THE INTEGRAL K-THEORETIC NOVIKOV CONJECTURE FOR GROUPS WITH FINITE ASYMPTOTIC DIMENSION

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ABSTRACT. The integral assembly map in algebraic K-theory is split injective for any geometrically finite discrete group with finite asymptotic dimension.

The goal of this paper is to apply the techniques developed by the first author in [3] to verify the integral Novikov conjecture for groups with finite asymptotic dimension as defined by M. Gromov [9].

Recall that a finitely generated group Γ can be viewed as a metric space with the *word metric* associated to a given presentation.

Definition (Gromov)**.** A family of subsets in a general metric space X is called *d-disjoint* if $\mathrm{dist}(V, V') = \inf\{\mathrm{dist}(x, x') | x \in V,\ x' \in V'\} > d$ for all V, V'. The *asymptotic dimension* of X is defined as the smallest number n such that for any $d > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X by $n + 1$ d-disjoint families of subsets $\mathcal{U} = \mathcal{U}^0 \cup \ldots \cup \mathcal{U}^n$.

It is known that asymptotic dimension is a quasi-isometry invariant and so is an invariant of the finitely generated group, independent of the presentation. One says Γ has finite asymptotic dimension if it does as the metric space with a word metric.

Examples from this apparently very large class are the Gromov hyperbolic groups [9], Coxeter groups [8], various generalized products of these, including the groups acting on trees with vertex stabilizers of finite asymptotic dimension [2], and, more generally, fundamental groups of developable complexes of finite dimensional groups [1]. We proved in [5] that cocompact lattices in connected Lie groups also have finite asymptotic dimension.

Let $K(A)$ be the nonconnective K-theory spectrum of the ring A. A discrete group is called *geometrically finite* if its classifying space has the homotopy type of a finite complex. Our main result is the following theorem.

Main Theorem. *Let Γ be a geometrically finite group with finite asymptotic dimension and let R be an arbitrary ring. Then the assembly map $\alpha\colon h(\Gamma, K(R)) \to K(R[\Gamma])$ from the homology of the group Γ with coefficients in the K-theory spectrum $K(R)$ to the K-theory of the group ring $R[\Gamma]$ is a split injection.*

We should mention that the original Novikov conjecture on homotopy invariance of higher signatures has been verified for fundamental groups with finite asymptotic dimension by G. Yu [10]. Also, Gromov has constructed examples of geometrically finite groups with infinite asymptotic dimension, cf. [7], footnote to Problem 8 in section 9.

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First, we summarize in section 2 the conversion of the Novikov conjecture for assembly maps in algebraic K-theory to the statement that certain controlled assembly maps are weak homotopy equivalences and list the properties of groups and metric spaces required to prove the latter. In section 3 we verify that groups with finite asymptotic dimension satisfy those properties.

1. Coarse Locally Finite Homology

In this section, we will modify the definition of ${}^b h^{lf}$ from [3] to produce a coarse version of it. This theory will have the property that it captures the homology of a locally compact space "at infinity", as does ${}^b h^{lf}$, but does not see any of the ordinary homology of the space. Locally finite homology as defined in [3] is equivalent to ordinary homology for compact spaces, while our coarse version will be identical to its value on a point for all compact spaces. This is an advantage, since the comparison with bounded K-theory of metric spaces will be more direct.

Here metric spaces will be understood in the following generalized sense.

Definition 1.1. A *metric space* is a set X and a function $d\colon X \times X \to [0,\infty) \cup \{\infty\}$ which is reflexive, symmetric, and satisfies the triangle inequality in the obvious way. The metric space is *proper* if it is a countable disjoint union of metric spaces X_i where $\mathrm{im}(d|X_i \times X_i) \subset [0,\infty)$, and all closed metric balls in X are compact. The metric topology on a metric space is defined as usual.

We recall from [3] that h^{lf} is defined as follows, for any topological space. $S.X$ is defined to be the usual singular complex simplicial set attached to X, whose k-simplices are the continuous maps from the standard k-simplex to X. We say that a subset $\mathcal{A} \subseteq S_k X$ is *locally finite* if for every point $x \in X$, there is a neighborhood U of x so that $U \cap \mathrm{im}(\sigma)$ is non-empty for only finitely many $\sigma \in \mathcal{A}$. It is clear that if $\mathcal{A}$ is locally finite, then so are $d_i\mathcal{A} = \{d_i\sigma \mid \sigma \in \mathcal{A}\}$ and $s_i\mathcal{A} = \{s_i\sigma \mid \sigma \in \mathcal{A}\}$, and that $d_i|\mathcal{A}$ and $s_i|\mathcal{A}$ are proper maps of sets. Recall that a map of sets is *proper* if the inverse images of finite sets of points are finite. We let $\mathcal{L}_k X$ denote the partially ordered set of locally finite subsets of $S_k X$, where the partial ordering is via inclusions of sets. The face and degeneracy maps d_i and s_i induce maps of partially ordered sets $\mathcal{L}d_i$ and $\mathcal{L}s_i$. For a spectrum S, we define $\mathcal{I}_k(X,S)$ to be the colimit

$$\underset{A \in \mathcal{L}_k X}{\underrightarrow{\mathrm{colim}}}\ h^{lf}(\mathcal{A};S)$$

