[G] \to K^G E \cong KF$ induces the composite

$$R[G] \otimes K_*k \xrightarrow{\varepsilon \otimes id} K_*k \hookrightarrow K_*k \oplus K_{*-1}k \cong K_*F$$

which does not appear to carry much information about KF. However, we may use derived completion as follows. As usual, let $\mathbb{H}$ denote the mod-l Eilenberg-MacLane spectrum. We now have a commutative diagram of S-algebras

$$\begin{array}{ccc} KRep_k[G] & \xrightarrow{\hat{\alpha}} & K^G E \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{H} & \xrightarrow{id} & \mathbb{H} \end{array}$$

where ε in both cases denotes the augmentation map which sends any vector space or representation to its dimension. The naturality properties of the derived completion construction yields a map

$$\alpha^{rep} : KRep_k[G]^\wedge_\varepsilon \to K^G E^\wedge_\varepsilon \xrightarrow{\sim} KF^\wedge_\varepsilon$$

which we refer to as the *representational assembly* for F. The goal of this section is to show that despite the fact that $\hat{\alpha}$ carries little information about K_*F, α^{rep} is an equivalence of spectra.

Proposition 4.2.3 *$KF^\wedge_\varepsilon$ is just the l-adic completion of KF.*

Proof: This is just the definition of l-adic completion, as in section 3. □

It remains to determine the structure of $\pi_* KRep_k[G]^\wedge_\varepsilon$.

Proposition 4.2.4 *There is an equivalence of spectra*

$$KRep_k[G]^\wedge_\varepsilon \cong ku^\wedge_l \wedge S^1_+$$

In particular the homotopy groups are given by $\pi_i KRep_k[G]^\wedge_\varepsilon \cong \mathbb{Z}_l$ for all $i \geq 0$, and $\cong 0$ otherwise.

Proof: We first show that the S-algebra $KRep_k[G]$ may be identified with the group ring $Kk[\chi(\mathbb{Z}_l)] \cong Kk[\mathbb{Z}/l^\infty\mathbb{Z}]$, where χ denotes the group of complex characters. We observe that because k is algebraically closed of characteristic zero, the irreducible representations are in bijective correspondence with the characters of $\mathbb{Z}_l$. We now recall the category $C[\chi(\mathbb{Z}_l)]$ whose objects are finite sets X equipped with a function $\varphi_X : X \to \chi(\mathbb{Z}_l)$. There is an evident functor from $C[\chi(\mathbb{Z}_l)]$ to $Rep_k[\mathbb{Z}_l]$, which respects the sum and multiplicaton operations defined in section 2, and which sends an object $(X, \varphi_X$ in $C[\chi(\mathbb{Z}_l)]$ to the free k-vector space on the set X, with the representation on each element $x \in X$ specified by the character $\varphi_X(x)$. Consequently we obtain a homomorphism of commutative S-algebras

$$S[\chi(\mathbb{Z}_l)] \longrightarrow KRep_k[\mathbb{Z}_l]$$

Since $KRep_k[\mathbb{Z}_l]$ is an algebra over the commutative S-algebra Kk, we obtain a homomorphism $Kk[\chi(\mathbb{Z}_l)] \to KRep_k[\mathbb{Z}_l]$, which is readily verified to be an equivalence of spectra.

There is further a homomorphism $\chi(\mathbb{Z}_l) \to S^1_.$, where $S^1_.$ denotes the singular complex of the circle group, and where the homomorphism is induced by the inclusion of topological groups

$$\chi(\mathbb{Z}_l) \cong \mathbb{Z}/l^\infty\mathbb{Z} \hookrightarrow S^1$$

. The induced map of classifying spaces induces isomorphisms on mod-l homology groups, and consequently by 3.8 it induces a weak equivalence on derived completions.

□

We have now shown that the derived completion $KRep_k[G]^\wedge_\varepsilon$ has the same homotopy groups as the K-theory spectrum KF. We must now show that the representational assembly induces an isomorphism on homotopy groups. We now recall the localization sequence [26] which allows us to compute $K_*(F)$. Consider the category $\mathcal{M} = Mod(k[[x]])$ of finitely generated modules over $k[[x]]$. Let $\mathcal{T} \subseteq \mathcal{P}$ denote the full subcategory of x-torsion modules. Then the category $Vect(F)$ may be identified with the quotient abelian category $\mathcal{M}/\mathcal{T}$, and we have the usual localization sequence

$$K\mathcal{T} \to K\mathcal{M} \to K\mathcal{P}/\mathcal{T}$$

which is identified with the fiber sequence

$$Kk \to Kk[[x]] \to KF$$

We now observe that there is a twisted version of this construction. Let $\mathcal{O}_E$ denote the integral closure of $k[[x]]$ in E. $\mathcal{O}_E$ is closed under the action of the group G. We now define $\mathcal{E}^G$ to be the category whose objects are finitely generated $\mathcal{O}_E$ modules equipped with a G-action so that $g(em) = e^g g(m)$. Further, let $\mathcal{T}^G$ denote the full subcategory of torsion modules.

Proposition 4.2.5 *The quotient abelian category $\mathcal{E}^G/\mathcal{T}^G$ can be identified with the category of linear descent data $V^G(E)$.*

Proof: Straightforward along the lines of Quillen's proof that $\mathcal{M}/\mathcal{T} \cong VectF$.
□

Corollary 4.2.6 *There is up to homotopy a fiber sequence of spectra*

$$K\mathcal{T}^G \longrightarrow K\mathcal{E}^G \longrightarrow KV^G(E) \cong KF$$

We note that for each of the categories $\mathcal{T}^G$, $\mathcal{E}^G$, and $V^G(E)$, objects may be tensored with finite dimensional k-linear representations of G to give new objects in the category. It follows that each of the K-theory spectra are module spectra over $KRep_k[G]$.

Lemma 4.2.7 *The homotopy fiber sequence of 4.2.6 is a homotopy fiber sequence of $KRep_k[G]$-module spectra.*

It follows from its definition that derived completion at ε preserves fiber sequences of module spectra, so we have a homotopy fiber sequence

$$(K\mathcal{T}^G)^\wedge_\varepsilon \longrightarrow (K\mathcal{E}^G)^\wedge_\varepsilon \longrightarrow (KV^G(E))^\wedge_\varepsilon \cong KF$$

The last equivalence follows from 4.2.3. We will now show that

- $(K\mathcal{T}^G)^\wedge_\varepsilon \simeq *$
- $(K\mathcal{E}^G)^\wedge_\varepsilon \simeq KRep_k[G]^\wedge_\varepsilon$

The result will follow immediately.

