Reductions for Homology Computations

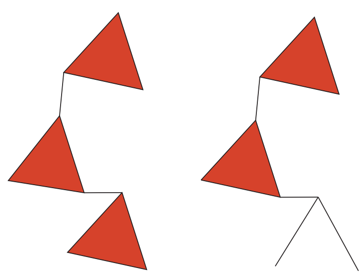
January 23, 2014

The purpose of this note is just to record the two results we showed in class.

Proposition 0.1 *Let X be an abstract simplicial complex. Suppose that σ is a maximal simplex of X, i.e. so that it is not contained in any other simplex. Let d denote the dimension of σ. Suppose further that $\sigma' \subseteq \sigma$ is a face, of dimension $d-1$, and suppose further that σ' is not contained in any simplex other than σ and itself. Then we may form the subcomplex $X' \subseteq X$, by removing the simplices σ and σ'. Then the result is that the linear transformation*

$$i_l : H_l(X') \longrightarrow H_l(X)$$

induced by the inclusion $X' \hookrightarrow X$ is an isomorphism.



X is on the left and X' is on the right.

Proposition 0.2 *Let X be any abstract simplicial complex, and let σ be any maximal simplex in X. The maximality of σ means that we may remove σ from X, and obtain from it a subcomplex X'. Let d denote the dimension of σ. We observe that $\partial^d(\sigma)$ is a cycle in $C_{d-1}(X')$, because all the $(d-1)$-simplices in X are also in X'. Consequently, the equivalence class $[\partial^d(\sigma)]$ is an element $\xi_\sigma \in H_{d-1}(X')$. Then the relationship between the homology of X and the homology of X' is given as follows.*

1. *$H_l(X) \cong H_l(X')$ whenever $l \neq d, d-1$.*

2. $H_{d-1}(X) \cong H_{d-1}(X')/(\xi_\sigma)$, *where* (ξ_σ) *denotes the span of the vector* ξ_σ. *Note that if* ξ_σ *is the zero element in* $H_{d-1}(X')$, *then* $H_{d-1}(X) \cong H_{d-1}(X')$.

3. *If* $\xi_\sigma \neq 0$ *in* $H_{d-1}(X')$, *then* $H_d(X) \cong H_d(X')$. *If* $\xi_\sigma = 0$, *then* $H_d(X) = H_d(X') \oplus F$, *where* F *is the ground field.*