The locally finite homology h^{lf} is defined on the category of sets and proper maps as in [3], section II. Now $h^{lf}(X,S)$ is defined to be the total spectrum $|\mathcal{I}.(X,S)|$. When X is a proper metric space, we restrict ourselves to sets $\mathcal{A}$ of singular simplices of uniformly bounded diameter (i.e. the sets $\mathrm{im}(\sigma)$ as σ varies over $\mathcal{A}$ have diameter bounded by some fixed number N), and obtain the related theory ${}^b h^{lf}(X,S)$. This is the theory which was used to prove the results in [3] where it was possible to define an assembly map from ${}^b h^{lf}(X,K(R))$ to $K(X,R)$.

The observation we now make that once one has made the restriction to families of singular simplices of uniformly bounded diameter, one can construct a locally finite homology and corresponding assembly map which does not require that the singular simplices be continuous maps. So, we will let $\mathcal{A}^C$ denote a collection of (possibly non-continuous) maps from Δ^k to X of uniformly bounded diameter. The local finiteness criterion still makes sense as stated, and we may construct a spectrum ${}^C h^{lf}(X,S)$ as in [3]. Moreover, when one examines the construction of the assembly map, it is easily observed that continuity of σ is never used, only the uniform boundedness of the diameters of the sets $\mathrm{im}(\sigma)$. Thus, we obtain an assembly map

$$A\colon {}^C h^{lf}(X, K(R)) \longrightarrow K(X,R).$$

It is now easy to check the following two properties of the theory ${}^C h^{lf}(X,S)$.

Proposition 1.2. *Suppose that a metric space X is a disjoint union of subsets X_α, and that for any $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$, with $\alpha \neq \beta$, we have $d(x_\alpha, x_\beta) = +\infty$. Suppose further that each of the subsets X_α has diameter uniformly bounded by a fixed number $N \geq 0$. Then ${}^C h^{lf}(X,S) \cong \prod_\alpha S$, and the assembly map is an equivalence for X.*

We recall that the theory h^{lf} is excisive for locally finite coverings of locally compact spaces. This means that for any locally finite covering $\{U_\alpha\}_{\alpha\in A}$ of X, we construct the simplicial spectrum which in level k is the space

$$h^{lf}\left(\coprod_{\{\alpha_0,\alpha_1,\ldots,\alpha_k\}\in A_k} U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k}\right)$$

and that the evident map from this simplicial spectrum to the constant simplicial spectrum with value $h^{lf}(X,S)$ is an equivalence of spectra. The analogous result for ${}^C h^{lf}$ is as follows.

Proposition 1.3. *Given a locally finite covering of a metric space X as above, and a parameter d, we construct the simplicial metric space which in level k is given by*

$${}^C h^{lf}\left(\coprod_{\{\alpha_0,\alpha_1,\ldots,\alpha_k\}\in A_k} N_d U_{\alpha_0}\cap N_d U_{\alpha_1}\cap\ldots\cap N_d U_{\alpha_k}\right),$$

where N_d denotes the d-neighborhood of the set in question. Note that the coproduct means that distances between points in different coproduct factors are always infinite. There is an evident map from this simplicial spectrum to the constant simplicial spectrum with value ${}^C h^{lf}(X,S)$, and this map becomes a weak equivalence of spectra after passage to colimits over d.

This proof is similar to the corresponding result in [3], Proposition II.20.

Whenever X is a proper metric space with a group action by isometries, the spectrum ${}^C h^{lf}(X,S)$ is equivariant. Recall that the fixed point spectrum of a Γ-spectrum R can be defined as $R^\Gamma = \mathrm{Map}_\Gamma(S^0, R_+)$. The homotopy fixed point spectrum can be defined similarly as $R^{h\Gamma} = \mathrm{Map}_\Gamma(X_+, R_+)$. The collapse $\rho\colon X_+ \to S^0$ induces the canonical maps $\rho^*\colon R^\Gamma \to R^{h\Gamma}$.

