Structured Stable Homotopy Theory and the Descent Problem for the Algebraic K-theory of Fields

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1 Introduction

In his seminal 1972 paper [44], D. Quillen introduced his definition of higher algebraic K-theory. This work was the culmination of a long search for a suitable definition, and has been the basis for an enormous amount of deep mathematics which has been developed since that time. Quillen's definition produced a functor

$$K : \underline{Rings} \longrightarrow \underline{Spectra}$$

and the group $K_i(A)$ of a ring A is defined to be the homotopy group $\pi_i(KA)$. These groups were show to agree with algebraic definitions already made in dimensions 0,1, and 2. In particular, $\pi_o(KA)$ is isomorphic to the *projective class groups* of isomorphism classes of finitely generated projective modules over A. In addition to the definition, Quillen produced a number of important formal properties of his K-theory spaces and groups, which permit the simplification of calculations. In particular, in the case of many commutative rings, the *localization sequence* permits the reduction via long exact sequences of the computation of the algebraic K-theory to the case of fields. In this case, the fields involved are the quotient fields of residue class rings of the ring in question at prime ideals of the ring. However, the calculation of the K-theory of the fields is a fundamental problem which is much more difficult. An important step was taken by A.A. Suslin in [57], where he showed that a great deal of information is available in the case of algebraically closed fields.

Theorem 1.1 (Suslin; [57]) *Let $F_0 \subseteq F$ be an inclusion of algebraically closed fields. Suppose l is a prime, different from the characteristic of F_0. Then the induced map of spectra*

$$K(F_0)^{\wedge}_l \longrightarrow K(F)^{\wedge}_l$$

is a weak equivalence of spectra. Here, for a spectrum X, $X^{\wedge}_l$ denotes the l-adic completion of X, as defined in [6].

This is a relative result which asserts that the K-theory spectra of any two algebraically closed fields of a given characteristic are equivalent to each other after completion at a prime not equal to the characteristic. In order to give a description of the K-theory spectrum of any algebraically closed field, we need to describe KF for some algebraically closed field of each characteristic. We have two theorems which achieve exactly this result.

Theorem 1.2 (Quillen; [45]) *Let $p \neq l$ be a primes, and let $\underline{ku}$ denote the conncective complex K-theory spectrum. Then there is a weak equivalence of spectra*

$$K(\overline{\mathbb{F}}_p)^{\wedge}_l \longrightarrow \underline{ku}^{\wedge}_l$$

Theorem 1.3 (Suslin; [58]) *Let l be any prime. There is a weak equivalence of spectra*

$$K(\mathbb{C})^{\wedge}_l \longrightarrow \underline{ku}^{\wedge}_l$$

Thus, the K-theory spectrum of any algebraically closed field is equivalent to the connective complex K-theory spectrum away from the characteristic of the field. The key question now is how to obtain information about the K-theory of a field which is not algebraically closed. An attempt to do this was made in the form of a conjecture of Qullen and Lichtenbaum. Let F be any field, and let $\overline{F}$ denote the algebraic closure of F. Let G_F denote the absolute Galois group $Gal(\overline{F}/F)$; this is a profinite group. G_F acts on the the spectrum $K\overline{F}$ from functoriality, and it is easy to see from the construction of the K-theory spectrum that the fixed point spectrum is KF. In order to make use of this information, we recall that when we have a group G acting on a space (or spectrum) X, we define its *homotopy fixed point set* X^{hG} to be the space (or spectrum) of equivariant maps $F^G(EG, X)$, where EG is a contractible free G complex. (In the case of profinite groups, one makes a definition which takes into account the topology on the profinite group). There is an obvious map from the actual fixed point set X^G to X^{hG}. In the case of a based action, he homotopy groups $\pi_*(X^{hG})$ are computable via a spectral sequence whose E_2-term has the form

$$E_2^{p,q} \cong H^{-p}(G, \pi_q(X))$$

In other words, the homotopy groups of the homotopy fixed point set are computable from the homotopy groups of the total space of the action together with the group action on those groups. The study of homotopy fixed point sets has generated a number of intersting problems, some of which have been resolved in recent years. See [7] and [8] for information about homotopy fixed point sets. If it were the case that the composite

$$\theta \colon KF \cong K\overline{F}^{G_F} \longrightarrow K\overline{F}^{hG_F}$$

were a homotopy equivalence, we would be able to compute $\pi_*(K(F))$ from knowledge of $\pi_*(K(\overline{F}))$ as a G_F-module. From the work of Suslin, we understand $\pi_*(K(\overline{F}))$, at least after l-adic completion. Unfortunately, θ is easily seen not to be a weak equivalence, even after l-adic completion. However, computational evidence suggested the following.

Conjecture 1.4 (Quillen-Lichtenbaum conjecture; see [46], [32]) *The map*

$$\hat{\theta}_l : KF_l^\wedge \longrightarrow (K\overline{F}_l^\wedge)^{hG_F}$$

induces an isomorphism on π_i for i greater than the cohomological dimension d of G_F. In particular, there is a spectral sequence with

$$E_2^{p,q} \cong H^{-p}(G_F, \pi_q(K\overline{F}_l^\wedge))$$

converging to $\pi_{p+q}(K(F)_l^\wedge)$ in dimensions greater than d

This attractive conjecture has been the subject of a great deal of work over the last 30 years; see [62], [14] for examples of this work. The last five years have seen an especially large amount of progress due to the introduction of motivic methods [63], [65]. The case of p-adic fields, at $l = p$, has also been resolved in the last five years [24].

One defect of the Quillen-Lichtenbaum conjecture (QLC) is that it is only a conjecture about some of the homotopy groups. It does not provide a spectral sequence which is expected to converge to *all* the K-groups. Computational evidence also suggested the following refinement of part of QLC.

Conjecture 1.5 (Beilinson-Lichtenbaum conjecture) *There is a spectral sequence with*

$$E_2^{p,q} \cong H^{-p}(G_F, \pi_q(K(\overline{F})_l^\wedge))$$

for $p + 2q \geq 0$, and with $E_2^{p,q} \cong 0$ otherwise, converging to $\pi_{p+q}(K(F)_l^\wedge)$. The E_2-term of this spectral sequence is obtained by truncating the Quillen-Lichtenbaum spectral sequence below the line $p + 2q = 0$.

This conjecture, although it gives a spectral sequence which converges to the graded group $\pi_*(K(F)_p^\wedge)$, there is no known model for the K-theory spectrum together with filtration inducing the spectral sequence. This lack of control will make the evaluation of differentials, as well as functorial behavior of the spectral sequence, difficult. Although we have an interpretation of a part of the E_2-term of the spectral sequence in terms of group cohomology, it does not give us understanding of the equivariant behavior of the K-theory spectra. Further, there is no interpretation of the entire E_2-term as an appropriate Ext-group. It is not really a descent spectral sequence.

One could then ask the question, "What is meant by a solution to a K-theoretic descent problem?". I will propose two possible answers to this question.

- The creation of a spectral sequence whose E_2-term is given by derived Hom of some "family of operators" on an ambient space.
- The creation of a homotopy theoretic model of a space whose ingredients involve only the absolute Galois group and the spectrum $\underline{ku}_l^\wedge$

In this paper, I will propose conjectural solutions to both of these problems. With regard to the first kind of descent, we first reinterpret the homotopy fixed point set in the context of the theory of *structured homotopy theory*, by which we mean the theory of ring spectra (referred to as S-modules in [17]), module spectra, and the Hom and smash product constructions which are defined on these objects. We first observe that if X is any spectrum with action by a group G, the homotopy fixed point spectrum is defined as the spectrum of equivariant maps

$$EG_+ \longrightarrow X$$

where as before EG is a contractible and free G-complex. Suppose now that X is actually a module over an S-algebra R, that R is equipped with an action by the group G. It is possible to construct a spectrum level analogue of the twisted group ring $R\langle G\rangle$, which is itself an S-algebra. It is not difficult to see that $\pi_*(R\langle G\rangle) \cong \pi_*(R)\langle G\rangle$. We remark that in a standard algebraic situation, if we are given a ring R which is acted on by a group G, a G-set Z and an R-module M with G-action suitable compatible with the action of R (i.e. $g(rm) = r^g gm$, so that M becomes an $R\langle G\rangle$-module), there is a canonical identification

$$F^G(Z, M) \cong Hom_{R\langle G\rangle}(R(Z), M)$$

where $R(Z)$ denotes the free R-module on the set Z, regarded as a $R\langle G\rangle$-module. The analogue to this statement in the case of S-algebras is that there is a natural weak equivalence

$$F^G(EG_+, X) \cong Hom_{R\langle G\rangle}(R \wedge EG_+, X)$$

There are spectral sequences for both sides of this equivalence, and on the E_2-level, it corresponds to the natural change of rings isomorphism

$$Ext^*_{\mathbb{Z}[G]}(\mathbb{Z}, \pi_*X) \cong Ext^*_{(\pi_*R)\langle G\rangle}(\pi_*R, \pi_*X)$$

In the case of the K-theory spectrum of a field F, we may consider the spectrum $K\overline{F}$, which is a ring spectrum with action by the absolute Galois group G_F of F. The above analysis now shows that $K\overline{F}$ becomes a module over the twisted group ring $K\overline{F}\langle G\rangle$, and that we may interpret the homotopy fixed spectrum $K\overline{F}^{hG}$ as the Hom-spectrum

$$Hom_{K\overline{F}\langle G\rangle}(K\overline{F}, K\overline{F})$$

There is a spectral sequence converging to the homotopy groups of this spectrum with E_2-term

$$Ext_{K_*\overline{F}\langle G\rangle}(K_*\overline{F}, K_*\overline{F})$$

It is not hard to verify, using the above change of rings isomorphism, that this spectral sequence is just the Quillen-Lichtenbaum spectral sequence described above, converging to the homotopy groups of the homotopy fixed point set of the G_F-action on $K\overline{F}$.