Proposition 4.2.8 $(K\mathcal{T}^G)^\wedge_\varepsilon \simeq *$

Proof: Let $\mathcal{T}^G_n$ denote the subcategory of finitely generated torsion $k[[t^{\frac{1}{n}}]]$-modules M equipped with G-action so that $g(fm) = f^g g(m)$ for $g \in G$, $f \in k[[t^{\frac{1}{n}}]]$, and $m \in M$. We have obvious functors

$$\tau^k_n = k[[t^{\frac{1}{k}}]] \underset{k[[t^{\frac{1}{n}}]]}{\otimes} - : \mathcal{T}^G_n \to \mathcal{T}^G_k$$

whenever n divides k, and an equivalence of categories

$$\underset{\rightarrow}{lim}\, \mathcal{T}^G_n \longrightarrow \mathcal{T}^G$$

A straightforward devissage argument now shows that the inclusion of the full subcategory of objects of $\mathcal{T}^G_n$ on which $x^{\frac{1}{n}}$ acts trivially induces an equivalence on K-theory spectra. It follows that $K_*\mathcal{T}^G_n \cong R[G] \otimes K_*k$, and moreover that $K\mathcal{T}^G_n \simeq KRep_k[G]$ as module spectra over $KRep_k[G]$. We will now need to analyze the map of K-theory spectra induced by the functors τ^k_n. Recall that $R[G] \cong \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. For all positive integeres n and s, let $\nu_{n,s}$ denote the element in $R[G]$ corresponding to the element $\sum_{i=0}^{s-1}[i/ns] \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. We claim that we have a commutative diagram

$$
\begin{array}{ccc}
K_*\mathcal{T}_n^G & \xrightarrow{\sim} & R[G]\otimes K_*k \\
\downarrow{\scriptstyle \tau_n^{ns}} & & \downarrow{\scriptstyle (\cdot\nu_{n,s})\otimes id} \\
K_*\mathcal{T}_{ns}^G & \xrightarrow{\sim} & R[G]\otimes K_*k
\end{array}
$$

That we have such a diagram is reduced to a π_0 calculation, since the diagram is clearly a diagram of $R[G]\otimes K_*k$-modules. It therefore suffices to check the commuting of this diagram on the element $1\in K_*\mathcal{T}_n^G$. But 1 under τ_n^{ns} clearly goes to the module $k[[x^{\frac{1}{ns}}]]/(x^{\frac{1}{n}})$. It is easily verified that as an element in the representation ring, this element is $\nu_{n,s}$.

We are now ready to describe $\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}^G$. Since smash products commute with filtering direct limits, we see that

$$\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}^G \simeq \underset{\rightarrow}{lim}\, \mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}_n^G$$

But since $K_*\mathcal{T}_n^G$ is a free $R[G]\otimes K_*k$-module of rank 1, $\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}_n^G \cong \mathbb{H}$. It is readily checked that the induced map

$$id_{\mathbb{H}} \underset{KRep[G]}{\wedge} (\cdot\nu_{ns})\otimes id$$

is multiplication by s on $\pi_*\mathbb{H}$. It now clearly follows that $\pi_*\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}^G = 0$, and consequently that

$$\pi_* \underbrace{\mathbb{H} \underset{KRep[G]}{\wedge} \cdots \underset{KRep[G]}{\wedge} \mathbb{H}}_{k \text{ factors}} \underset{KRep[G]}{\wedge} K\mathcal{T}^G = 0$$

for all k. It now follows directly from the definition of the completion, together with the fact that the total spectrum of a cosimplicial spectrum which is level-wise contractible is itself contractible, that $(K\mathcal{T}^G)^\wedge_\varepsilon \simeq *$. □

Corollary 4.2.9 *The map $(K\mathcal{E}^G)^\wedge_\varepsilon \to (KV^G(E))^\wedge_\varepsilon$ is a weak equivalence of spectra.*

Proof: Completion preserves fibration sequences of spectra. □

We next analyze $K_*\mathcal{E}^G$ in low degrees. Let $\mathbb{N}^\cdot$ be the partially ordered set of positive integers, where $m\le n$ if and only if $m|n$. We define a directed system of $R[G]$-modules $\{A_n\}_{n>0}$ parametrized by $\mathbb{N}^{\cdot}$ by setting $A_n = R[G]$, and where whenever $m|n$, we define the bonding map from A_m to A_n to be multiplication by the element $\nu_{m,\frac{n}{m}}$. We let $QR[G]$ denote the colimit of this module. It follows from the proof of 4.2.8 that $K_*\mathcal{T}^G \cong QR[G] \otimes K_*k$. In particular, $K_*\mathcal{T}^G = 0$ in odd degrees. On the other hand, we know that $K_*F \cong \mathbb{Z}_l$ in odd degrees, but that in these odd elements map non-trivially in the connecting homomorphism in the localization sequence. Consequently, we have

Proposition 4.2.10 *$K_*\mathcal{E}^G = 0$ in odd degrees. In particular, $K_1\mathcal{E}^G = 0$.*

Proposition 4.2.11 *$K_0\mathcal{E}^G \cong R[G]$.*

Proof: Objects of $\mathcal{E}^G$ are the same thing as modules over the twisted group ring $\mathcal{O}_E\langle G\rangle$, and a devissage argument shows that the inclusion

$$Proj(\mathcal{O}_E\langle G\rangle) \hookrightarrow Mod(\mathcal{O}_E\langle G\rangle)$$

induces an isomorphism, so that we may prove the corresponding result for projective modules over $\mathcal{O}_E\langle G\rangle$. Note that we have a ring homomorphism

$$\pi : \mathcal{O}_E\langle G\rangle \longrightarrow k[G]$$

given by sending all the elements $x^{\frac{1}{n}}$ to zero. $\mathcal{O}_E\langle G\rangle$ is complete in the I-adic topology, where I is the kernel of π. As in [38], isomorphism classes of projective modules over $\mathcal{O}_E\langle G\rangle$ are now in bijective correspondence via π with the isomorphism classes of projective modules over $k[G]$, which gives the result. □

Corollary 4.2.12 *The functor $\mathcal{O}_E \underset{k}{\otimes} - : Rep_k[G] \to \mathcal{E}^G$ induces a weak equivalence on spectra. Consequently, the map $KRep_k[G]^{\wedge}_{\varepsilon} \to (K\mathcal{E}^G)^{\wedge}_{\varepsilon}$ is a weak equivalence.*

Proof: 4.2.10 and 4.2.11 identify the homotopy groups of $K\mathcal{E}^G$ in degrees 0 and 1. From the localization sequence above, together with the fact that multiplication by the Bott element induces isomorphisms $K_i\mathcal{T}^G \to K_{i+2}\mathcal{T}^G$ and $K_iF \to K_{i+2}F$, it follows that it also induces an isomorphism $K_i\mathcal{E}^G \to K_{i+2}\mathcal{E}^G$. So, the map of the statement of the corollary induces an isomorphism on homotopy groups, which is the required result. □

Theorem 4.2.13 *The map $\alpha^{rep} : KRep_k[G]^{\wedge}_{\varepsilon} \to KV^G(E)^{\wedge}_{\varepsilon} \cong KF^{\wedge}_{\varepsilon}$ is a weak equivalence of spectra.*

Proof: α^{rep} is the composite of the maps of Corollary 4.2.9 and Corollary 4.2.12, both of which we have shown are weak equivalences. □

We will now show how to extend this result to the case of geometric fields (i.e. containing and algebraically closed subfield) F of characteristic zero with G_F a free pro-l abelian group. Let F be geometric, and let $k \subseteq F$ denote an algebraically closed subfield. Let $A = k[t_1^{\pm 1}, t_2^{\pm 1}, \ldots t_n^{\pm 1}]$. We let B denote the ring obtained by adjoining all p-th power roots of unity of the variables t_i to A, so

$$B = \bigcup_l k[t_1^{\pm \frac{1}{p^l}}, t_2^{\pm \frac{1}{p^l}}, \ldots t_n^{\pm \frac{1}{p^z eta_n l}}]$$

We choose for a sequence of elements ζ_n in k so that ζ_n is a primitive p^n-th root of unity, and so that $\zeta_n^p = \zeta_{n-1}$. We define an action of the group $\mathbb{Z}_p^n$ on B by $\tau_i(t_i^{\frac{1}{p^n}}) = \zeta_n t_i^{\frac{1}{p^n}}$, and $\tau_j(t_i^{\frac{1}{p^n}}) = t_i^{\frac{1}{p^n}}$ for $i \neq j$, where $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is a set of topological generators for $\mathbb{Z}_p^n$.