Proposition 1.4. *Let X be a locally finite simplicial complex with a free action by a torsion-free group Γ. We assume that there is a contractible finite dimensional complex with a free action of Γ. Then the canonical map $\rho^*\colon\ h^{lf}(X,S)^\Gamma \to$*

$h^{lf}(X,S)^{h\Gamma}$ is an equivalence. If X is equipped with a metric so that all simplices of X have uniformly bounded diameter, then the map $\rho^*\colon {}^C h^{lf}(X,S)^\Gamma \to {}^C h^{lf}(X,S)^{h\Gamma}$ is an equivalence.

The first statement is proved in [3]. The second follows similarly by observing that comparison with the auxiliary simplicial theory ${}^C h^{lf}(X,S)$ again does not require continuity of singular simplices.

2. Controlled Assembly Maps and the Novikov Conjecture

Given a discrete group Γ and a ring R, one may view an element $\gamma \in \Gamma$ as an isomorphism of the trivial $R[\Gamma]$-module with the inverse γ^{-1}. Following Loday, to each isomorphism f of finitely generated R-modules there corresponds an isomorphism $\gamma \otimes f$ of finitely generated $R[\Gamma]$-modules. This functor induces the assembly map $\alpha\colon B\Gamma_+ \wedge K(R) \to K(R[\Gamma])$ from the Main Theorem. Here $B\Gamma$ is the compact universal space of Γ and $B\Gamma_+ \wedge K(R)$ is the homology spectrum of the group Γ with coefficients in the nonconnective K-theory spectrum of R. The target is the nonconnective K-theory spectrum of the group ring $R[\Gamma]$. The *integral Novikov conjecture* predicts that this map is a split injection for any ring R and any group Γ with $B\Gamma$ a finite CW-complex.

The method here is to interpret α as the fixed point map A^Γ of an equivariant assembly map of spectra

$$A\colon {}^C h^{lf}(X, K(R)) \longrightarrow K(X,R)$$

where X is the universal cover of $B\Gamma$, ${}^C h^{lf}$ is the coarse locally finite homology theory from section 1, and $K(X,R)$ is the nonconnective bounded K-theory of geometric R-modules on X. The latter construction requires a proper metric on X which can be chosen to be the lifting of any bounded metric on the compact classifying space $B\Gamma$.

Given a proper metric space X and a ring R, recall that the category of *geometric modules* $\mathcal{B}(X,R)$ associated to X has objects triples (F,B,ϕ) where F is a free R-module on the basis B, and $\phi\colon B \to X$ is the labelling function such that $\phi^{-1}(S)$ is finite for a bounded $S \subset X$. A morphism $f\colon (F,B,\phi) \to (F',B',\phi')$ in $\mathcal{B}(X,R)$ is an R-linear homomorphism $f\colon F \to F'$ with associated number $D \geq 0$ such that for any $b \in B$, its image $f(b)$ is generated by those elements b' in B' with the property $d(\phi(b),\phi(b')) \leq D$. The category $\mathcal{B}(X,R)$ is clearly additive. Its nonconnective algebraic K-theory spectrum $K(X,R)$ is usually called the *bounded K-theory* of geometric R-modules over X.

Definition 2.1. A map between metric spaces $\phi\colon (M_1,d_1) \to (M_2,d_2)$ is *eventually continuous* if there is a real function g such that

$$d_2(\phi(x),\phi(y)) \leq g(d_1(x,y))$$

for all pairs of points x, y in M_1. The map ϕ is *proper* if for any bounded subset $S \subset M_2$, the preimage $\phi^{-1}(S)$ is bounded in M_1.

It is easy to see that proper eventually continuous maps induce maps of K-theory spectra $\phi_*\colon K(M_1,R) \to K(M_2,R)$.

If X is a proper metric space with a free action of Γ by isometries so that the quotient X/Γ is compact, then $K(X,R)$ is a Γ-equivariant spectrum. A different weakly equivalent spectrum $K_{\Gamma,0}(X,R)$ defined in [3] has good equivariant properties but we will suppress the distinction in this paper always assuming the latter construction.

The following is a sketch of the approach to the integral Novikov conjecture. The assembly map α is related to the map induced by A on the Γ-fixed point spectra via the commutative diagram

$$\begin{array}{ccc} B\Gamma_+ \wedge K(R) & \xrightarrow{\alpha} & K(R[\Gamma]) \\ \simeq\downarrow & & \simeq\downarrow \\ {}^{C}h^{lf}(X, K(R))^{\Gamma} & \xrightarrow{A^{\Gamma}} & K(X,R)^{\Gamma} \end{array}$$

where the vertical arrows are both weak equivalences. Further, there is a commutative square

$$\begin{array}{ccc} {}^{C}h^{lf}(X, K(R))^{\Gamma} & \xrightarrow{A^{\Gamma}} & K(X,R)^{\Gamma} \\ \rho^*\downarrow & & \downarrow \\ {}^{C}h^{lf}(X, K(R))^{h\Gamma} & \xrightarrow{A^{h\Gamma}} & K(X,R)^{h\Gamma} \end{array}$$