Given any morphism of S-algebras $A \to B$, it is possible to construct another S-algebra $End_A(B) = Hom_A(B, B)$. B is a left module over this new S-algebra, and we may also construct the spectrum $Hom_{End_A(B)}(B, B)$. There is a natural inclusion $A \to Hom_{End_A(B)}(B, B)$, since multiplication by elements in A commutes with any A-linear morphism from B to itself. One can show that if one considers suitably l-completed version of this construction (denoted by $End^\wedge$), one finds that $Hom_{End^\wedge_A(B)}(B, B)$, one obtains the so-called *derived completion* of A along the homomorphism $A \to B$. This derived completion is a homotopy invariant version of the usual completion of rings. Its properties are outlined in Section 3 of the present paper. It is now easy obtain an inclusion

$$(K\overline{F})\langle G\rangle \hookrightarrow End^\wedge_{KF}(K\overline{F})$$

from the fact that elements of G act as KF-linear automorphisms of $K\overline{F}$, and that "multiplication by elements of $K\overline{F}$" are also KF-linear endomorphisms of $K\overline{F}$. This yields a map

$$Hom_{End^\wedge_{KF}(K\overline{F})}(K\overline{F}) \longrightarrow Hom_{(K\overline{F})\langle G\rangle}(K\overline{F})$$

which can be identified with the map $KF \to (K\overline{F})^{hG_F}$, or equivalently with the map from algebraic K-theory to etale K-theory. There is a spectral sequence converging to $\pi_* Hom_{End^\wedge_{KF}(K\overline{F})}(K\overline{F})$ whose E_2-term is of the form

$$Ext_{\pi_* End^\wedge_{KF}(K\overline{F})}(K_*\overline{F}, K_*\overline{F})$$

Moreover, there is a map of spectral sequences between the two situations, which is induced by the map of graded rings

$$\pi_*(K\overline{F})\langle G\rangle \longrightarrow \pi_* End^\wedge_{KF}(K\overline{F})$$

Thus, an understanding of $\pi_* End^{\wedge}_{KF}(K\overline{F})$ is what is required to obtain control over this "on the nose" descent spectral sequence. In this paper, we will propose a conjecture on the structure of $\pi_* End^{\wedge}_{KF}(K\overline{F})$, and we will prove that this conjecture holds if a very strong form of the Bloch-Kato conjecture proposed by Positselskii and Vishik in [43] holds. Our conjecture is as follows. Fix a prime l, and consider the homotopy groups l-completed K-theory of $\overline{F}$. By a theorem of Suslin, these homotopy groups are isomorphic as a graded ring to $A_* = \mathbb{Z}_l[x]$, where x is a generator in dimension 2 (the Bott element). We obtain an action of G_F on A_*, and therefore compatible actions of G_F on the quotient rings $A_*/l^k A_*$. The action of G_F on $A_*/l^k A_*$ factors through a finite quotient of G_F. We define $(A_*/l^k A^*)^{Iw}\langle G\rangle$ to be the inverse limit over all normal subgroups N of finite index which act trivially on $A_*/l^k A^*$ of the twisted group rings $(A_*/l^k A^*)\langle G/N\rangle$, and define $A_*^{Iw}\langle G\rangle$ to be the inverse limit $\underleftarrow{lim}$ $(A_*/l^k A^*)^{Iw}\langle G\rangle$. Letting B_* denote the graded ring $\mathbb{Z}_l[x, x^{-1}]$, we obtain a similar algebra $B_*^{Iw}\langle G\rangle$. We define the graded ring $R\langle G\rangle$ to be the $A_*^{Iw}\langle G\rangle$-subalgebra of $B_*^{Iw}\langle G\rangle$ generated by $\frac{1}{x}I$, where I denotes the augmentation ideal in the The conjecture on the structure of $\pi_* End^{\wedge}_{KF}(K\overline{F})$ is now as follows.

Conjecture 1.6 *There is an isomorphism of graded rings*

$$R_*\langle G\rangle \cong \pi_* End^{\wedge}_{KF}(K\overline{F})$$

The rationale for this conjecture is that one can make an argument that elements in I, regarded as self maps of $K\overline{F}$, are left divisible by the Bott element. This heuristic argument is given in Section 4.4. A main result of this paper is the following.

Theorem 1.7 *Conjecture 1.6 holds if the mod-l Bloch-Kato conjecture holds for all fields, and the mod-l Milnor K-theory is a Koszul duality algebra for all fields. The last condition is the conjecture of Positselskii and Vishik ([43]). Consequently, there will be a spectral sequence with E_2-term given by*

$$Ext^{**}_{R\langle G_F\rangle}(\mathbb{Z}_l[x], \mathbb{Z}_l[x])$$

converging to the l-completed K-theory of a field F.

One benefit of this kind of description is that one can now also obtain results about finite Galois descent in algebraic K-theory. It is known that if we have an inclusion of rings $A \hookrightarrow B$, and left B-modules M and N, then there is a spectral sequence whose E_2-term is the Hochschild cohomology of A with coefficients in the the graded B-bimodule $Ext^*_A(M, N)$ and converging to $Ext^*_B(M, N)$. The same result should be applicable to S-algebras, in particular to the inclusion

$$End^{\wedge}_{KE}(K\overline{F}) \to End^{\wedge}_{KF}(K\overline{F})$$

where $F \subset E$ is an inclusion of fields. Applying homotopy groups, one should obtain information about finite descent by studying Hochshild cohomology of $R\langle G_F\rangle$ with coefficients in the Ext groups of $R\langle G_E\rangle$ with coefficients in $\mathbb{Z}_l[x]$.

For the second type of descent, i.e. for constructing an explicit homotopy theoretic model of the K-theory spectrum KF, we construct a homomorphism of S-algebras from a "derived version" of completion of the S-algebra attached to the symmetric monoidal category of complex representations of the absolute Galois group G_F into the spectrum KF. The idea of this construction is as follows. We assume for simplicity that F contains an algebraically closed subfield. A more involved and more general version of the construction is discussed in the body of the paper. Let F be a field, with algebraic closure $\overline{F}$, and define the category of *descent data* for the extension $F \subset \overline{F}$ (denoted by $V^G(\overline{F})$) to have objects the finite dimensional $\overline{F}$-vector spaces V with G_F-action satisfying $\gamma(\overline{f}v) = \overline{f}^{\gamma}\gamma(v)$, with equivariant $\overline{F}$-linear isomorphisms. It is standard descent theory that this category is equivalent to the category of finite dimensional F-vector spaces. On the other hand, let $Rep_F[G_F]$ denote the category of finite dimensional continuous representations of G_F. There is a canonical homomorphism from $Rep_F[G_F]$ to $V^G(\overline{F})$ obtained by applying $\overline{F} \underset{F}{\otimes} -$ and extending the action via the Galois action of G_F on $\overline{F}$. We then have the composite

$$KRep_k[G_F] \longrightarrow KRep_F[G_F] \longrightarrow V^G(\overline{F})$$

and we are able to form derived completions for all these spectra attached to the homomorphism to the mod-l Eilenberg-MacLane spectrum which simply counts the dimension mod l of a vector space. We conjecture that a suitably equivariant version of this construction gives an equivalence of spectra, which provides the desired homotopy theoretic model for KF. We will refer to this map as the *representational assembly* for the field F. Note that the construction depends only on G_F and its complex representation theory.

The derived completion is obtained as the total spectrum of a cosimplicial spectrum, and there is therefore a spectral sequence defined by Bousfield and Kan [6] involving the homotopy groups of the spectrum $KRep_k[G_F]$. This spectral sequence appears closely related to the descent spectral sequence in 1.7, and to the Bloch-Lichtenbaum spectral sequence as described in [63]. This relationship suggests a relationship between the Milnor K-theory on the one hand and groups defined from the representation ring $R[G_F]$ as derived functors of completion. The suggestion is that the mod-l higher derived functors of completion should be identified with the mod-l Milnor K-groups. It seems that understanding this relationship better would be very interesting. In the case of dimension one, the homomorphism is obtained as follows. Recall that for a field F containing the p-th roots of unity, with absolute Galois group $G = G_F$, Kummer theory asserts that there are perfect pairings

$$G^{ab}/p^kG^{ab} \times F^{\cdot}/p^kF^{\cdot} \to \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \cong C_{p^\infty}$$

and consequently adjoint isomorphisms

$$Hom(G^{ab}/p^kG^{ab}, C_{p^\infty}) \longrightarrow F^{\cdot}/p^kF^{\cdot}$$

. These maps are related via commutative diagrams

$$\begin{array}{ccc} Hom(G^{ab}/p^{k+1}G^{ab}, C_{p^\infty}) & \longrightarrow & F^{\cdot}/p^{k+1}F^{\cdot} \\ \tau \downarrow & & \downarrow \\ Hom(G^{ab}/p^kG^{ab}, C_{p^\infty}) & \longrightarrow & F^{\cdot}/p^kF^{\cdot} \end{array}$$

where τ is the map given by multiplication by p. Consequently, we have an isomorphism

$$\underset{\leftarrow}{lim}\ Hom(G^{ab}/p^kG^{ab}, C_{p^\infty}) \to \underset{\leftarrow}{lim}\ F^{\cdot}/p^kF^{\cdot}$$