Proposition 4.2.14 *The p-completed K-theory groups of A and B are given by*

- $K_*A \cong \Lambda_{K_*k}(\theta_1, \theta_2, \ldots, \theta_n)$
- $K_*B \cong K_*k$

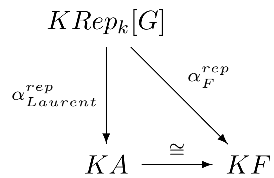
Proof: The first result is a direct consequence of the formula for the K-groups of a Laurent polynomial ring. The second also follows from this fact, together with analysis of the behavior of the p-th power map on such a Laurent extension. □

We now may construct a representational assembly in this case as well, to obtain a representational assembly

$$\alpha^{rep}_{Laurent} : KRep_k[G]^{\wedge}_{\varepsilon} \longrightarrow KA^{\wedge}_{\varepsilon} \cong KA$$

Proposition 4.2.15 $\alpha^{rep}_{Laurent}$ *is an equivalence of spectra.*

Proof: We first consider the case $n = 1$, and the field $F = k((x))$, where we have already done the analysis. It is immediate that the inclusion $A \to F$ defined by $t \to x$ extends to an equivariant homomorphism $B \to E$ of k-algebras, and consequently that we get a commutative diagram



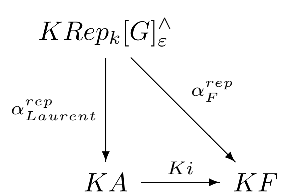
It is readily checked that the inclusion $KA \to KF$ is a weak equivalence, and so that the horizontal arrow in the diagram is a weak equivalence. On the other hand, we have already shown that α^{rep}_F is a weak equivalence of spectra, which shows that $\alpha^{rep}_{Laurent}$ is an equivalence. For larger values of n, the result follows by smashing copies of this example together over the coefficient S-algebra K_*k.

□

Now consider any geometric field F of characteristic zero, with $G_F \cong \mathbb{Z}_l^n$, and let k be an algebraically closed subfield. It is a direct consequence of Kummer theory that there is a family of elements $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ so that $E = \overline{F}$ can be written as

$$E = \bigcup_n F(\sqrt[l^n]{\alpha_1}, \sqrt[l^n]{\alpha_2}, \ldots, \sqrt[l^n]{\alpha_n})$$

It is therefore further clear that we may define a k-algebra homomorphism $i : B \to E$ by setting $i(t_i^{\frac{1}{l^n}}) = \sqrt[l^n]{\alpha_i}$, where $\sqrt[l^n]{\alpha_i}$ is chosen as the choice of l^n-th root so that the action of τ_i is via multiplication by ζ_n. It follows that we obtain a commutative diagram



Since we now know that $\alpha^{rep}_{Laurent}$ is a weak equivalence, it will now suffice to show that Ki is a weak equivalence of spectra. But the work of Geisser-levine [13] shows that if the Bloch-Kato conjecture holds, then the l-completed version of the Bloch-Lichtenbaum spectral sequence converging to the algebraic K-theory has as its E_2-term an exterior algebra on generators in degree one, corresponding to the elements $\alpha_1, \alpha - 2, \ldots, \alpha_n$, from which this result follows directly.

Proposition 4.2.16 *If the Bloch-Kato conjecture holds, then α^{rep}_F is an equivalence of spectra for geometric fields of characteristic zero with finitely generated abelian absolute Galois group.*

Remark: We believe that the techniques used here will also extend to the finite characteristic case, for characteristic prime to l. The techniques of section 4.5 allows us to extend to the general case of a field containing all the l-th power roots of unity.

4.3 Representational assembly in the geometric case

Suppose now that we are in the case of a *geometric field*, i.e. a field F containing an algebraically closed subfield k. Let E denote the algebraic closure of F, and let G denote the absolute Galois group. As in the case of the example of the previous section, we have a composite functor

$$Rep_k[G] \longrightarrow Rep_F[G] \cong V^G(F) \longrightarrow V^G(E)$$

and the induced map of completed spectra

$$KRep_k[G]^{\wedge}_{\varepsilon} \longrightarrow KV^G(E)^{\wedge}_{\varepsilon} \cong KF^{\wedge}_{\varepsilon}$$

We may ask whether this map is an equivalence as occurred in the case of the example in the previous section. In order to discuss the plausibility of this kind of statement, let us examine what occurs in the example, when $G = \mathbb{Z}_l$. In this case, $\pi_0 KRep_k[G] \cong \mathbb{Z}_l[\mathbb{Q}/\mathbb{Z}]$. This is a very large, non-Noetherian ring. However, after derived completion, we find that $\pi_0 KRep_k[G]^{\wedge}_{\varepsilon} \cong \mathbb{Z}_l$. On the other hand, we have from 3.7 that $\pi_0 KRep_k[G]^{\wedge}_{\varepsilon}$ is isomorphic to the ordinary (not derived) completion of the ring $R[G]$ at the ideal $J = (l) + I$, where I denotes the augmentation ideal in $R[G]$. So the conclusion is that $\underset{\leftarrow}{lim}\ R[G]/J^n \cong \mathbb{Z}_l \cong (R[G]/I)^{\wedge}_l$. The reason this happens is the following result about the ideal I.

Proposition 4.3.1 *We have that I/I^2 is a divisible group, and that $I^k = I^{k+1}$ for all k.*

Proof: It is a standard result for any group G that in the group ring $\mathbb{Z}[G]$, the augmentation ideal $I[G]$ has the property that $I[G]/I[G]^2 \cong G^{ab}$. Since $\mathbb{Q}/\mathbb{Z}$ is abelian, we find that $I/I^2 \cong \mathbb{Q}/\mathbb{Z}$, which is divisible. For the second statement, we note that I^k/I^{k+1} is a surjective image of $\underbrace{I/I^2 \otimes \ldots \otimes I/I^2}_{k \text{ factors}}$. But it is clear that $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0$, so $I^k/I^{k+1} = 0$. This gives the result. □

Corollary 4.3.2 *We have that $((l) + I)^k = (l^k) + I$. Equivalently, for any element $\theta \in I$, and any positive integers s and t, there is an $\eta \in I$ and $\mu \in I^k$ so that*

$$\theta = l^s\eta + \mu$$

Proof: We have

$$((l)+I)^k = \sum_t l^t I^{k-l} = (l^k) + l^{k-1}I + I^2$$

But 4.3.1 implies that $l^{k-1}I + I^2 = I$, which gives the result. $\square$

It now follows that $R[G]/J^k \cong R[G]/(I + (l^k)) \cong \mathbb{Z}/l^k\mathbb{Z}$, and therefore that the completion of $R[G]$ at the ideal J is just $\mathbb{Z}_l$.

This result extends to torsion free abelian profinite groups. That is, any such group also has the property that if I is the augmentation ideal in the representation ring, then $I^k = I^2$ for $k \geq 2$, and that I/I^2 is divisible. However, for non-abelian Galois groups, there appears to be no obvious reason why such a result should hold, and indeed we believe that it does not. However, there is a modification of this statement which is true for absolute Galois groups and which suggests a conjecture which can be formulated for any geometric field.