By Proposition 1.4, the left-hand vertical map ρ^* is an equivalence whenever the group Γ is torsion-free. As soon as the lower fixed-point map $A^{h\Gamma}$ is an equivalence, the two combined commutative squares show that α induces a split injection. The second equivalence would follow from the observation that

$$A\colon {}^{C}h^{lf}(X, K(R)) \longrightarrow K(X,R)$$

is a nonequivariant equivalence and the general fact that in this case

$$A^{h\Gamma}\colon {}^{C}h^{lf}(X, K(R))^{h\Gamma} \longrightarrow K(X,R)^{h\Gamma}$$

is always a weak equivalence. For example, this was verified in [3] for torsion-free discrete cocompact subgroups of a connected Lie group. In this paper, we show that A is a weak equivalence for groups of finite asymptotic dimension.

One of the basic results in bounded K-theory is the *controlled excision* theorem for finite coverings of a proper metric space X. For a subset $S \subset X$, let $S[D]$ stand for the metric *D-enlargement* $\{x \in X \mid \mathrm{dist}(x,S) \le D\} \subset X$. Now suppose X is a union of subspaces Y and Z and let $\mathcal{B}(Y,Z;R)$ stand for the full additive subcategory of $\mathcal{B}(X,R)$ on objects (F,B,ϕ) such that there are numbers D_Y, $D_Z \ge 0$ with $\mathrm{im}(\phi) \subset Y[D_Y] \cap Z[D_Z]$.

Theorem 2.2 (IV.1 [3]). *The commutative diagram*

$$\begin{array}{ccc} K(Y,Z;R) & \longrightarrow & K(Y,R) \\ \downarrow & & \downarrow \\ K(Z,R) & \longrightarrow & K(X,R) \end{array}$$

is a homotopy pushout.

We also need a controlled excision result for certain infinite one-dimensional coverings of X. Instead of proving an excision theorem for a single covering of X, one is forced to look at a directed system of coverings.

Definition 2.3. Let $\mathcal{U} = \{U_\alpha\}$, $\alpha \in A$, be a covering of a metric space X. It is *locally finite* if it has the following two properties:

(1) for any $\alpha \in A$, the set $\{\alpha' \in A \mid U_{\alpha'} \cap U_\alpha \neq \emptyset\}$ is finite,

(2) for any bounded subset $U \subset X$, the set $\{\alpha \in A \mid U \cap U_\alpha \neq \emptyset\}$ is finite.

Associated to a locally finite covering, one has a simplicial metric space $N.\mathcal{U}$ where

$$N_k \mathcal{U} = \coprod_{(\alpha_0,\ldots,\alpha_k)\in A^{k+1}} \bigcap_{i=0}^{k} U_{\alpha_i},$$

the face maps $d_i | U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}$ are the inclusions onto the disjoint union factors corresponding to $(\alpha_0, \ldots, \widehat{\alpha_i}, \ldots, \alpha_k)$, and the degeneracy maps $s_i | U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}$ are the identity maps onto the factors corresponding to $(\alpha_0, \ldots, \alpha_i, \alpha_i, \ldots, \alpha_k)$. It is easy to see that the face and degeneracy maps are proper and eventually continuous with respect to the induced generalized proper metrics in the sense of Definition 1.1.

A *map of coverings* $\Theta \colon \mathcal{U} \to \mathcal{U}'$ is a function of the indexing sets $\theta \colon A \to B$ such that $U_\alpha \subset U_{\theta(\alpha)}$. Such maps induce maps of simplicial spectra

$$K(N.\Theta, R) \colon K(N.\mathcal{U}, R) \longrightarrow K(N.\mathcal{U}', R).$$

The inclusions of the multiple intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}$ in X induce the assembly map

$$A(\mathcal{U}) \colon |K(N.\mathcal{U}, R)| \longrightarrow K(X, R)$$

associated to the covering $\mathcal{U}$. If $\mathcal{V}$ is a covering, then the covering by the d-enlargements of the elements of $\mathcal{V}$ is denoted by $\mathcal{V}[d]$. For a family of coverings $\mathcal{U}(\ell)$ parametrized by integers ℓ with maps $\mathcal{U}(\ell) \to \mathcal{U}(\ell')$ for $\ell \leq \ell'$, the associated family $\mathcal{U}(\ell)[d]$ determines the canonical assembly

$$\underset{\overrightarrow{\ell,d}}{\operatorname{hocolim}}\, A(\mathcal{U}(\ell)[d]) \colon \underset{\overrightarrow{\ell,d}}{\operatorname{hocolim}}\, |K(N.\mathcal{U}(\ell)[d], R)| \longrightarrow K(X, R).$$

Our goal is to find conditions on the coverings $\mathcal{U}(\ell)$ so that on one hand this assembly map may be identified with the controlled assembly A and on the other is a weak equivalence. This was done for coverings parametrized by the integers $\mathbb{Z}$ in [3]. In order for the result to apply to metric spaces of finite asymptotic dimension, we need to generalize to parametrizations by vertices in a locally finite tree T.