On the other hand, we may consider the representation ring of the group G^{ab}, obtained by taking the direct limit of its finite quotients. It is isomorphic to the group ring of the character group $Hom(G, C_{p^\infty}) \cong \underset{\rightarrow}{lim}\ Hom(G/p^kG, C_{p^\infty})$. A standard fact is therefore that if I is the augmentation ideal in this representation ring, that $I/I^2 \cong Hom(G, C_{p^\infty})$. The derived completion construction which we describe in this paper has the effect of taking the derived p-completion if I/I^2. From the work of Bousfield and Kan, the homotopy groups of the p-completion of a space are computable using an exact sequence involving the p-completion of the homotopy groups of the space itself, together with the derived functor lim^1 applied to the homotopy groups. For any abelian group A, lim^1 applied to the pro abelian group

$$\cdots \to A \stackrel{\times p}{\to} A \stackrel{\times p}{\to} \cdots$$

is defined to be the inverse limit of the system

$$\cdots \to A^{p^{i+1}} \stackrel{\times p}{\to} A^{p^i} \stackrel{\times p}{\to} \cdots$$

When this is applied to the abelian group I/I^2, we obtain exactly the left hand side of the Kummer theory isomorphism above. The case of higher dimensions can therefore be viewed as a higher dimensional or non-abelian generalization of Kummer theory.

In this paper, we are able to prove that the representational assembly at a prime l is an equivalence of spectra in the case of fields with abelian absolute Galois groups, and which contain the l-th power roots of unity. Further progress will require a better understanding of the derived representation theory of non-abelian groups, which does not seem to be out of range of current technique.

The ultimate hope is that the clarification of the relationship between arithmetically defined descriptions of algebraic K-theory, such as the motivic spectral sequence, with descriptions which involve the Galois group and its representation theory directly, will shed more light on arithmetic and algebraic geometric questions.

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2 Preliminaries

We assume the reader to be familiar with the category of spectra, as developed in [16] and [17] or [25]. In either of these references, the category of spectra is shown to possess a coherently commutative and associative monoidal structure, called smash product. Of course, on the level of homotopy categories, this monoidal structure is the usual smash product. The presence of such a monoidal structure makes it possible to define *ring spectra* as monoid objects in $\underline{Spectra}$. It is also possible to define the notion of a commutative ring spectrum, with appropriate higher homotopies encoding the commutativity as well as associativity and distributivity. In line with the terminology of [17], we will refer to these objects as S-algebras and commutative S-algebras, respectively. For the relationship with earlier notions of A_∞ and E_∞ ring spectra, see [17].

The monoidal structure on $\underline{Spectra}$ also makes it possible to define the notion of a left or right R-module, where R is either an A_∞ or E_∞ ring spectrum. By analogy with the commutative algebra setting, it is also possible to define spectra $Hom_R(M, N)$ where M and N are left or right R-modules, and $M \underset{R}{\wedge} N$ when M is a right R-module and N is a left R-module. Of course, $\underset{R}{\wedge}$ is analogous to algebraic $\underset{R}{\otimes}$. The usual adjoint relationships familiar from the algebraic setting hol, and composition of maps is associative.

For certain classes of modules, it is possible to construct spectral sequences for computing the homotopy groups of Hom and smash product spectra. To understand this, we first observe that since the category of modules over an S-algebra R admits mapping cylinder and mapping cones, it should be thought of as analogous to the *derived category* of chain complexes of modules over a ring R. In the derived category of modules, a special role is played by by projective complexes. The analogous condition condition for modules is the condition of *cofibrancy*. Just as it is always possible to replace a any chain complex over a ring with a weakly equivalent projective one (a weak equivalence here means a chain map inducing an isomorphism on homology), so it is always possible to replace any R-module with a weakly equivalent cofibrant one, where a weak equivalence is an R-module map inducing an isomorphism on homotopy groups.

Proposition 2.1 *Let R be an S-algebra, and suppose that M and N are both left or right R-modules. Suppose further that M is cofibrant. Then there is a spectral sequence with*

$$E_2^{p,q} \cong Ext_{\pi_* R}^{-p,q}(\pi_*(M), \pi_*(N))$$

converging to $\pi_{p+q}(Hom(M,N))$. This spectral sequence will be known as the **Universal coefficient spectral sequence** *for this situation. Similarly, if M is a right R-module and N is a left R-module, and one of M and N is cofibrant, there is a spectral sequence with*

$$E_2^{p,q} \cong Tor_{p,q}^{\pi_*(R)}(\pi_*(M), \pi_*(N))$$

converging to $\pi_{p+q}(M \underset{R}{\wedge} N)$. In both cases, the q-index refers to the internal grading coming from the graded structure on π_*, and the p-index refers tothe homological degree. This spectral sequence is referred to as the **Künneth** *spectral sequence for this situation.*

Remark: Throughout this paper, all Hom and smash product spectra will be computed using only cofibrant modules. If a module is not cofibrant by construction, we will always replace it with a weakly equivalent cofibrant model. We will sometime do this without comment.

An important method for constructing spectra from combinatorial data is via *infinite loop space machines* (see [35] or [49]), which are functors from the category of symmetric monoidal categories to spectra. The algebraic K-theory functor is a prime example of this construction, since it can obtained by applying an infinite loop space machine to the symmetric monoidal category of finitely generated projective modules over a ring. When the symmetric monoidal category has a coherently associative and distributive second monoidal structure, such as tensor product of modules, the spectrum constructed by an infinite loop space machine will have the structure of an S-algebra. If in addition the second monoidal structure is coherently commutative, the spectrum will be a commutative S-algebra. See [17] or [25] for these results. Since the tensor product of finitely generated projective modules over a commutative ring R is coherently commutative, we have

Proposition 2.2 *For any commutative ring A, the spectrum KA is equipped with a commutative S-algebra structure in a canonical way.*

We also want to remind the reader of the construction by Bousfield-Kan of the l-completion of a space (simplicial set) at a prime l, $X_l^\wedge$. Bousfield and Kan construct a functorial cosimplicial space $T_l^\cdot X$, and define the l-completion of X to be $Tot(T_l^\cdot)$. This construction gives rise to a functorial notion of completion. It has long been understood in topology, though, that it is not always advantageous to pass to the total space, but to view the completion as a functor to the category of *pro-spaces*, via sending X to the pro-space $\{Tot^k(T_l^\cdot X)\}_{k\geq 0}$. The reasons for this are many, but an important one is that the homotopy groups of a completion depend not only on the l-completion of the homotopy groups of the space, but of a derived functor of completion known as lim^1. This functor depends only on the structure of the pro-groups obtained by applying π_j to the pro-space $\{Tot^k(T_l^\cdot X)\}_{k\geq 0}$. For this reason, we will adopt the following notational conventions concerning completions. For us, the prime at which the completion is taking place will always be denoted l, and for that reason we can remove it from the notation. For a space X, we will write $X^\wedge$ for the pro-space $\{Tot^k(T_l^\cdot X)\}_{k\geq 0}$, and no confusion should arise. When we write $X_l^\wedge$, we will mean the total space of the pro-space $X^\wedge$. There is a standard result concering this construction, which is the following. See Bousfield-Kan [6].

Proposition 2.3 *For any space, the pro-group $\pi_i X^\wedge$ is isomorphic to the pro-group $\{\pi_i X/l^s\pi_i X\}_{s\geq 0}$.*

This theory extends in a simple way to the category of spectra, and we will use this extension without comment. Further, when applied to an S-algebra or a commutative S-algebra, the $^\wedge$ and $^\wedge_l$ constructions produce new S-algebras and commutative S-algebras, and given an S-algebra R and left modules M and N, $M^\wedge$ and $N^\wedge$ become pro-modules over $R^\wedge$ in an obvious way.

As mentioned, throughout the paper, l will be used for a prime at which spaces and spectra are being completed. Given an S-algebra R and left R-modules M and N, we define a cosimplicial spectrum $\mathcal{T}^\cdot$ by $\mathcal{T}^k = Hom_{T_l^k R}(T_l^k M, T_l^k N)$. There are evident notions of coface and degeneracy maps, and we define the pro-spectrum $Hom_R^\wedge(M,N)$ to be $\{Tot^k\mathcal{T}^\cdot\}_{k\geq 0}$. In particular, $Hom_R^\wedge(M,M)$ becomes a pro-S-algebra, and we write $End_R^\wedge(M)$ for this pro-S-algebra. The Universal coefficient spectral sequence extends to this situation as well. For any $\mathbb{Z}$-algebra R and left R-modules M and N, we define the *continuous Ext-groups* $Ext_R^{cont}(M,N)$ to be

$$\underset{\overleftarrow{s}}{lim}\,\underset{\overrightarrow{t}}{lim}\; Ext_{R/l^tR}(M/l^tM, N/l^sN)$$

This construction clearly makes sense in the graded setting as well.

Proposition 2.4 *Suppose that R is an S-algebra, and M and N are left R-modules, with M cofibrant. Then there is a spectral sequence with E_2-term $Ext^{cont}_{\pi_*R}(\pi_*M, \pi_*N)$ converging to $\pi_*Hom^{\wedge}_R(M, N)$.*

It follows from these considerations that for any ring A, we obtain a pro-S-algebra $KA^{\wedge}$. We refer the reader to [44] for results concerning K-theory spectra, notably the localization, devissage, and reduction by resolution theorems. These theorems apply equally well to the completed versions $KA^{\wedge}$. We will also recall Suslin's theorem.