Recall that the representation ring of a finite group is not just a ring, but is actually a part of a *Green functor*. This was discussed in section 2, where there is a discussion of Mackey and Green functors. See [3] for a thorough discussion of these objects. A Mackey functor for a group G is a functor from a category of orbits to abelian groups. In the case of the representation ring, there is a functor $\mathcal{R}$ given on orbits by $\mathcal{R}(G/K) = R[K]$. The maps induced by projections of orbits induce restriction maps on representation rings, and transfers induce inductions. This functor is actually a commutative *Green functor*, in the sense that there is a multiplication map $\mathcal{R}\square\mathcal{R} \to \mathcal{R}$, which is associative and commutative. Also for any finite group G, there is another Green functor $\mathcal{Z}$ given on objects by $\mathcal{Z}(G/K) = \mathbb{Z}$, and for which projections of orbits induce the identity map and where transfers associated to projections induce multiplication by the degree of the projection. We may also consider $\mathcal{Z}/l\mathcal{Z}$, which is obtained by composing the functor $\mathcal{Z}$ with the projection $\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z}$. The *augmentation* is the morphism of Green functors $\mathcal{R} \to \mathcal{Z}$ which is given on an object G/K by the augmentation of $R[K]$. The mod-l augmentation ε is the composite $\mathcal{R} \to \mathcal{Z} \to \mathcal{Z}/l\mathcal{Z}$. The theory of Green functors is entirely parallel with the theory of rings, modules, and ideals. We may therefore speak of the *augmentation ideal* $\mathcal{I}$ and the ideal $\mathcal{J} = (l) + \mathcal{I}$, as well as the powers of these ideals. We can therefore also speak of completion at a Green functor ideal. We also observe that the theory of Mackey and Green functors estends in an obvious way to profinite groups, by considering the catgory of finite G-orbits. We want to identify a class of profinite groups G for which

$$\mathcal{R}_l^{\wedge} = \lim_{\leftarrow} \mathcal{R}/\mathcal{J}^n$$

is isomorphic to $\mathcal{Z}_l^\wedge = \mathbb{Z}_l \otimes \mathcal{Z}$.

Definition 4.3.3 *Let G be a profinite group. We say G is* totally torsion free *if every subgroup of finite index has a torsion free abelianization.*

Example: Free profinite groups and free profinite l-groups are totally torsion free.

Example: Free profinite abelian and profinite l-abelian groups.

Example: Let Γ denote the integral Heisenberg group, i.e. the group of upper triangular integer matrices with ones along the diagonal. Then the profinite and pro-l completion of Γ is not totally torsion free.

We have

Proposition 4.3.4 *Let G be the absolute Galois group of a geometric field F. Let G_l denote the maximal pro-l quotient of G. Then G_l is totally torsion free.*

Proof: Consider any subgroup K of finite l-power index in G. Then let E denote the extension of F corresponding to K. Then the abelianization of K corresponds to the maximal pro-l Abelian extension of E. Let $\mathbb{N}^{\cdot}$ denote the partially ordered set of positive integers, where $m \leq n$ if and only if $m|n$. Define a functor Φ from $\mathbb{N}^{\cdot}$ to abelian groups by $\Phi(n) = k^*/(k^*)^n$, and on morphisms by $\Phi(m \to mn) = k^*/(k^*)^m \overset{(-)^n}{\to} k^*/(k^*)^{mn}$. Let $\mathcal{K}$ denote the direct limit over $\mathbb{N}^{\cdot}$ of Φ. Kummer theory then asserts the existence of a perfect pairing

$$K^{ab} \times \mathcal{K} \to \mathbb{Q}/\mathbb{Z}$$

So $K^{ab} \cong Hom(\mathcal{K}, \mathbb{Q}/\mathbb{Z})$. The group $\mathcal{K}$ is clearly divisible, and it is easily verified that the $\mathbb{Q}/\mathbb{Z}$-dual of a divisible group is torsion free. □

We have the following result concerning totally torsion free profinite groups.

Proposition 4.3.5 *Let G be a totally torsion free l-profinite group. Then the natural map $\mathcal{R}_l^\wedge \to \mathcal{Z}_l^\wedge$ is an isomorphism. Consequently, for G the maximal pro-l quotient of the absolute Galois group of a geometric field, we have that $\mathcal{R}_l^\wedge \cong \mathcal{Z}_l^\wedge$.*

Proof: We will verify that $\mathcal{R}_l^\wedge(G/G) \to \mathcal{Z}_l^\wedge(G/G)$ is an isomorphism. The result at any G/K for any finite index subgroup will follow by using that result for the totally torsion free group K. It will suffice to show that for every finite dimensional representation ρ of G/N, where N is a normal subgroup of finite index, and every choice of positive integer s and t, there are elements $x \in \mathcal{R}(G/G)$ and $y \in \mathcal{I}^t(G/G)$ so that $[dim\rho] - [\rho] = l^s x + y$. We recall *Blichfeldt's theorem* [33], which asserts that there is a subgroup L of G/N and a one-dimensional representation ρ_L of L so that ρ is isomorphic to the representation of G/N induced up from ρ_L. It follows that $[dim\rho] - [\rho] = i_L^{G/N}(1-\rho_L)$. Let $\overline{L}$ denote the subgroup $\pi^{-1}L \subseteq G$, where $\pi : G \to G/N$ is the projection, and let $\rho_{\overline{L}} = \rho_L \circ \pi$. Then we clearly also have $[dim\rho] - [\rho] = i_{\overline{L}}^G(1-\rho_{\overline{L}})$. Now, $1 - \rho_{\overline{L}}$ is in the image of $R[\overline{L}^{ab}] \to R[\overline{L}]$, and let the corresponding one-dimensional representation of $\overline{L}^{ab}$ be $\rho_{\overline{L}^{ab}}$. Since $\overline{L}^{ab}$ is abelian and torsion free (by the totally torsion free hypothesis), we may write $1 - \rho_{\overline{L}^{ab}} = l^s\xi + \eta$, where $\eta \in I^t(\overline{L}^{ab})$ and where $\xi \in R[\overline{L}^{ab}]$, by 4.3.2. This means that we may pull ξ and η back along the homomorphism $\overline{L} \to \overline{L}^{ab}$, to get elements $\overline{\xi} \in R[\overline{L}]$ and $\overline{\eta} \in I^t(\overline{L})$ so that $\rho_{\overline{L}} = l^s\overline{\xi} + \overline{\eta}$. Since $\mathcal{I}^t$ is closed under induction, we have that $i_{\overline{L}}^G(\overline{\eta}) \in \mathcal{I}^t(G/G)$. Now we have that $[dim\rho] - \rho = i_{\overline{L}}^G(1-\rho_{\overline{L}}) = p^s i_{\overline{L}}^G\overline{\xi} + i_{\overline{L}}^G\eta$. The result follows. □

We recall the relationship between Mackey functor theory and equivariant stable homotopy theory. The natural analogue for homotopy groups in the world of equivariant spectra takes its values in the category of Mackey functors. The Adams spectral sequence and the algebraic-to-geometric spectral sequence have E_2-terms which are computed using derived functors of Hom and □-product, which we will denote by *Ext* and *Tor* as in the non-equivariant case. We will now observe that the constructions we have discussed are actually part of equivariant spectra.