Suppose we are given a locally finite covering where the parameter set is viewed as vertices of a locally finite tree T with the property $U_\alpha \cap U_{\alpha'} \neq \emptyset$ if and only if $\{\alpha, \alpha'\}$ is an edge in T, which makes the nerve $N.\mathcal{U}$ a subcomplex of a simplicial tree. References to the natural order on $\mathbb{Z}$ must be replaced with the references to the partial order on vertices in a tree induced by choosing and fixing a vertex $\alpha_0 \in T$.

Definition 2.4. If $[t, t']$ stands for the unique geodesic segment connecting t to t' in T then the relation $\alpha \leq \alpha'$ on vertices corresponds to $\alpha \in [\alpha_0, \alpha']$. An *adjacent pair* of vertices consists of two vertices α and α' in T connected by an edge. When the adjacent vertices are related as in $\alpha \leq \alpha'$, we denote this relationship by the ordered pair $\langle \alpha, \alpha' \rangle$.

A simplicial complex is called *C-1* if it is a subcomplex of a locally finite simplicial tree.

Theorem 2.5. *If $\mathcal{U}(\ell)$ is a sequence of locally finite coverings of X indexed by $\ell \geq 0$ together with maps of coverings $\mathcal{U}(\ell) \to \mathcal{U}(\ell + 1)$ and the properties:*

(1) *for any index* ℓ *the nerve* $N.\mathcal{U}(\ell)$ *is C-1,*
(2) *for any number* $d \geq 0$ *there is an index* ℓ *so that* $N.\mathcal{U}(\ell)[d]$ *is C-1,*

then the assembly map $\operatorname{hocolim} A(\mathcal{U}(\ell)[d])$ *is a weak equivalence.*

Proof. The proof of Theorem IV.19 that occupies most of section IV in [3] can be repeated with necessary modifications in terminology and notations. Recall that there is a partial order on vertices in the tree determined by the fixed vertex v_0. The pairs of adjacent vertices replace the sets of adjacent integers in the proof. Thus in the definition of disjoint unions on page 82 of [3], one defines the union $Y_\ell = \coprod_\gamma U_\alpha \cap U_{\alpha'}$ where the sets U_* are members of $\mathcal{U}(\ell)$ and γ ranges over the set of all adjacent pairs of vertices $\langle \alpha, \alpha' \rangle$. The terminology 'even' and 'odd' is adapted to mean vertices whose distance to v_0 is even or odd respectively. $\triangledown$

3. Proof of the Main Theorem

We will apply Theorem 2.5 to spaces that asymptotically embed in products of trees. The *trees* we consider are the contractible one-dimensional simplicial complexes which are *locally finite* in the sense that the star of any vertex is finite. Our interest in locally finite trees is justified by the following characterization of asymptotic dimension.

Definition 3.1. A map between metric spaces $\phi\colon (M_1, d_1) \to (M_2, d_2)$ is a *uniform embedding* if there are two real functions f and g with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ such that

$$f(d_1(x,y)) \leq d_2(\phi(x), \phi(y)) \leq g(d_1(x,y))$$

for all pairs of points x, y in M_1.

Theorem 3.2 (Dranishnikov [6, 7]). *The asymptotic dimension of a metric space* M *is finite if and only if there is a uniform embedding of* M *in a finite product of locally finite simplicial trees.*

First observe that we may assume that Γ or the space X is a metric subspace of a product of trees Π.

Proposition 3.3. *A surjective map* $\phi\colon M_1 \to M_2$ *is a uniform embedding if and only if it is eventually continuous and there is an eventually continuous map* $\psi\colon M_2 \to M_1$ *which is an asymptotic inverse, in the sense that the compositions* $\psi\phi$ *and* $\phi\psi$ *are bounded. In other words, there are real functions* g *and* $\overline{g}$ *such that* $d_2(\phi(x), \phi(y)) \leq g(d_1(x,y))$ *and* $d_1(\psi(t), \psi(s)) \leq \overline{g}(d_2(t,s))$ *for all pairs of points* x, y *in* M_1 *and* t, s *in* M_2.