Theorem 2.5 *Let $k \to F$ be an inclusion of algebraically closed fields of characteristic $p \neq l$ (p may be 0). The the natural map $Kk \to KF$ induces an equivalence $Kk^{\wedge}_l \to KF^{\wedge}_l$. In fact, the proof shows that the map of pro-spectra $Kk^{\wedge} \to KF^{\wedge}$ is a weak equivalence in the sense that it induces an isomorphism of homotopy pro-groups.*

We also will free to use the standard results concerning higher algebraic K-theory, such as localization sequences, devissage, reduction by resolution, etc., as presented in [44]. We also recall the theorem of Suslin concerning the algebraic K-theory of algebraically closed fields from the introduction.

We also recall the ideas of equivariant stable homotopy theory. See for example [22] or [10] for information about this theory. In summary, for a finite group G, there is a complete theory of G-equivariant spectra which includes suspension maps for one-point compactifications of all orthogonal representations of G. The proper analogue of homotopy groups takes its values in the abelian category of *Mackey functors*, which is a suitably defined category of diagrams over a category whose objects are finite G-sets, and whose morphisms include maps of G-sets, and also transfer maps attached to orbit projections. The category of Mackey functors admits a coherently commutative and associative tensor product, which is denoted by $\square$. Consequently, by analogy with the theory of rings, we define a *Green functor* to be a monoid object in the category of Mackey functors. The theory of ring functors has analogues for most of the standard theorems and constructions of ring theory. In particular, it is possible to define ideals, modules, completions, tensor products of modules over a Green functor, and modules $Hom_R(M, N)$ for any Green functor and modules M and N over R. The category of modules over a Green functor also has enough projectives, so homological algebra can be carried out in this category. It is also easy to verify that by passing to direct limits, it is possible to directly extend the ideas about Mackey and Green functors to profinite groups. See [5] for background material about Mackey and Green functors.

As mentioned above, the Mackey functor valued analogue of homotopy groups plays the same role for equivariant spectra that ordinary homotopy groups play for spectra. For example, it is shown in [31] that there exist Eilenberg-MacLane spectra attached to every Mackey functor. Moreover, a theory of ring and module spectra in the equivariant category has been developed by May and Mandell [33] and [37] in such a way that the Mackey-functor valued homotopy group becomes a Green functor in an evident way, and that the usual spectral sequences (Künneth and Universal coefficient) hold for modules in this category.

We also will assume the reader familiar with standard objects in stable homotopy theory, such as the Steenrod algebra and its dual at a prime l (see [42] and [41]) and the Eilenberg-Moore spectral sequence for a loop space (see [55]). For an abelian group A, we will use the notation $H(A)$ for the Eilenberg-MacLane spectrum attached to the group A. We will also use the notation $\mathbb{H}$ and $\mathbb{L}$ for $H(\mathbb{Z}/l\mathbb{Z})$ and $H(\mathbb{Z}_l)$ respectively, throughout the paper.

3 Completions

In this section we will discuss the properties of a completion construction in the category of ring spectra and modules, which is analogous to the completion construction in the category of rings. We will first motivate the construction.

In [6], Bousfield and Kan defined the completion of a space (actually simplicial set) at a prime l. The construction went as follows. Recall first that a *monad* in a category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$, equipped with natural transformations $\epsilon : Id \to T$ and $\mu : T^2 \to T$, satisfying various compatibility conditions. See [35] for details on these conditions.

Example 1: $\mathcal{C}$ is the category of based sets, and $T = G$, the *free group functor*, which assigns to any set the free group on that set, with the base point set equal to the identity.

Example 2: $\mathcal{C}$ is the category of based sets, and $T = F^{ab}$, the *free abelian group functor* which assigns to any set the free abelian group on that set, with the base point set equal to zero.

Example 3: $\mathcal{C}$ is the category of based sets, and $T = F_{\mathbb{F}_l}$, the *free* $\mathbb{F}_l$*-vector space functor*.

Example 4: Let $\mathcal{C}$ be the category of left R-modules, for a ring with unit R. Let A be an R-algebra. Let T be the functor $M \to A \underset{R}{\otimes} M$.

Example 5: Let R be an S-algebra, and let $\mathcal{C}$ be the category of left R-modules. Let A be an R-algebra. Then the functor $M \to A \underset{R}{\wedge} M$ is a monad on the category of left R-module .

In all cases, it is easy to see what ϵ and μ should be. Note that all functors can be extended to monads on the associated category of simplicial objects in the relevant category. Given any monad on a category $\mathcal{C}$ and object $c \in \mathcal{C}$, we can associate to it a cosimplicial object in $\mathcal{C}$, $T^{\cdot}(c)$. The object in codimension k is the $(k+1)$-fold iterate of the functor T, the coface maps are given by applying natural transformations of the form $T^l(\epsilon)$ to the object $T^{k-l+1}(c)$, and the codegeneracies are given by applying natural transformations of the form $T^l(\mu)$ to objects of the form $T^{k-l-1}(c)$ There is an evident natural transformation from the constant cosimplicial object with value c to $T^{\cdot}(c)$, induced by ϵ. See [8] for details of this construction.

Definition 3.1 Bousfield-Kan; [6]) *Let $X_.$ be a simplicial set, and let l be a prime. Then by the l-completion of $X_.$, $X_l^{\wedge}$, we will mean the total space of the cosimplicial simplicial set attached to the monad $F_{\mathbb{F}_l}$. l-competion becomes a functor from the category of simplicial sets to itself, and it is equpped with a natural transformation $X_. \to X_l^{\wedge}$. We will also write $X_.^{\wedge}$ for the pro-simplicial set obtained by applying Tot^k to this cosimplicial simplicial set for each k.*

We can extend this completion construction to spectra by first applying the construction to each space in the spectrum to obtain a prespectrum, and then applying the "spectrification" functor to obtain an Ω-spectrum. This creates a completion functor on the category of spectra. Let $\mathbb{H}$ denote the mod-l Eilenberg-MacLane spectrum. $\mathbb{H}$ is an algebra over the sphere spectrum S^0, and consequently we may consider the monad $X \to \mathbb{H} \underset{S^0}{\wedge} X$, as in example 5 above. There is a canonical equivalence between these two monads, since mod l spectrum homology of a spectrum X can be computed either by computing the homotopy groups of the functor $F_{\mathbb{F}_l}(X)$ or by computing the homotopy groups of the spectrum $\mathbb{H} \underset{S^0}{\wedge} X$. We may now define a second completion functor, which assigns to a spectrum X the total space of the cosimplicial object $S^{\cdot}(X)$, where S is the monad $X \to \mathbb{H} \underset{S^0}{\wedge} X$. These two completion constructions are equivalent, and suggests to us the following definition.

Definition 3.2 *Let R be an S-algebra, and let A be an R-algebra, with defining ring spectrum homomorphism $f\colon R \to A$. Let T_A denote the monad on the category of left R-module spectra defined by*

$$M \longrightarrow A \underset{R}{\wedge} M$$

We now define the completion of a left R-module M along the map f, $M_f^{\wedge}$, to be the total space of the cosimplicial spectrum $T_A^{\cdot}(M)$. This construction is clearly functorial in M, and is functorial for maps of S-algebra homomorphisms in the following sense. Suppose that we have a commutative square of S-algebra homomorphisms

$$\begin{array}{ccc} R & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ R^{'} & \xrightarrow{f^{'}} & A^{'} \end{array}$$

Then for any R-module M, we have a canonical homomorphism $M_f^{\wedge} \longrightarrow (R^{'} \underset{R}{\wedge} M)^{\wedge}_{f^{'}}$. In particular, there is a canonical map $\theta\colon R \to R_f^{\wedge}$. When both R and A are commutative S-algebras, and the homomorphism f is a homomorphism of commutative S-algebras, then the completion $R_f^{\wedge}$ is itself a commutative S-algebra, and the map θ is a homomorphism of commutative S-algebras.

The completion construction has a number of important properties.

Proposition 3.3 *Let $f : R \to A$ be a homomorphism of S-algebras. Suppose that both R and A are (-1)-connected, that $\pi_0(f)$ is an isomorphism, and that $\pi_1(f)$ is surjective. Then the canonical homomorphism $R \to R_f^{\wedge}$ is a homotopy equivalence.*

Next, we note that the completion construction may be applied to rings, rather than S-algebra, by viewing a ring as an S-algebra via the Eilenberg-MacLane spectrum construction. For any map $f : R \to A$, we have the induced map of Eilenberg-MacLane spectra $H(f) : H(R) \to H(A)$, and we refer to $H(R)^{\wedge}_{H(f)}$ as the *derived completion* of the homomorphism f. Its homotopy groups are interesting invariants of the ring homomorphism f. It is not hard to see that one can define this derived completion in the graded setting as well. The following result concerns the **algebraic to geometric** spectral sequence for computing the homotopy groups of derived completions. It generalizes the lim^1 exact sequence for computing the homotopy groups of the l-completion of a space X.