Proposition 4.3.6 *For any profinite group G, there is a stable category of G-spectra, which has all the important properties which the stable homotopy theory of G-spectra for finite groups has. In particular, there are fixed point subspectra for every subgroup of finite index as well as transfers for finite G-coverings. The "tom Dieck" filtration holds as for finite groups. The homotopy groups of a G-spectrum form a Mackey functor. Moreover, the homotopy groups of a G S-algebra are a Green functor, and the homotopy groups of a G-module spectrum are a module over this Green functor.*

Proof: For any homomorphism $f : G \to H$ of finite groups, there is a pullback functor f^* from the category of H-spectra to the category of G-spectra. Hence, for a profinite group g, we get a direct limit system of categories parametrized by the normal subgroups of finite index, with the value at N being the category of G/N-spectra. A G spectrum can be defined as a family of spectra $\mathcal{S}_N$ in the category of G/N-spectra, together with isomorphisms

$$(G/N_1 \to G/N_2)^*(\mathcal{S}_{N_2}) \xrightarrow{\sim} \mathcal{S}_{N_1}$$

This general construction can be made into a theory of G spectra with the desired properties. □

Proposition 4.3.7 *Let F be a field, with E its algebraic closure, and G the absolute Galois group. Then there is a G-S-algebra with total spectrum $KV(G,E)$, with fixed point spectra $KV(G,E)^H \cong KV^H(E)$. The attached Green functor is given by*

$$G/L \to \pi_* KV(G,E)^L \cong \pi_* KV^L(E) \cong K_*(E^L)$$

Similarly, there is a G-S-algebra with total spectrum $KV(G,F)$, and with fixed point spectra $KV(G,F)^L \cong KV^L(F) \cong KRep_F[L]$. (Note that the G-action on F is trivial.) In this case, the associated Green functor is given by

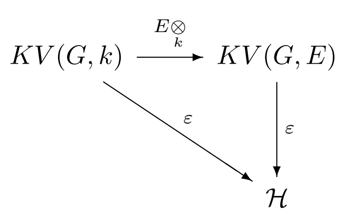
$$G/L \to \pi_* KV(G,F)^L \cong \pi_* KV^L(F) \cong K_* Rep_F[L]$$

In the case when F contains all the l-th power roots of unity, we find that the Mackey functor attached to $KV(G,F)$ is $ku_* \otimes \mathcal{R}$. The functor $E \underset{F}{\otimes} -$ induces a map of G-S-algebra $KV(G,F) \to KV(G,E)$, which induces the functors of Proposition 4.1.6 on fixed point spectra.

Proof: This result is essentially a consequence of the equivariant infinite loop space recognition principle [34]. □

We also recall the results of [18], where it was shown that in the category of G-equivariant spectra, there is an Eilenberg-MacLane spectrum for every Mackey functor. We will let $\mathcal{H}$ denote the Eilenberg-MacLane spectrum attached to the Green functor $\mathcal{Z}/l\mathcal{Z}$. Derived completions of homomorphisms of S-algebras are defined as in the non-equivariant case.

Proposition 4.3.8 *Let F be a geometric field, with the algebraically closed subfield k. There is a commutative diagram of G-S-algebras*



where, the maps ε are given by "dimension mod l". Consequently, there is a map of derived completions.

$$\alpha^{rep} : KV(G,k)^\wedge_\varepsilon \longrightarrow KV(G,E)^\wedge_\varepsilon$$

It follows from the equivariant algebraic-to-geometric spectral sequence that $\pi_0 KV(G,k)^\wedge_\varepsilon \cong \mathcal{R}^\wedge_l$, and this is isomorphic by Proposition 4.3.5 to $\mathcal{Z}^\wedge_l$, which is in turn isomorphic to the Green functor $\pi_0 KV(G,E)$. This makes the plausible the conjecture the following conjecture.

Conjecture 4.3.9 *Let F be a geometric field. The representational assembly map*

$$\alpha^{rep} : KV(G,k)^\wedge_\varepsilon \to KV(G,E)^\wedge_\varepsilon$$

is an equivalence of G-S-algebras. On G-fixed point spectra, we have an equivalence of S-algebras

$$(\alpha^{rep})^G : (KV(G,k)^\wedge_\varepsilon)^G \longrightarrow (KV(G,E)^\wedge_\varepsilon)^G \cong K(E^G) \cong KF$$

4.4 Representational assembly in the twisted case

In the case of a field F which does not contain an algebraically closed subfield, it is not as easy to construct a model for the K-theory spectrum of the field. As usual, let F denote a field, E its algebraic closure, and G its absolute Galois group. We observe first that for any subfield of $F' \subseteq E$ which is closed under the action of G, we obtain a map

$$KV(G,F')^\wedge_\varepsilon \longrightarrow KV(G,E)^\wedge_\varepsilon$$

We may conjecture (as an extension of Conjecture 4.3.9 above) that this map is always an equivalence, and indeed we believe this to be true. In the case when F' is algebraically closed, we have believe that the domain of the map has what we would regard as a simple form, i.e. is built out of representation theory of G over an algebracially closed field. In general, though, the K-theory of the category $V^G(F')$ may not be a priori any simpler than the K-theory of F. However, when F' is a field whose K-theory we already understand well, then we expect to obtain information this way. In addition to algebraically closed fields, we have an understanding of the K-theory of finite fields from the work of Quillen [27]. So, consider the case where F has finite characteristic p, distinct from l. We may in this case let F' be the maximal finite subfield contained in F, which is of the form $\mathbb{F}_q$, where $q = p^n$ for some n. In this case we have, in parallel with Conjecture 4.3.9,

Conjecture 4.4.1 *For F of finite characteristic $p \neq l$, with $\mathbb{F}_q$ the maximal finite subfield of F, the map*

$$KV(G, \mathbb{F}_q)^{\wedge}_{\varepsilon} \longrightarrow KV(G, E)^{\wedge}_{\varepsilon}$$

is an equivalence of G-S-algebras. In particular, we have an equivalence

$$(KV(G, \mathbb{F}_q)^{\wedge}_{\varepsilon})^G \simeq KV(G, E)^{\wedge}_{\varepsilon} \cong KF$$

We now argue that this is a reasonable replacement for Conjecture 4.3.9 in this case. We note that the fixed point category of $V(G, \mathbb{F}_q)$ is the category $V^G(\mathbb{F}_q)$ of linear descent data. We have the straightforward observation

Proposition 4.4.2 *For any finite group G acting on $\mathbb{F}_q$ by automorphisms, the category of linear descent data $V^G(\mathbb{F}_q)$ is equivalent to the category of left modules over the twisted group ring $\mathbb{F}_q\langle G\rangle$. Similarly, for a profinite group G acting on $\mathbb{F}_q$, we find that the category of linear descent data is equivalent to the category of continuous modules over the Iwasawa algebra version of $\mathbb{F}_q\langle G\rangle$.*

Note that $\mathbb{F}_q\langle G\rangle$ is a finite dimensional semisimple algebra over $\mathbb{F}_p$, so it is a product of matrix rings over field extensions of $\mathbb{F}_p$. Consequently, we should view its K-theory as essentially understood, given Quillen's computations for finite fields. For this reason, we regard the above conjecture as a satisfactory replacement for Conjecture 4.3.9.

In the characteristic 0 case, though, we have more difficulties, since in this case we do not know the K-theory of the prime field $\mathbb{Q}$. Indeed, we would like to make conjectures about the K-theory of $\mathbb{Q}$ involving the representation theory of $G_{\mathbb{Q}}$. We do, however, thanks to the work of Suslin [36], understand the K-theory of the field $\mathbb{Q}_p$ and $\mathbb{Z}_p$ where $p \neq l$.

Theorem 4.4.3 (Suslin; see [36]) *Let K denote any finite extension of $\mathbb{Q}_p$, and let $\mathcal{O}_K$ denote its ring of integers. Let (π) denote its unique maximal ideal. The quotient homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ induces an isomorphism on l-completed K-theory.*

Now, suppose we have any field F containing $\mathbb{Q}_p$, containing the l-th roots of unity. and let E denote its algebraic closure. Let L be $\bigcup_n \mathbb{Q}_p(\zeta_{l^n}) \subseteq E$. L is of course closed under the action of the absolute Galois group $G = G_F$. By abuse of notation, we will write $V(G, \mathcal{O}_L)$ for the category of twisted $G - \mathcal{O}_L$-modules over $\mathcal{O}_L$, i.e. finitely generated $\mathcal{O}_L$-modules M equipped with a G-action so that $g(rm) = r^g g(m)$ for all $r \in \mathcal{O}_L$, $m \in M$, and $g \in G$. The following is an easy consequence of the Theorem 4.4.3 above.