Proof. If ϕ is a uniform embedding, we may choose g for g and define

$$\overline{g}(z) = \sup\{z' \mid f(z') \leq z\}.$$

If $\psi\colon M_2 \to M_1$ is any function such that $\phi\psi(x) = \phi(x)$ for all $x \in M_1$, then since $f(d_1(\psi(t), \psi(s))) \leq d_2(t,s)$,

$$d_1(\psi(t), \psi(s)) \leq \sup\{z' \mid f(z') \leq d_2(t,s)\} = \overline{g}(d_2(t,s)).$$

Notice that $\phi\psi = \mathrm{id}$. Let $D > \sup\{r \mid f(r) = 0\}$, then for any pair of points x, $y \in M_1$ with $\phi(x) = \phi(y)$ we have $d_1(x,y) < D$. This shows that $\psi\phi$ is bounded by D.

To see that ϕ with an asymptotic inverse is a uniform embedding, we may again choose g for one of the bounding functions and define

$$f(z) = \inf\{z' \mid z \le \overline{g}(z') + 2D\},$$

where D is a bound for $\psi\phi$. Then

$$f(d_1(x,y)) = \inf\{z' \mid d_1(x,y) \le \overline{g}(z') + 2D\} \le d_2(\phi(x),\phi(y))$$

since

$$d_1(x,y)) \le \overline{g}(d_2(\phi(x),\phi(y))) + 2D.$$

We have $\lim_{z\to\infty} f(z) = \infty$ because X is not compact. ▽

Corollary 3.4. *If $\phi\colon M_1 \to M_2$ is a surjective uniform embedding then the induced map of spectra $K(M_1,R) \to K(M_2,R)$ is an equivalence.*

Proof. Both ϕ and ψ induce maps of the K-theory spectra since both are eventually continuous and clearly proper. Since bounded endomorphisms induce equivalences on the bounded K-theory, the induced maps are equivalences. ▽

Proposition 3.5. *If $\phi\colon M_1 \to M_2$ is a metric embedding onto a commensurable subspace, in the sense that there is a number D such that* $\mathrm{im}(\phi)[D] = M_2$*, then the induced map of spectra $K(M_1,R) \to K(M_2,R)$ is an equivalence.*

Proof. This follows from Corollary 3.4 since any function $M_2 \to \mathrm{im}(\phi)$ which is identity on the subspace $\mathrm{im}(\phi)$ and is bounded by D is a uniform embedding of M_2 in M_1. ▽

Corollary 3.6. *If there is a uniform embedding of proper metric spaces X in M then there is an open subset V of M such that the assembly $A(X)$ is an equivalence if and only if $A(M)$ is an equivalence.*

Proof. If $\phi\colon X \to M$ is the uniform embedding then the subset V can be taken to be the interior of $\mathrm{im}(\phi)[D]$ for some $D > 0$. ▽

The proof of the Main Theorem will require a family of coverings of trees with specific properties.

Proposition 3.7. *There is a family of coverings $\{\mathcal{U}(\ell)\}$ of a locally finite tree T by uniformly bounded subsets that satisfies the hypotheses of Theorem* 2.5.

Proof. Fix a vertex v_0 in the geometric realization of the tree T. Given another vertex $v \in T$, we define its *shadow* as the subset

$$\mathrm{Sh}(v) = \{t \in T \mid v \in [v_0,t)\}.$$

Let $B(v,d)$ stand for the open metric ball of radius d centered at v and $S(v,d)$ stand for its boundary sphere. If $l \ge 0$ then also define

$$\mathrm{Sh}(v,l) = \mathrm{Sh}(v) \cap B(v,l)$$

and

$$\mathrm{Sh}(v;l_1,l_2) = \mathrm{Sh}(v,l_2) - B(v,l_1) - S(v,l_1)$$

for $l_2 > l_1 > 0$.

For a number $d > 1$, consider the collection of open subsets of T consisting of the ball $B(v_0,2d)$ and the differences $\mathrm{Sh}(v;d-1,3d)$ where the vertices v vary over $S(v_0,(2n-1)d)$ for arbitrary natural numbers $n \in \mathbb{N}$. It is easy to see that this collection is a covering of T. Its nerve is a locally finite tree where the vertices are v_0 and the vertices $v \in S(v_0,(2n-1)d)$, the edges are the pairs (v,v') where $v' \in \mathrm{Sh}(v,2(n+1)d)$. The diameter of each set in the covering is bounded by $6d$.