Proposition 3.4 (Algebraic to geometric spectral sequence) *Let $f : R \to A$ be a homomorphism of S-algebras. Then we may consider the graded rings π_*R and π_*A as well as the ring homomorphism π_*f. There is a spectral sequence with $E_2^{p,q}$-term*

$$\pi_p(\pi_* R^{\wedge}_{\pi_* f})_q$$

(Note that $\pi_p(\pi_* R^{\wedge}_{\pi_* f})$ is a graded group, using the internal grading of the rings π_*R and π_*A).

Given any cosimplicial space (or spectrum), Bousfield-Kan [6] produce a spectral sequence converging to the homotopy groups of the total space (spectrum) of the cosimplicial object. This spectral sequence can be applied in this situation, and the E_2-term can in some cases be identified in terms of Ext-groups over the graded homotopy ring of a ring spectrum of operators.

Definition 3.5 *For any homomorphism $R \to A$, of S-algebras, we define $End_R(A)$ to be the spectrum $Hom_R(A, A)$, where A is viewed as a left R-module. It is shown in [17] that $End_R(A)$ becomes an S-algebra in a natural way under the composition product, and that A becomes a left $End_R(A)$-module. Moreover, there is a canonical homomorphism*

$$\eta : R \longrightarrow Hom_{End_R(A)}(A, A)$$

Remark: The map η is formally similar to the isomorphism

$$R \longrightarrow Hom_{End_R(P)}(P, P)$$

of Morita theory (see [29]), which applies when P is a finitely generated projective module left R-module.

The map η is certainly not in general an equivalence. It is possible to formulate useful criteria for when it is, but for our purposes it is more useful to study an l-completed version where a simple general condition suffices. We first observe that $A^\wedge$ is in a natural way a module over the pro-S-algebra $End_R^\wedge(A)$. We also consider the composite $R \to A \to \mathbb{H} \wedge A$, and denote it by $\hat{f}$. We define a new pro-S-algebra $R^{\wedge f}$ to be $\{Tot^k T^{\cdot}_{\mathbb{H}\wedge A}(R)\}_{k \geq 0}$. We next observe that we have an analogue $\eta^{\wedge f}$ of η, which gives a map of pro-spectra

$$\eta^{\wedge f} : R^{\wedge f} \to Hom_{End_R^\wedge(A)}(A^\wedge, A^\wedge)$$

The following will be proved in [11].

Proposition 3.6 *In the above situation, suppose that R and A are -1-connected S-algebras, that the homotopy groups of A are finitely generated, and that the map $\pi_0 f : \pi_0 R \to \pi_0 A$ is surjective. Then $\eta^{\wedge f}$ is a weak equivalence in the sense that it induces an isomorphism of homotopy groups, viewed as pro-groups.*

In order to obtain spectral sequences in from this result, we must make some observations about pro-rings. The category of pro-abelian groups is equipped with a coherently associative and commutative tensor product. For any pro-ring R, we define the category of left (right) modules over R as the category of pro-abelian groups M equipped with a homomorphism of pro-abelian groups $R \otimes M$ or $M \otimes R \to M$ for which the usual diagrams commute. There is also a functor $\underleftarrow{lim}$ from the category of pro-rings to the category of rings, and a corresponding inverse limit functor from the category of R-modules to the category of modules over the ring $\underleftarrow{lim}\ R$. The category also admits the construction of modules of the form $Hom_R(M, N)$, and has enough projectives. One can therefore define Ext functors in the category. A useful special case of pro-rings is the case where $R = \{A/I_n\}_{n \geq 0}$, where A is a ring and $\{I_n\}$ is a decreasing chain of two-sided ideals in A. For any two left A-modules M and N, we have the corresponding modules $\hat{M} = \{M/I_nM\}_{n \geq 0}$ and $\hat{N} = \{N/I_nN\}_{n \geq 0}$. We define the *continuous Ext-modules* of M and N to be

$$Ext_A^{cont}(M, N) = \underleftarrow{lim}_n \underrightarrow{lim}_{k \geq l}\ Ext_{R/I_k}(M/I_lM, N/I_nN)$$

We recall that the pro-S-algebra $End_R^\wedge(A)$ is obtained by applying the functors Tot^k to a cosimplicial S-algebra. We write $End_R^\wedge(A)[k]$ for the result of applying Tot^k. Then we obtain a decreasing sequence of ideals in $\mathcal{J}_k \subseteq \underleftarrow{lim}\ \pi_* End_R^\wedge(A)$

Proposition 3.7 *Let R be the pro-ring $\{A/I_n\}_{n \geq 0}$, for a decreasing chain of two-sided ideals $I_n \subseteq A$, and let M and N be A-modules. Define $\hat{M}$ and $\hat{N}$ as above. Then we have an isomorphism*

$$\underleftarrow{lim}\ Ext_R(\hat{M}, \hat{N}) \cong Ext_A^{cont}(M, N)$$

Corollary 3.8 (Descent spectral sequence) *Let R and A be as in the statement of Proposition 3.6. Then there is a spectral sequence with E_2-term*

$$Ext_{\pi_* End_R^\wedge(A)}(\pi_* A^\wedge, \pi_* A^\wedge)$$

converging to $\pi_*(R^{\wedge f})$. We also have

$$\underleftarrow{lim}\ Ext_{\pi_* End_R^\wedge(A)}(\pi_* A^\wedge, \pi_* A^\wedge) \cong Ext^{cont}_{\underleftarrow{lim}(\pi_* End_R^\wedge(A))}(\underleftarrow{lim}\ \pi_* A^\wedge, \underleftarrow{lim}\ \pi_* A^\wedge)$$

where the continuous Ext-modules are with respect to the decreasing sequence of $\{\mathcal{J}_k\}_{k \geq 0}$

This spectral sequence will be referred to as the *descent spectral sequence* for the map $R \to A$. We will write $\mathcal{E}_*$ for the pro-graded ring $\pi_* End_R^\wedge(A)$.

We now give some examples of this construction. Fix a prime l.

Example: $A = S^0$, and $B = H(\mathbb{F}_l)$, the mod-l Eilenberg-MacLane spectrum. Then $\mathcal{E}_* = \mathcal{A}(l)_{-*}$, where $\mathcal{A}(l)_*$ denotes the mod-l Steenrod algebra. It is a the constant pro-graded ring. The associated spectral sequence is the Adams spectral sequence. $R_f^\wedge$ is the l-adically completed sphere.

Example: $A = H(\mathbb{Z})$, and $B = H(\mathbb{F}_l)$, as above. In this case, $\mathcal{E}_* \cong \Lambda_{\mathbb{F}_l}(\beta)_*$, the exterior algebra over $\mathbb{F}_l$ on a single generator β of dimension -1. This generator is the first Bockstein operator in mod-l homology, and the associated spectral sequence is the Bockstein spectral sequence converging to $\pi_* H(\mathbb{Z}_l)$. In this case too, the algebra is pro-constant.

Example: $A = ku$, the connective complex K-theory spectrum, $B = H(\mathbb{Z})$, and the map $f : A \to B$ is induced by the evident functor which sends a vector space to its dimension. In this case, $A_f^\wedge \cong ku$, and $\mathcal{E}_* \cong \Lambda_{\mathbb{Z}_l}[[\theta]]_*$, where the generator θ has dimension -3. The spectral sequence collapses in this case, and is the homotopy spectral sequence attached to the Postnikov tower for $ku_l^\wedge$. In this case, the pro-ring is the one attached to the l-adic topology on $\mathcal{E}_*$.

Example: Let Γ denote a discrete group, and let $A = H(\mathbb{Z}[\Gamma])$ and $B = H(\mathbb{Z})$. Let the homomorphism f of ring spectra be induced by the augmentation $\mathbb{Z}[\Gamma] \to \mathbb{Z}$, given by $\gamma \to 1$. In this case, $\mathcal{E}_* \cong H^{-*}(B\Gamma, \mathbb{Z}_l)$, and the spectral sequence is the homology Eilenberg-Moore spectral sequence associated to the fiber square

$$\begin{array}{ccc} \Gamma & \longrightarrow & E\Gamma \\ \downarrow & & \downarrow \\ * & \longrightarrow & B\Gamma \end{array}$$

So, the Eilenberg-Moore spectral sequence converges the derived completion of the l-adic group ring of Γ at its augmentation ideal. Again, the topology on the $\mathcal{E}_*$ is the p-adic topology.

Example: Let F be a field, and $\overline{F}$ its algebaic closure. Let $A = K(F)_l^\wedge$ and $B = K(\overline{F})_l^\wedge$, with f induced by the inclusion $F \hookrightarrow \overline{F}$. Since $\pi_0(f)$ is an isomorphism and $\pi_1(f)$ is surjective ($\pi_1(K(\overline{F})_l^\wedge = 0$ by the earlier mentioned theorem of Suslin), $R_f^\wedge \cong K(F)_l^\wedge$. One goal of this paper is to make a conjecture about the structure of $\pi_* End_{KF}^\wedge(K\overline{F})$.