Proposition 4.4.4 *The functor $V(G, \mathcal{O}_L) \to V(G, \mathcal{O}_L/\pi\mathcal{O}_L)$ induces an equivalence $K^G\mathcal{O}_L \to K^G\mathcal{O}_L/\pi\mathcal{O}_L$.*

Since $\mathcal{O}_L/\pi\mathcal{O}_L$ is a semisimple algebra over a finite field, we will regard it as an understood quantity. This means that in this case, we also have a version of the representational assembly via the diagram

$$(K^G\mathcal{O}_L/\pi\mathcal{O}_L)^\wedge_\varepsilon \xleftarrow{\sim} (K^G\mathcal{O}_L)^\wedge_\varepsilon \xrightarrow{E \underset{\mathcal{O}_L}{\otimes}} (K^G E)^\wedge_\varepsilon \cong KF$$

where as usual ε is the natural map defined to the Eilenberg-MacLane spectrum attached to the Green functor $\mathcal{Z}$. This kind of result extends to the case where F contains a Henselian local ring which residue class field algebraic over a finite field.

4.5 The ascent map and assembly for the case $\mu_{l^\infty} \subseteq F$

The constructions of the previous section provide explicit maps from derived completions of spectra attached to representation categories of the absolute Galois group G of a field F to the KF. However, they only apply to the geometric case, i.e. where F contains an algebraically closed subfield. We believe, though, that the statement relating the K-theory of the field to the derived representation theory of the absolute Galois group should be true for all fields containing the l-th power roots of unity. The purpose of this section is to develop a criterion which when satisfied will produce an equivalence between these spectra. The criterion will involve a map which we call the *ascent* map. The terminology "ascent" refers to the fact that the method gives a description of KE coming from information about KF, rather than describing KF in terms of KE, as is done in the descent. In the interest of clarity, we will first describe the approach as it works in the abelian case, where it is not necessary to pass to the generality of Mackey and Green functors, and then indicate the changes necessary when we deal with the more general situation. Finally, we suppose that F contains all the l-th power roots of unity, and as usual we let E denote the algebraic closure of F and G denote the absolute Galois group of F.

Consider the forgetful functor $V^G(E) \xrightarrow{\phi} V(G, E) \cong Vect(E)$, which simply forgets the G-action. It induces a map

$$KF \cong KV^G(E) \xrightarrow{K\phi} KV(G, E) \cong KE$$

We now have a commutative diagram

$$\begin{array}{ccc}
KRep_F[G] \wedge KV^G(E) & \longrightarrow & KV^G(E) \\
\Big\downarrow {\scriptstyle \varepsilon \wedge id} & & \Big\downarrow {\scriptstyle K\phi} \\
KF \wedge KV^G(E) & \longrightarrow & KE \\
\Big\downarrow {\scriptstyle id \wedge K\phi} & & \Big\downarrow {\scriptstyle id} \\
KF \wedge KE & \longrightarrow & KE
\end{array}$$

of spectra, where the horizontal map are multiplication maps defining algebra structures, and where the vertical maps are built out of the augmentation map ε and the forgetful functor ϕ. Note that the middle horizontal map exists because ε is induced by the forgetful functor $Rep_F[G] \to Vect(F)$ which forgets the G action. Considering the four corners of this diagram, we obtain a map of spectra

$$asc_F : KF \underset{KRep_F[G]}{\wedge} KV^G(E) \longrightarrow KV(G,E) \cong KE$$

It is readily verified that asc_F is a homomorphism of commutative S-algebras. We note further that both sides of this map are KF-algebras, and that asc_F is a homomorphism of KF-algebras. As usual, let $\mathbb{H}$ denote the mod-l Eilenberg-MacLane spectrum, so we have an obvious morphism $KF \to \mathbb{H}$ of S-algebras. We will now explore the consequences of the hypothesis that the homomorphism

$$id_{\mathbb{H}} \underset{KF}{\wedge} KF \underset{KRep_F[G]}{\wedge} KV^G(E) = \mathbb{H} \underset{KRep_F[G]}{\wedge} KV^G(E) \to \mathbb{H} \underset{KF}{\wedge} KE$$

is a weak equivalence.

For any homomorphism of S-algebras $f : D \to E$, we let $T^{\cdot}(D \to E)$ denote the cosimplicial S-algebra given by

$$T^k(D \to E) = \underbrace{E \underset{D}{\wedge} E \underset{D}{\wedge} \cdots \underset{D}{\wedge} E}_{k+1 \text{ factors}}$$

so $Tot(T^{\cdot}(D \to E)) \cong D^{\wedge}_f$. Now suppose that D and E are both k-algebras, where k is a commutative S-algebra, and that we are given a homomorphism $k \to \mathbb{H}$. We can now construct a bisimplicial spectrum $\Sigma^{\cdot\cdot} = \Sigma^{\cdot\cdot}(D \to E, k \to \mathbb{H})$ by setting

$$\Sigma^{kl} = \underbrace{\mathbb{H} \underset{k}{\wedge} \mathbb{H} \underset{k}{\wedge} \cdots \underset{k}{\wedge} \mathbb{H}}_{k+1 \text{ factors}} \underset{k}{\wedge} T^{l}(D \to E)$$

Proposition 4.5.1 *Suppose that k, D, and E are all (-1)-connected, and that $\pi_0 k \cong \pi_0 E \cong \mathbb{Z}$. Then $Tot(\Sigma^{\cdot\cdot}) \cong D^{\wedge}_{\varepsilon}$, where ε is the composite $D \to E \to \mathbb{H} \underset{k}{\wedge} E$. If in addition, $\pi_0 D \cong \mathbb{Z}$, then $Tot(\Sigma^{\cdot\cdot})$ is equivalent to the l-adic completion of D.*

Proof: The diagonal bisimplicial spectrum is canonically equivalent to the cosimplicial spectrum $T^{\cdot}(D \to E \underset{k}{\wedge} \mathbb{H})$. By use of the Künneth spectral sequence, it is clear that $E \underset{k}{\wedge} \mathbb{H}$ is (-1) connected and that $\pi_0 E \underset{k}{\wedge} \mathbb{H} \cong \mathbb{F}_l$. The first result now follows from 3.5. The second follows immediately from 3.6. $\square$

Whenever we have a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

of k-algebras, we have an induced map

$$\Sigma^{\cdot\cdot}(A \to B) \to \Sigma^{\cdot\cdot}(A' \to B')$$

Now suppose we have two k-algebra homomorphisms $A \to B$ and $A \to C$, so that each of the composites $k \to A \to B$ and $k \to A \to C$ are weak equivalences. We then obtain a commutative diagram of commutative k-algebras

$$\begin{array}{ccc} B & \longrightarrow & B \underset{A}{\wedge} C \\ \downarrow & & \downarrow \\ B \underset{A}{\wedge} C \underset{k}{\wedge} A & \longrightarrow & B \underset{A}{\wedge} C \end{array}$$

The upper horizontal arrow and the left hand vertical arrow are obvious inclusions, the right hand arrow is the identity, and the lower horizontal arrow is the multiplication map obtained by regarding $B \underset{A}{\wedge} C$ as a right module over the k-algebra A. We consequently obtain a map of bicosimplicial k-algebras

$$\Sigma^{\cdot\cdot}(B \to B \underset{A}{\wedge} C) \longrightarrow \Sigma^{\cdot\cdot}(B \underset{A}{\wedge} C \underset{k}{\wedge} A \to B \underset{A}{\wedge} C)$$