Fix a number $D > 1$. We define coverings $\mathcal{U}(\ell)$, $\ell \geq 0$, by applying the construction with $d = 2^\ell D$. To see that $\mathcal{U}(\ell)$ is subordinate to $\mathcal{U}(\ell+1)$, notice that

$$U_{\ell+1}(v) = \mathrm{Sh}(v; 2^{\ell+1}D - 1, 2^{\ell+1}(3D))$$

in $\mathcal{U}(\ell+1)$ is the union of elements from $\mathcal{U}(\ell)$. Indeed, if

$$v \in S(v_0,(2n-1)2^{\ell+1}D)$$

then the corresponding subset is the union of

$$U_\ell(v') = \mathrm{Sh}(v'; 2^\ell D - 1, 2^\ell(3D))$$

for all vertices v' in $S(v_0, 2^\ell(4n-1)D) \cap \mathrm{Sh}(v)$ or in $S(v_0, 2^\ell(4n+1)D) \cap \mathrm{Sh}(v)$.

To check property (2), one only needs to choose $\ell > \log_2 d/D + 2$. $\triangledown$

Notation 3.8. In addition to the notation

$$U_\ell(v) = \mathrm{Sh}(v; 2^\ell D - 1, 2^\ell(3D))$$

introduced in Proposition 3.7 for elements of $\mathcal{U}(\ell)$, we will use $U_\ell(v,v')$ for the nonempty pairwise intersections $U_\ell(v) \cap U_\ell(v')$.

Explicitly, if $v \in S(v_0,(2n-1)d)$ and $v' \in S(v_0,(2n+1)d)$ so that $v \in [v_0,v']$, then

$$U_\ell(v,v') = \mathrm{Sh}(v'; 2^\ell D - 1, 2^\ell D).$$

Let $\Pi = \prod_{i=1}^n T_i$ be a product of n locally finite trees. Let Z be a countable discrete metric space, in the sense that the metric function d takes only values 0 and ∞.

Theorem 3.9. *If there is a uniform embedding of X in a product of the form $\Pi \times Z$ then the canonical assembly map $A(X)$ is a weak equivalence.*

We are interested in this theorem when Z is a single point, but the setup for the following inductive proof requires this general statement.

Proof. The proof proceeds by induction on $\mathrm{asdim}(X)$. Suppose $\mathrm{asdim}(X) = n$, then X has a uniform embedding ϕ in a product $\Pi \times Z = \prod_{i=1}^n T_i \times Z$. By Corollary 3.6, we may assume that X is an open metric subspace of $\Pi \times Z$, and ϕ is the embedding. Let $\pi \colon \Pi \to T_1$ be the first coordinate projection. Using the coverings $\mathcal{U}(\ell)$ of T_1 from Theorem 2.5, construct the associated coverings $\mathcal{U}'(\ell) = \{U' = \pi^{-1}(U) \mid U \in \mathcal{U}(\ell)\}$ of Π. Now there are product coverings $\mathcal{U} \times Z$, $\mathcal{U}(\ell) \times Z$, and $\mathcal{U}(\ell)[d] \times Z$ of $\Pi \times Z$ defined as $\{U' \times Z\}$, $\{U'(\ell) \times Z\}$, and $\{U'(\ell)[d] \times Z\}$ for $U' \in \mathcal{U}'$. There are also associated coverings $\mathcal{U}_X$, $\mathcal{U}_X(\ell)$, and $\mathcal{U}_X(\ell)[d]$ of X defined as $\{X \cap U \times Z\}$, $\{X \cap U(\ell) \times Z\}$, and $\{X \cap U(\ell)[d] \times Z\}$. These coverings of X satisfy the conditions of Proposition 3.7.

The coverings $\mathcal{U}_X(\ell)$ satisfy the conditions of Theorem 2.5, so the induced assembly map

$$\operatorname{hocolim} A(\mathcal{U}_X(\ell)[d]) \colon \operatorname*{hocolim}_{\overrightarrow{\ell,d}} |K(N.\mathcal{U}_X(\ell)[d], R)| \longrightarrow K(X,R)$$

is an equivalence. For the identification of $\operatorname{hocolim} A(\mathcal{U}_X(\ell)[d])$ with

$$A(X) \colon {}^{C}h^{lf}(X, K(R)) \longrightarrow K(X,R),$$

one observes that by Proposition 1.3 the coverings $\mathcal{U}_X(\ell)[d]$ are excisive, in addition to the properties in Theorem 2.5. So there is an equivalence

$$\underset{\overrightarrow{\ell,d}}{\operatorname{hocolim}}\, |\, {}^C h^{lf}(N.\mathcal{U}_X(\ell)[d], K(R))| \longrightarrow {}^C h^{lf}(X, K(R)).$$