It will also be important for us to have tools for computing $\pi_* End_R^\wedge(A)$ based on homology. Let R be an S-algebra, and let M and N denote left S-modules. We can complete the target module N to obtain the pro-spectrum $Hom_R(M, N^\wedge)$. We may also consider the spectrum $Hom_R(M, \mathbb{H} \wedge N)$, where $\mathbb{H}$ denotes the mod-l Eilenberg-MacLane spectrum. $\pi_* Hom_R(M, \mathbb{H} \wedge N)$ becomes a module over the mod-l Steenrod algebra $\mathcal{A}_*$. $\mathcal{A}_*$ is regarded as negatively graded. We now have the standard adjunction

$$Hom_R(M, \mathbb{H} \wedge N) \cong Hom_{\mathbb{H} \wedge R}(\mathbb{H} \wedge M, \mathbb{H} \wedge N)$$

Consequently, the graded group $\pi_* Hom_{\mathbb{H}\wedge R}(\mathbb{H}\wedge M, \mathbb{H}\wedge N)$ carries a graded $\mathcal{A}_*$-module structure. The graded group $\pi_* Hom_{\mathbb{H}\wedge R}(\mathbb{H}\wedge M, \mathbb{H}\wedge N)$ is computable via a universal coefficient spectral sequence.

Proposition 3.9 *There is a spectral sequence with E_2-term*

$$Ext_{H_*(R,\mathbb{F}_l)}(H_*(M,\mathbb{F}_l), H_*(N,\mathbb{F}_l))$$

converging to $\pi_* Hom_{\mathbb{H}\wedge R}(\mathbb{H}\wedge M, \mathbb{H}\wedge N)$

Proof This is a straightforward application of the universal coefficient spectral sequence, since $\pi_*({}_\wedge\mathbb{H}) \cong H_*(.\mathbb{F}_l)$. □

We now have the following result, which will be proved in [11].

Proposition 3.10 *There is a spectral sequence with E_2-term given by*

$$Ext_{\mathcal{A}_*}(\mathbb{F}_l, \pi_* Hom_{\mathbb{H}\wedge R}(\mathbb{H}\wedge M, \mathbb{H}\wedge N))$$

converging to $\pi_* Hom_R^\wedge(M,N)$.

We will refer to this spectral sequence as the **Adams-Hom spectral sequence**.

4 Endomorphism algebras for K-theory spectra

Let l be a prime. In this section, we will consider the inclusion of S-algebras

$$A = KF^\wedge \to K\overline{F}^\wedge = B$$

where F is a field and l is a prime. As we saw in the previous section, if we can compute the homotopy groups of the endomorphism ring spectrum

$$End_A^\wedge(B)$$

we will be able to compute the homotopy groups of $KF^\wedge$ using an analogue of the Adams spectral sequence.

We will describe the ring in $\pi_* End_A^\wedge(B)$ for some fields. What we must first do, though, is to make some general observations about $\pi_* End_A^\wedge(B)$ when A and B are as above.

4.1 Some algebraic constructions

Let G be a group and a A a ring. We write $A[G]$ for the A-group ring of G. We may also consider the *twisted group ring* $A\langle G, \rho\rangle$ attached to a homomorphism $G \to Aut(A)$, where the elements are formal linear combinations $\sum_{g\in G} \alpha_g g$, with the relations given by the usual multiplicative relations among the elements of G, and $g\alpha = \rho(g)(\alpha)g$. When G is a profinite group, the natural construction analogous to the group ring is the so-called *Iwasawa algebra*, which is defined to be $\lim_{\overleftarrow{K}} A[G/K]$, where K ranges over all normal subgroups of G of finite index. When we have a homomorphism $\rho : G \to Aut(A)$ which factors through a finite quotient of G, we may make the analogous inverse limit construction. We will write $A^{Iw}[G]$ and $A^{Iw}\langle G, \rho\rangle$ for these two constructions. Suppose now that the ring of coefficients is actually an *Ind*-ring, i.e. a directed system of rings $\{A_\beta\}_{\beta\in B}$, where B is a right filtering partially ordered set. Suppose further that a profinite group acts on each ring A_β via a representation $\rho_\beta : G \to Aut(A_\beta)$, and that the maps in the directed system are equivariant relative to the two actions. Suppose further that for each β, ρ_β factors through a finite quotient of the group G. We can then form the direct limit

$$\lim_{\overrightarrow{\beta}} A_\beta^{Iw}\langle G, \rho_\beta\rangle$$

which we will denote by $\underset{\sim}{A}\langle G, \rho\rangle$.

This construction extends in an obvious way to the case of a graded ring of coefficients. Under certain circumstances, this direct limit can be interpreted as a twisted Iwasawa algebra with coefficients in the ring $A = \lim_{\overrightarrow{\beta}} A_\beta$

Proposition 4.1.1 *Let G, A_β, and ρ_β be as above, with $A = \lim_{\overrightarrow{\beta}} A_\beta$. Suppose further that the directed system $\{A_\beta\}_{\beta\in B}$ is* co-Mittag-Leffler, *i.e. that for every fixed β, the group $Ker(A_\beta \to A_{\beta'})$ stabilizes with β'. This happens for example if each A_β is Noetherian. Suppose also that there exists a β so that the homomorphism $A_\beta \to A$ is surjective. Then the action of G on A factors through a finite quotient of G, and the canonical homomorphism $\lim_{\overrightarrow{\beta}} A_\beta^{Iw}\langle G, \rho_\beta\rangle \to A^{Iw}\langle G, \rho\rangle$ is an isomorphism. In the case of a graded ring, we need only verify that the co-Mittag Leffler and surjectivity conditions hold in each grading, not necessarily that one β works for all gradings.*

Proof This is a straightforward verification. □

Next, suppose that we are given a profinite G as above, and a ring (possibly graded) A, on which G acts via $\rho : G \to Aut(A)$. Suppose further that each of the induced actions $\rho_i : G \to Aut(A/l^i A)$ factors through a finite quotient of G. Then we define

$$A^{Iw}\langle G, \rho, l\rangle = \varprojlim_i (A/l^i A)^{Iw}\langle G, \rho_i\rangle$$

This will be called the *p-complete twisted Iwasawa algebra* attached to the data (G, ρ).

The case of particular interest to us is the case where $A = K_*$ is the graded ring $\pi_* \underline{ku}_l^\wedge \cong \mathbb{Z}_l[x]$, where x has grading 2. In this case, any homomorphism $G \to \mathbb{Z}_l^*$ defines an action on K_*, and each of the associated actions ρ_i factor through a finite quotient of G, since $(\mathbb{Z}/l^i\mathbb{Z})^*$ is finite.

We also wish to perform a construction related to the *Rees ring* construction in algebraic geometry (see [15]). From this point on, K_* will denote the graded polynomial given above, and K_*^{per} will denote the graded ring obtained by inverting the element x. We also make a general definition.

Definition 4.1.2 *Let R be any ring, and let M denote any $R - R$ bimodule over R, equipped with a homomorphism of bimodules $R \to M$. Then by the* enveloping algebra *of M over R, written $E(R; M)$, is the quotient of the tensor algebra*

$$T_R(M) \cong R \oplus M \oplus M \underset{R}{\otimes} M \oplus M \underset{R}{\otimes} M \underset{R}{\otimes} M \oplus \cdots$$

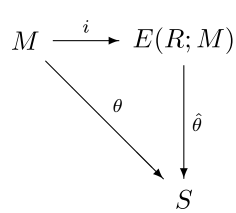
by the two-sided ideal generated by elements of the form

$$m_1 \otimes m_2 \otimes \cdots m_{i-1} \otimes r \otimes m_{i+1} \cdots \otimes m_s - m_1 \otimes m_2 \otimes \cdots m_{i-1} \otimes rm_{i+1} \cdots \otimes m_s$$

and

$$m_1 \otimes m_2 \otimes \cdots m_{i-1} \otimes r \otimes m_{i+1} \cdots \otimes m_s - m_1 \otimes m_2 \otimes \cdots m_{i-1}r \otimes m_{i+1} \cdots \otimes m_s$$

The enveloping algebra is the solution to a universal problem. In the diagram



we are given an R-algebra S and a homomorphism of R-bimodules $\theta : M \to S$, and i denotes the obvious inclusion $M \to E(R; M)$. Then there is a unique *homomorphism of R-algebras $\hat{\theta} : E(R; M) \to S$ making the diagram commute. This definition is easily extended to the graded case. Moreover, if the ring R is complete relative to a sequence of ideals, i.e. $R \cong \lim_{\leftarrow} R/I_n$, where I_n is a nested sequence of two sided ideals, we may make a completed version of the construction which relies on completed tensor products instead of ordinary tensor products. This construction has the universal property described above for maps into R-algebras which are also complete with respect to this sequence of ideals.*

We now specialize to the case where R is an algebra over the graded ring K_*, to make another construction.

Definition 4.1.3 *Let R be a graded K_*-algebra. We define $\overline{E}_*(R; M)$ to be the quotient of $E(R; M)$ by the two-sided ideal generated by elements which are annihilated by x^n for some n. It is the solution for the universal problem described above, where S is restricted to be an x-torsion free R-algebra.*

Now consider the case $R = K_*^{Iw}\langle G, \rho, l\rangle$. We set $S = (K^{per})_*^{Iw}\langle G, \rho, l\rangle$, and we have the evident inclusion $R \to S$ of graded R-algebras, which are complete under a sequence of ideals. We now define an R-subbimodule M of S by letting M be the R-bimodule generated by the element $\frac{1}{x}$ together with R in S. We describe this bimodule in terms of generators and relations. Let I denote the augmentation ideal in the Iwasawa algebra $\mathbb{Z}_l^{Iw}\langle G, \varepsilon\rangle \cong \mathbb{Z}_l^{Iw}[G]$, where ε indicates the trivial homomorphism from G to $Aut(\mathbb{Z}_l)$. We define a new bimodule I^{mod} whose underlying group is I, and whose right action of $\mathbb{Z}_l^{Iw}[G]$ coincides with the right action on I, but where the new left action $*$ is instead given on g by $g * i = \rho(g)^{-1}gi$. The action given on group elements clearly extends uniquely to an action of the full $\mathbb{Z}_l^{Iw}[G]$. We now form the completed tensor product

$$W = K_*^{Iw}\langle G, \rho, l\rangle \underset{\mathbb{Z}_l[G]}{\hat{\otimes}} I^{mod}[-2] \underset{\mathbb{Z}_l[G]}{\hat{\otimes}} K_*^{Iw}\langle G, \rho, l\rangle$$

The bimodule structure is now given by multiplication on either side. Note that as an abelian group, W is isomorphic to

$$K_* \underset{\mathbb{Z}_l}{\hat{\otimes}} I^{mod}[-2] \underset{\mathbb{Z}_l}{\hat{\otimes}} K_*$$

Since S is x-torsion free, we obtain a homomorphism of graded algebras $\overline{E}(R; M) \to S$, so that the composite $R \to \overline{E}(R; M) \to S$ is the inclusion $R \to S$. In order to describe M in terms of generators and relations, we will need to define an automorphism α of R by setting $\alpha(x) = x$ and $\alpha(g) = \rho(g)g$. It is now not difficult to verify the following.