Proposition 4.5.2 *Under the hypotheses above, i.e. that $k \to B$ and $k \to C$ are weak equivalences, this map of bicosimplicial k-algebras is a levelwise weak equivalence, and hence induces an equivalence taking total spectra.*

Proof: There is a commutative diagram

$$\begin{array}{ccc} B \underset{A}{\wedge} T^{\cdot}(A \to C) & \xrightarrow{\sim} & T^{\cdot}(B \to B \underset{A}{\wedge} C) \\ \downarrow & & \downarrow \\ B \underset{A}{\wedge} T^{\cdot}(C \underset{k}{\wedge} A \to C) & \xrightarrow{\sim} & T^{\cdot}(B \underset{A}{\wedge} C \underset{k}{\wedge} A \to B \underset{A}{\wedge} C) \end{array}$$

where the horizontal arrows are levelwise weak equivalences. This follows from the standard isomorphism

$$B \underset{A}{\wedge} \underbrace{(C \underset{A}{\wedge} C \underset{A}{\wedge} \cdots \underset{A}{\wedge} C)}_{k \text{ factors}} \to \underbrace{(B \underset{A}{\wedge} C) \underset{B}{\wedge} (B \underset{A}{\wedge} C) \underset{B}{\wedge} \cdots \underset{B}{\wedge} (B \underset{A}{\wedge} C)}_{k \text{ factors}}$$

It now follows that the map $T^{\cdot}(A \to C) \to T^{\cdot}(C \underset{k}{\wedge} A \to C)$ is a levelwise weak equivalence, since the composite

$$A \cong k \underset{k}{\wedge} A \longrightarrow C \underset{k}{\wedge} A$$

is a weak equivalence. Applying the functors

$$\underbrace{\mathbb{H} \underset{k}{\wedge} \mathbb{H} \underset{k}{\wedge} \cdots \underset{k}{\wedge} \mathbb{H}}_{k \text{ factors}} \underset{k}{\wedge} -$$

to this equivalence, we obtain the result for $\Sigma^{\cdot\cdot}$ $\square$

We now examine the consequences of this result in our case. The spectra will be $k = KF$, $A = KRep_F[G]$, $B = KV^G(E)$, and $C = KF$, viewed as an A-algebra via the augmentation which forgets the action. This data clearly satisfies the hypotheses above. If $id_{\mathbb{H}} \underset{KF}{\wedge} asc_F$ is a weak equivalence, then the arrow $B \to B \underset{A}{\wedge} C$ can be identified with the arrow $KV^G(E) \to KV(G, E)$, which in turn can be identified canonically with the arrow $KF \to KE$. It follows that

$$Tot(\Sigma^{\cdot\cdot}(B \to B \underset{A}{\wedge} C)) \cong Tot(\Sigma^{\cdot\cdot}(KF \to KE)) \cong KF_l^{\wedge}$$

On the other hand,

$$\Sigma^{\cdot\cdot}(B \underset{A}{\wedge} C \underset{k}{\wedge} A \to B \underset{A}{\wedge} C) \cong$$

$$\Sigma^{\cdot\cdot}(KF \underset{KRep_F[G]}{\wedge} KV^G(E) \underset{KF}{\wedge} KRep_F[G] \to KF \underset{KRep_F[G]}{\wedge} KV^G(E))$$

and asc_F induces a map of bicosimplicial spectra

$$\Sigma^{\cdot\cdot}(KF \underset{KRep_F[G]}{\wedge} KV^G(E) \underset{KF}{\wedge} KRep_F[G] \to KF \underset{KRep_F[G]}{\wedge} KV^G(E)) \longrightarrow$$

$$\Sigma^{\cdot\cdot}(KE \underset{KF}{\wedge} KRep_F[G] \to KE)$$

By 3.4, if $id_{\mathbb{H}} \underset{KF}{\wedge} asc_F$ is a weak equivalence, this map of bicosimplicial spectra is a weak equivalence. The natural homomorphism $KE \underset{KF}{\wedge} KRep_F[G] \to KRep_E[G]$ now induces another map of bicosimplicial spectra

$$\Sigma^{\cdot\cdot}(KE \underset{KF}{\wedge} KRep_F[G] \to KE) \longrightarrow \Sigma^{\cdot\cdot}(KRep_E[G] \to KE)$$

We now have

Proposition 4.5.3 *When all l-th power roots of unity are in F, the evident map $KRep_F[G] \underset{KF}{\wedge} KE \to KRep_E[G]$ is a weak equivalence.*

Proof: It is a standard fact in representation theory that for any field F which contain the l-th power roots of unity, the isomorphism classes of representations of any finite l-group are in bijective correspondence with the isomorphism classes of representations of the same group in $\mathbb{C}$. Moreover, the endomorphism ring of any irreducible representation is a copy of F. We therefore have isomorphisms

$$\pi_* KRep_F[G] \cong K_*F \otimes R[G]$$

and

$$\pi_* KRep_E[G] \cong K_*E \otimes R[G]$$

where $R[G]$ denotes the complex representation ring. The E_2-term of the Künneth spectral sequence for $\pi_* KRep_F[G] \underset{KF}{\wedge} KE$ now has the form

$$Tor_{K_*F}(K_*F \otimes R[G], K_*E) \cong K_*Rep_E[G]$$

which gives the result. □

We now have the following conclusion.

Theorem 4.5.4 *Suppose that F contains all the l-th power roots of unity. Suppose further that $id_{\mathbb{H}} \underset{a}{KF} sc_F$ is a weak equivalence. Then we have a weak equivalence*

$$KF_l^\wedge \cong KRep_E[G]_\varepsilon^\wedge$$

where $\varepsilon : KRep_E[G] \to KE$ denotes the augmentation.

We observed in Section 4.3 that we did not expect to have an equivalence

$$KF \cong KRep_k[G]^{\wedge}_{\varepsilon}$$

as stated except in the abelian case, but that there is an analogous "fully equivariant version" involving equivariant spectra and Green functors which we expect to hold in all cases. This suggests that we should not expect the ascent map as formulated above to hold except in the abelian case. However, it is possible to modify the ascent map to extend it to the fully equivariant version of the completion, and we indicate how.

We fix the group $G = G_F$, and we define various equivariant S-algebras. We let $\underline{KV}$ denote the equivariant spectrum obtained using the category of descent data described above. For any subgroup $H \subseteq G$, the fixed point spectrum $\underline{KV}^H$ is equivalent to the K-theory spectrum $K(E^H)$. For any field F, we let $\underline{KRep_F}$ denote the equivariant spectrum corresponding to the trivial action of G on F, so for any $H \subseteq G$, we have that $\underline{KRep_F}^H \cong KRep_F[H]$. Also for any field F, we will write $\underline{KF}$ for the equivariant spectrum $KF \wedge S^0$, where S^0 denotes the G-equivariant sphere spectrum. We have that $\underline{KF}^H \cong KF \wedge G/H_+$ for any subgroup of finite index $H \subseteq G$. Finally, we let $\underline{\mathcal{H}}$ denote the G-spectrum associated to the Green functor $\mathcal{Z}/l\mathcal{Z}$ described above in section 4.3. We note that there is an obvious homomorphism of G-equivariant S-algebras $\underline{KF} \to \underline{\mathcal{H}}$. We can easily verify that there is a natural analogue of the ascent map in this equivariant setting, i.e. a map of G-equivariant S-algebras $asc_F : \underline{KF} \underset{\underline{KRep_F}}{\wedge} \underline{KV} \to \underline{KE}$ We now have the following equivariant generalization of 4.5.4, whose proof is identical to the nonequivariant version given above.