Now the vertical maps in the commutative diagram

$$\begin{array}{ccc} \underset{\overrightarrow{\ell,d}}{\operatorname{hocolim}}\, |\, {}^C h^{lf}(N.\mathcal{U}_X(\ell)[d], K(R))| & \longrightarrow & \underset{\overrightarrow{\ell,d}}{\operatorname{hocolim}}\, |K(N.\mathcal{U}_X(\ell)[d], R)| \\ \big\downarrow & & \big\downarrow {\scriptstyle \operatorname{hocolim} A(\mathcal{U}_X(\ell)[d])} \\ {}^C h^{lf}(X, K(R)) & \xrightarrow{A(X)} & K(X, R) \end{array}$$

are weak equivalences. It remains to show that the assembly map

$$\underset{\overrightarrow{\ell,d}}{\operatorname{hocolim}}\, |\, {}^C h^{lf}(N.\mathcal{U}_X(\ell)[d], K(R))| \longrightarrow \underset{\overrightarrow{\ell,d}}{\operatorname{hocolim}}\, |K(N.\mathcal{U}_X(\ell)[d], R)|$$

is also an equivalence. It will suffice to prove that levelwise

$${}^C h^{lf}(N_k\mathcal{U}_X(\ell)[d], K(R)) \longrightarrow K(N_k\mathcal{U}_X(\ell)[d], R)$$

is an equivalence for all k. This can be shown for a cofinite family of pairs (ℓ, d). We take the family to be the pairs such that $N.\mathcal{U}(\ell)[d]$ is C-1.

Let $P = P(\ell, D)$ be the set of all pairs (n, v) with $v \in S(v_0, (2n-1)2^\ell D)$. In this case the metric space $N_k\mathcal{U}_X(\ell)[d]$ is a finite disjoint union of metric spaces which are either

$$\coprod_P X \cap (\pi^{-1}U_\ell(v)[d] \times Z), \tag{1}$$

where $v \in S(v_0, (2n-1)2^\ell D)$, or of the form

$$\coprod_P X \cap (\pi^{-1}U_\ell(v'', v)[d] \times Z), \tag{2}$$

where $v'' \in [v_0, v]$ is uniquely determined by v. Here each space of type (2) corresponds to a choice of an integer $1 \le m \le 2^{k+1}$. For a given m, written in base 2, the subsets $U_\ell(v'', v)$ involved in the expression (2) should be viewed as the $k+1$-fold intersections of the kind $U_\ell(v'') \cap \ldots \cap U_\ell(v)$ where the occurrences of $U_\ell(v'')$ correspond to zeros in m and the occurrences of $U_\ell(v)$ correspond to ones in m.

Let f be a function from P to vertices in T_1 such that $f(n, v) \in U_\ell(v'', v)$. The image of f, which we denote by F, is a countable discrete metric space. There are inclusions of metric spaces

$$\pi^{-1}F \times Z \subset \coprod_P \pi^{-1}U_\ell(v)[d] \times Z,$$

and

$$\pi^{-1}F \times Z \subset \coprod_P \pi^{-1}U_\ell(v'', v)[d] \times Z.$$

Let F'_X be either of the metric disjoint unions (1) or (2), then the orthogonal projection

$$p\colon F'_X \longrightarrow \pi^{-1}F \times Z = \Pi_{i=2}^n T_i \times F \times Z$$

is a bounded map with image F_X. We have a commutative diagram

$$\begin{array}{ccc} {}^{C}h^{lf}(F'_X, K(R)) & \longrightarrow & K(F'_X, R) \\ \downarrow p_* & & \downarrow p_* \\ {}^{C}h^{lf}(F_X, K(R)) & \longrightarrow & K(F_X, R) \end{array}$$

It can be shown that the left hand vertical arrow is an equivalence, cf. V.7 in [3]. Since F_X is commensurable to F'_X, the map $p\colon F'_X \to F_X$ is a coarse equivalence, so the right hand vertical arrow is also an equivalence. Now to show that the upper horizontal arrow is an equivalence, it suffices to show that the lower one is an equivalence. But F_X is a subspace of $\Pi_{i=2}^n T_i \times F \times Z$, where $\Pi_{i=2}^n T_i$ is the product of $n-1$ locally finite trees, and $F \times Z$ is a countable discrete space.

We may conclude by induction on n that

$${}^{C}h^{lf}(F'_X, K(R)) \longrightarrow K(F'_X, R)$$

is an equivalence if this is true for $n = 0$. In this case X is a disjoint union of possibly infinitely many uniformly bounded components. Therefore, the category of geometric modules on X and bounded homomorphisms is equivalent to an infinite product, parametrized by the components, of copies of the category of finitely generated projective modules over the coefficient ring R. It now follows from [4] that the K-theory is equivalent to the infinite product of copies of the spectrum $K(R)$. The same is true for the homology theory ${}^{C}h^{lf}(X, K(R))$ by Proposition 1.2. This completes the proof of Theorem 3.9 and the Main Theorem. ▽

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