Proposition 4.1.4 *The bimodule M is isomorphic to the quotient $R \oplus W/V$, where V is the sub-bimodule generated by the elements $(-i, x \otimes i \otimes 1)$ and $(-\alpha(i), 1 \otimes i \otimes x)$.*

We now have the following description of $\overline{E}(R;M)$.

Proposition 4.1.5 *The natural homomorphism $\overline{E}(R;M) \to S$ is injective, and its image is the R-subalgebra generated by the elements $\frac{1}{x}i$, where $i \in I$.*

Proof That the image is as stated is clear, since the elements $\frac{1}{x}i$ together with elements in R generate M. To prove injectivity, we first note that any element in $\overline{E}(R;M)$ is in the image of

$$T_R^n(M) = \underbrace{M \underset{R}{\otimes} M \underset{R}{\otimes} \cdots \underset{R}{\otimes} M \underset{R}{\otimes} M}_{n \text{ terms}}$$

for some n. Using the relations defining $\overline{E}(R;M)$, it is easy to check that $x^n \cdot T_R^n(M) \subseteq R$ in $\overline{E}(R;M)$. Suppose, therefore, that $f \in T_R^n(M)$ is in the kernel of the map $\overline{E}(R;M) \to S$. Then $x^n f$ is also in the kernel of this map, and it is contained in R. However, the map $R \to \overline{E}(R;M) \to S$ is injective, so $x^n f = 0$. Since all x-torsion elements in $\overline{E}(R;M)$ are $= 0$, $f = 0$, which gives the result. $\Box$

We have therefore identified the R-algebra $\overline{E}(R;M)$ with a familiar object. We will now analyze the quotient $\overline{E}(R;M)/p\overline{E}(R;M)$. Recall first that for any ring R and and a two-sided ideal $I \subseteq R$, we may construct the *graded Rees construction* $\mathcal{R}(R,I)$ on I as the subalgebra of $R[x,x^{-1}]$, where x has grading 2, given by

$$\mathcal{R}(R,I) = \sum_n x^{-n} I^n$$

This is clearly a graded ring, and if the original ring R was graded, we impose the usual "total degree" grading on $\mathcal{R}(R,I)$. Note that in the case of a complete algebra, the notation I^n will mean the closure of the n-th power of the ideal I in the algebra.

Proposition 4.1.6 *The graded Rees ring construction has the following propoerties.*

- *The construction commutes with inverse limits of rings, in the sense that if we are given an inverse system of rings*

$$\cdots \longrightarrow R_n \longrightarrow R_{n-1} \longrightarrow \cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0$$

  together with ideals I_n compatible in the obvious way under the homomorphisms in the inverse system of rings, then

$$\mathcal{R}(\varprojlim R_n, \varprojlim I_n) \cong \varprojlim \mathcal{R}(R_n, I_n)$$

- *If R is complete under a system of ideals, i.e. $R \cong \varprojlim R/I_n$, then $\mathcal{R}(R, I)$ is complete under the corresponding system of ideals in $\mathcal{R}(R, I)$.*

- $\mathcal{R}(R, I)/x\mathcal{R}(R, I) \cong \bigoplus_n I^n/I^{n+1}[-2n]$*, where for a graded module M, $M[k]$ denotes the graded module $M[k]_l = M[k-l]$.*

We now define the algebra $\hat{R}_*\langle G, \rho\rangle$ to be the completion of the algebra $\overline{E}(R; M)$ at the two-sided ideal $I = (l, x)$. This algebra will in many cases be isomorphic to the algebra $\pi_* End^{\wedge}_{KF}(KE)$. It enjoys the following universal property.

Proposition 4.1.7 *Suppose $\mathcal{A}_*$ is a graded $K^{Iw}_*\langle G, \rho\rangle$-algebra, which is a (l, x) complete module, and which is l and x-torsion free. Let $I \subseteq K^{Iw}_*\langle G, \rho\rangle$ denote the closure in $K^{Iw}_*\langle G, \rho\rangle$ of the span of the set of elements of the form $1 - g$, with $g \in G$. Suppose that I is left divisible by x. Then there is a unique algebra homomorphism $\hat{R}_*\langle G, \rho\rangle \to \mathcal{A}_*$ making the diagram*

$$\begin{array}{ccc} K^{Iw}_*\langle G, \rho\rangle & & \\ \downarrow & \searrow & \\ \hat{R}_*\langle G, \rho\rangle & \longrightarrow & \mathcal{A}_* \end{array}$$

commute.

Proof Recall the notation in the discussion preceding Proposition 4.1.4. We must first show that there is a bimodule homomorphism $M \to \mathcal{A}_*$. We first note that there is a homomorphism $\nu : I^{mod}[-2] \to \mathcal{A}_*$ given by $\nu(i) = \frac{i}{x}$, where $\frac{i}{x}$ denotes the unique element in $\mathcal{A}_*$ with the property that $x \cdot \frac{i}{x} = i$. We claim that this homomorphism is a map of right $\mathbb{Z}^{Iw}_l[G]$-modules. To see this, we note that $x \cdot \frac{i}{x}\lambda = i\lambda$, so by the x-torsion freeness of $\mathcal{A}_*$, we find that since $x\nu(i\lambda)$ is also equal to $i\lambda$, $\nu(i)\lambda = \nu(i\lambda)$. This shows that the ν is a homomorphism of right modules. On the other hand, we consider $g\nu(i)$. We have $x \cdot g\nu(i) = \rho(g)^{-1}gx\nu(i) = \rho(g)^{-1}gi$, which shows that we have a map of left modules as well, since the action readily extends to the full ring $\mathbb{Z}^{Iw}_l[G]$. It is now clear that we obtain a map of $K^{Iw}_*\langle G, \rho\rangle$-bimodules $W \to \mathcal{A}_*$. It remains to show that the relations defining M in Proposition 4.1.4 are satisfied. But this follows from the equation

$$x\nu(i)x = ix = x\alpha(i)$$

We have now constructed a bimodule homomorphism $M \to \mathcal{A}_*$, and therefore an algebra homomorphism $\overline{E}(R; M) \to \mathcal{A}_*$. Now extend to $\hat{R}_*\langle G, \rho\rangle$ by completeness of $\mathcal{A}_*$ in the (l, x)-adic topology. □

In order to establish isomorphisms to homotopy groups of endomorphism spectra, it will also be important to understand the associated graded ring to the J-adic filtration, where $J = (l, x^{l-1})$.

We now consider the ring $\overline{E}(R; M)/l\overline{E}(R; M)$. Note that since the homomorphism $\rho : G \to \mathbb{Z}_l^*$ reduces to the identity mod l, the ring $\overline{E}(R; M)/l\overline{E}(R; M)$ may be identified with the subring of the Iwasawa algebra $(K_*^{per}/lK_*^{per})^{Iw}[G]$ topologically generated by elements of the form $\frac{1}{x}(1-g)$, where $g \in G$. It is now easy to verify the following.

Proposition 4.1.8 *We have an isomorphism*

$$\overline{E}(R; M)/l\overline{E}(R; M) \cong \mathcal{R}(\mathbb{F}_l^{Iw}[G], I)$$

where I denotes the augmentation ideal in the Iwasawa algebra $\mathbb{F}_l^{Iw}[G]$.

Corollary 4.1.9 $\overline{E}(R, M)/(x, l) \cong \Gamma_*(G, \mathbb{F}_l)$*, where $\Gamma_*(G, \mathbb{F}_l)$ denotes the graded $\mathbb{F}_l$-algebra given by $\Gamma_{-2k}(G, \mathbb{F}_l) \cong I^k/I^{k+1}$ for non-negative k, and where $\Gamma_i(G, \mathbb{F}_l) = 0$ for positive and odd values of i. Similarly,*

$$\overline{E}(R; M)/(l, x^{l-1}) \cong \Gamma_*(G, \mathbb{F}_l) \underset{\mathbb{F}_l}{\otimes} \mathbb{F}_l[x]/(x^{l-1})$$

We have the following general fact about associated graded algebras. Suppose that we have a graded algebra M_* equipped with a two-sided ideal J_*. Then we may construct the associated bigraded algebra $gr_{*,*}(M)$ defined by

$$gr^J_{s,t}(M) = I^s_t/I^{s+1}_t$$

Proposition 4.1.10 *Suppose that we have a homomorphism of graded algebras $L_* \to M_*$, where L_* is a (-1)-connected graded algebra. Let m denote the augmentation ideal of L_*, and suppose that mM_* is a twosided ideal of M_*, which we denote by J_*. Then $gr^J_{*,*}(M)$ is generated by the subalgebra $gr_{0,*}(M)$ together with the graded vector space $gr^J_{1,*}(L) \subseteq gr^J_{1,*}(M)$. In particular, there is a surjective homomorphism of bigraded algebras*

$$M_*/JM_* \otimes gr^J_{*,*}(L) \longrightarrow gr^J_{*,*}(M)$$

where M_*/JM_* denotes the graded algebra viewed as a bigraded algebra concentrated in bidegrees $(0, *)$.

