Math 149 - Problem Set 1 - Due Tuesday, January 28

1. For any finite partially ordered set $(X, \leq)$, we define the *nerve* of $(X, \leq)$ to be the abstract simplicial complex with vertex set X, and where a family of points $\{x_0, x_1, \ldots, x_n\}$ spans a simplex if and only if there is a permutation σ of the set $\{0, 1, \ldots, n\}$ so that

$$x_{\sigma(0)} < x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}$$

   Let Π_n denote the partially ordered set of proper non-trivial subsets of the set $\{0, 1, \ldots, n\}$. Determine the all the homology groups of the nerve of Π_n for $n = 1, 2, 3$ and 4, with coefficients in any field. Also, determine familiar spaces which are homeomorphic to $|\Pi_n|$. Make a conjecture for the homology for all values of n.

2. Find triangulations for the projective plane, the Klein bottle, and the two holed torus, and determine their homology directly without peeking at other sources.

3. For any finite metric space X and parameter $R > 0$, we define the *Vietoris-Rips complex* of X with respect to R, $V(X, R)$, to be the abstract simplicial complex with vertex set X, and so that a set $\{x_0, \ldots, x_n\}$ is an n-simplex if and only if $d(x_1, x_j) \leq R$ for all $0 \leq i, j \leq n$. Let X be the collections of points on the unit circle $\{(\pm 1, 0), (0, \pm 1), (\pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2})\}$. These are the points of a regular octagon, on the unit circle. Determine the homology of $V(X, R)$ for all values of R. Also, determine the effect on homology of the inclusion $V(X, R) \hookrightarrow V(X, R')$ whenever $R \leq R'$.

4. Let X be any set, with a covering by subsets $U_i \subseteq X$, so $X = \bigcup_i U_i$, where $0 \leq i \leq n$. By the *nerve* of the covering $\mathcal{U} = \{U_i\}_i$, we will mean the abstract simplicial complex whose vertices are the integers $\{0, 1, \ldots, n\}$, and where $\{i_0, i_1, \ldots, i_x\}$ spans an s-simplex if and only if $U_{i_0} \cap \cdots \cap U_{i_s} \neq \emptyset$. Describe the nerve of the covering of the sphere S^2 by the intersection of all the octants in $\mathbb{R}^3$ with S^2, and compute its homology.