Theorem 4.5.5 *Let F, E, and G be as above. Suppose further that the map of equivariant spectra*

$$id_{\mathcal{H}} \underset{\underline{KF}}{\wedge} asc_F : \mathcal{H} \underset{\underline{KRep_F}}{\wedge} \underline{KV} \to \mathcal{H} \underset{\underline{KF}}{\wedge} \underline{KE}$$

is a weak equivalence of spectra. Then there is a canonically defined equivalence of G-equivariant S-algebras

$$\underline{KV}^{\wedge}_{\varepsilon} \to \underline{KRep_E}^{\wedge}_{\varepsilon}$$

where in both cases the map ε denotes the obvious homomorphism to $\mathcal{H}$. Note that the fixed point sets of the equivariant spectrum $\underline{KV}^{\wedge}_{\varepsilon}$ are just the l-adic completions of the K-theory spectra of the corresponding subfields.

Remark: We expect that the equivariant map $id_{\mathcal{H}} \underset{\underline{KF}}{\wedge} asc_F$ will be an equivalence for all fields containing all the l-th power roots of unity.

4.6 Derived representation theory and deformation K-theory

So far, our understanding of the derived completion of representation rings is limited to the knowledge of spectral sequences for computing them, given information about Tor or Ext functors of these rings. In this (entirely speculative) section, we want to suggest that there should be a relationship between derived representation theory and deformations of representations.

We consider first the representation theory of finite l-groups G. In this case, the derived representation ring of G is just the l-adic completion of $R[G]$, as can be verified using the results of Atiyah [1]. In particular, the derived representation ring has no higher homotopy. However, when we pass to a profinite l-group, such as $\mathbb{Z}_l$, we find that the derived representation ring does have higher homotopy, a single copy of $\mathbb{Z}_l$ in dimension 1. This situation appears to be parallel to the following situation. Consider the infinite discrete group $\Gamma = \mathbb{Z}$. We may consider the category $Rep_{\mathbb{C}}[\Gamma]$ of finite dimensional representations of Γ, and its K-theory spectrum $KRep_{\mathbb{C}}[\Gamma]$. The homotopy of this spectrum is given by

$$\pi_* KRep_{\mathbb{C}}[\Gamma] \cong \mathbb{Z}[S^1] \otimes K_*\mathbb{C}$$

where S^1 is the circle regarded as a discrete group. This isomorphism arises from the existence of Jordan normal form, which (suitably interpreted) shows that every representation of Γ admits a filtration by subrepresentations so that the subquotients are one-dimensional, and are therefore given by multiplication by a uniquely defined non-zero complex number. This construction does not take into account the topology of $\mathbb{C}$ at all. We might take the topology into account as in the following definition.

Definition 4.6.1 *Consider any discrete group Γ. For each k, we consider the category $Rep^k_{\mathbb{C}}[\Gamma]$ whose objects are all possible continuous actions of Γ on $\Delta[k] \times V$ which preserve the projection $\pi : \Delta[k] \times V \to \Delta[k]$, and which are linear on each fiber of π. It is clear that the categories $Rep^k_{\mathbb{C}}[\Gamma]$ fit together into a simplicial symmetric monoidal category, and we define the* **deformation K-theory spectrum of** Γ*, $K^{def}[\Gamma]$, as the total spectrum of the simplicial spectrum*

$$k \to KRep^k_{\mathbb{C}}[\Gamma]$$

Remark: The terminology is justified by the observation that, for example, an object in the category of 1-simplices $Rep^1_{\mathbb{C}}[\Gamma]$ is exactly a path in the space of representations of Γ in $GL(V)$, or a deformation of the representation at $0 \times V$.

One can easily check that $\pi_0 K^{def}[\mathbb{Z}] \cong \mathbb{Z}$, with the isomorphism given by sending a represenatation to its dimension. This follows from the fact that any two representations of the same dimension of $\mathbb{Z}$ can be connected by a deformation. In fact, one can apply the functor π_0 levelwise to the simplicial spectrum $KRep^{\cdot}_{\mathbb{C}}[\mathbb{Z}]$, and attempt to compute the homotopy groups of the simplicial abelian group $k \to \pi_0 KRep^k_{\mathbb{C}}[\mathbb{Z}]$. As above, it is easy to see that π_0 of this simplicial abelian group is zero, and it appears likely that

$$\pi_*(\pi_0 KRep^{\cdot}_{\mathbb{C}}[\mathbb{Z}]) \cong \pi_* \mathbb{Z}[S^1] \cong H_*(S^1)$$

where S^1 is the circle regarded as a topological group. Note that S^1 is the character group of the original discrete group $\mathbb{Z}$.

Now consider the profinite group $\mathbb{Z}_l$. We may define the deformation K-theory of this group as the direct limit of its finite quotients, and in view of the rigidity of complex representations of finite groups we have that

$$KRep_{\mathbb{C}}[\mathbb{Z}_l] \cong K^{def}[\mathbb{Z}_l]$$

We have an inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_l$, and therefore a map of spectra $K^{def}[\mathbb{Z}_l] \to K^{def}[\mathbb{Z}]$. We now obtain a composite map of derived completions

$$KRep_{\mathbb{C}}[\mathbb{Z}_l]^{\wedge}_{\varepsilon} \simeq K^{def}[\mathbb{Z}_l]^{\wedge}_{\varepsilon} \longrightarrow K^{def}[\mathbb{Z}]^{\wedge}_{\varepsilon}$$

Since the homotopy groups on the two sides appear isomorphic, it appears likely that this composite is a homotopy equivalence. We now observe that for any discrete group Γ, we may consider its pro-l completion $\Gamma^{\wedge}_l$, and obtain a map $KRep_{\mathbb{C}}[\Gamma^{\wedge}_l] \to K^{def}[\Gamma]$. This map induces a map j of derived completions, and we ask the following question.

Question: For what discrete groups does the map j induce a weak equivalence of spectra after derived completion at the augmentation map $\varepsilon : K^{def}[\Gamma] \to K\mathbb{C}$? Note that the completion may have to be taken in the category of $\Gamma^{\wedge}_l$-equivariant spectra.

We can further ask if the effect of derived completion can be computed directly on the pro-l group, rather than by permitting deformations on a discrete subgroup. This cannot be achieved over $\mathbb{C}$ since representations of finite groups are rigid, but perhaps over some other algebraically closed field.

Question: Can one construct a deformation K-theory of representations of a pro-l group G, using an algebraic deformation theory like the one discussed in [25], so that it coincides with the derived completion of $KRep_{\mathbb{C}}[G]$?

We conclude by pointing out an analogy between our theory of the representational assembly and other well known constructions. Consider a $K(\Gamma, 1)$ manifold X. The category $VB(X)$ of complex vector bundles over X is a symmetric monoidal category, and we can construct its K-theory. On the other hand, we have a functor from the category of complex representations of Γ to $VB(X)$, given by

$$\rho \to (\tilde{X} \underset{\Gamma}{\times} \rho \to X)$$

where $\tilde{X}$ denotes the universal covering space of X. This functor produces a map of K-theory spectra. It is also easy to define a "deformation version" of

$KVB(X)$, and one obtains a map

$$K^{def}[\Gamma] \rightarrow K^{def}VB(X)$$

This map should be viewed as the analogue in this setting for the representational assembly we discussed in the case of K-theory of fields, using the point of view that fields are analogues of $K(G,1)$ manifolds in the algebraic geometric context.

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