The following corollary now is an immediate consequence of Corollary 4.1.9.

Corollary 4.1.11 *There is a surjective homomorphism of bigraded algebras*

$$\alpha_G : \Gamma_*(G) \otimes \mathbb{F}_l[\beta, x] \longrightarrow gr^J_{*,*}(\hat{R}_*\langle G, \rho\rangle)$$

where β has bidegree $(1,0)$ and x has bidgree $(1,2)$. Under α_G, $1 \otimes \beta$ is carried to the element of $gr^J_{1,0}(\hat{R}_*\langle G,\rho\rangle)$ corresponding to the element p, and x is carried to the element in $gr^J_{1,2}(\hat{R}_*\langle G,\rho\rangle)$ corresponding to the element $x \in A_*$. Finally, on $\Gamma_*(G)$, α_G is the isomorphism $\Gamma_*(G) \to gr_{0,*}(\hat{R}_*\langle G,\rho\rangle)$.

4.2 Space level constructions

The first observation is that group rings and twisted group rings are constructions which may be extended to the case of S-algebras, not only rings. Let G be any discrete group, and A any commutative S-algebra. The multiplication map

$$G_+ \wedge A \wedge G_+ \wedge A \cong G_+ \wedge G_+ \wedge A \wedge A \xrightarrow{m \wedge \mu} G_+ \wedge A$$

makes $G_+ \wedge A$ into an S-algebra, where m denotes the multiplication map in G and μ denotes the multiplication map in A. We will write $A[G]$ for this ring spectrum. More generally, suppose that G acts on A via automorphisms of S-algebras, i.e. via a homomorphism $G \to Aut(A)$, where A is an S-algebra. We then have the twist map $\tau\colon G_+ \wedge A \to A \wedge G_+$ defined by the formula $g \wedge \alpha \to \alpha^g \wedge g$, and the resulting multiplication map on $G_+ \wedge A$ given by

$$G_+ \wedge (A \wedge G_+) \wedge A \xrightarrow{id \wedge \tau \wedge id} G_+ \wedge G_+ \wedge A \wedge A \xrightarrow{m \wedge \mu} G_+ \wedge A$$

which makes $G_+ \wedge A$ into an S-algebra. We will denote this *twisted group ring* by $A\langle G,\rho\rangle$, as is conventional in algebra. Note that the action of G on A is part of the data required in the construction. The verification of these facts follows from the results of [17], which show that a large variety of constructions from the category of modules over a ring can be carried out in the category of modules over S-algebras.

Proposition 4.2.1 *Let a group G act on a commutative ring spectrum A. Then we have*

$$\pi_*(A\langle G\rangle) \cong (\pi_* A)\langle G\rangle$$

Proof This is clear from the definitions. □

We will be dealing primarily with profinite groups, since absolute Galois groups are profinite groups. We make a construction of a version the Iwasawa algebra in the context of S-algebras. Let

$$\cdots \subset K_{n+1} \subset K_n \subset \cdots \subset K_1 \subset K_0 = G$$

be a decreasing sequence of normal subgroups of G of finite index. Suppose that the profinite group G acts on an S-algebra A via an action which factors through a finite quotient of G. Then we write $A^{Iw}\langle G, \rho\rangle$ for the S-algebra

$$\underset{\overleftarrow{K}}{holim}\ A\langle G/K, \rho\rangle$$

where K varies over the normal subgroups of G of finite index.

It is easy to verify that $\pi_* A^{Iw}\langle G, \rho\rangle \cong (\pi_* A)^{Iw}\langle G, \pi_*\rho\rangle$.

We will also need to make a similar definition in the case of an Ind-S-algebra as coefficients. Suppose that $\underset{\sim}{B} = \{B_\alpha\}_{\alpha\in A}$ is an inductive system of S-algebras, each equipped with an action ρ_α by the group G which factors through a finite quotient of G. We now write $\underset{\sim}{B}\ \langle G, \underset{\sim}{\rho}\rangle$ for the colimit

$$\underset{\overrightarrow{\alpha}}{lim}\ B_\alpha^{Iw}\langle G, \rho_\alpha\rangle$$

Proposition 4.2.2 *Suppose that for each i the Ind-group $\pi_i\ \underset{\sim}{B}$ satisfies the co Mittag-Leffler property, and that that there is an $\alpha \in A$ so that $\pi_i B_\alpha \to \mathcal{B}_i = \underset{\rightarrow}{lim}\ \pi_i(B_\alpha)$ is surjective. Then the group action on $\mathcal{B}_i$ factors through a finite quotient of G, and we have*

$$\pi_*\ \underset{\sim}{A}^{Iw}\ \langle G, \underset{\sim}{\rho}\rangle \cong \mathcal{B}_*^{Iw}\langle G, \pi_*\rho\rangle$$

Proof Direct verification using 4.1.1 □

There are a number of situations in which we can verify the co-Mittag Leffler condition as well as the surjectivity condition. For any spectrum B, we let $T^i B$ denote $Tot^i T^{\cdot}_{\mathbb{F}_l}(B)$

Proposition 4.2.3 *Suppose that $\{B_\alpha\}_{\alpha\in A}$ is an inductive system of spectra so that each group $\pi_i B_\alpha$ is finitely generated, and that $\underset{\overrightarrow{\alpha}}{lim}\ \pi_i T^j B_\alpha$ is finitely generated for each i and j. Then the inductive system $\pi_* T^j B_\alpha$ is co-Mittag Leffler and satisfies the surjectivity condition.*

Proof Clear since all groups $\pi_i T^j B_\alpha$ are finite. □

We now consider two special cases. The spectrum $K\overline{\mathbb{Q}}$ is the colimit of spectra $K\mathcal{O}_\alpha$, as α varies over a cofinal set of rings of S-integers $\mathcal{O}_\alpha$ in finite Galois extensions of $\mathbb{Q}$, for finite sets of primes S. For a fixed prime p, the spectrum $K\overline{\mathbb{F}}_p$ is the colimit of the spectra $K\mathbb{F}_{p^r}$, as r varies over the partially ordered set of positive integers, with the ordering defined by divisibility. We write $\mathcal{K}\overline{\mathbb{Q}}$ and $\mathcal{K}\overline{\mathbb{F}}_p$ respectively for these Ind-spectra.

Proposition 4.2.4 *Let l be a prime. Then for each i and j the Ind-group $\pi_j T_l^i \mathcal{K}\overline{\mathbb{Q}}$ satisfies the co-Mittag Leffler condition and the surjectivity condition. If $l \neq p$, then the same holds for $\pi_j T_l^i \mathcal{K}\overline{\mathbb{F}}_p$*

Proof We first note that by a theorem of Quillen [47], $\pi_i K\mathcal{O}_\alpha$ is a finitely generated group for each i. By another calculation of Quillen [45], $\pi_i K\mathbb{F}_p$ is also finitely generated for each i. The co-Mittag Leffler condition for $\pi_i T_l^k \mathcal{K}\overline{\mathbb{Q}}$ and $\pi_i T_l^k \mathcal{K}\overline{\mathbb{F}}_p$ now follows, since both groups are finitely generated and l-torsion, hence finite. The surjectivity follows since it is an easy consequence of Suslin's theorem on the mod-l homotopy type of the K-theory spectrum of an algebraically closed field (see [57]) that $\pi_j T_l^i \mathcal{K}\overline{\mathbb{Q}}$ and $\pi_j T_l^i \mathcal{K}\overline{\mathbb{F}}_p$ are finitely generated for each i and j. □

Let $\mathcal{K}$ denote either $\mathcal{K}\overline{\mathbb{Q}}$ or $\mathcal{K}\overline{\mathbb{F}}_p$. We note that in the cofinal family of subspectra $K\mathcal{O}_\alpha$ and $K\mathbb{F}_{p^r}$ are acted on by the absolute Galois group G, via an action which factors through a finite quotient of G. Therefore, we may construct the twisted Iwasawa algebra

$$(T_l^i \mathcal{K})^{Iw}\langle G, \rho\rangle$$

for each i, and we have evident homomorphisms of S-algebras

$$(T^{i+1}\mathcal{K})^{Iw}\langle G, \rho\rangle \longrightarrow (T^i\mathcal{K})^{Iw}\langle G, \rho\rangle$$

We define $\mathcal{K}^{Iw}\ \langle G, \rho, l\rangle$ to be the homotopy inverse limit of this system.

Proposition 4.2.5 $\pi_* \mathcal{K}^{Iw}\langle G, \rho, l\rangle \cong (\mathbb{Z}_l[x])_*^{Iw}\langle G, \rho, l\rangle$

Proof It is easy to see that the pro-groups $\{\pi_*(T_l^i\mathcal{K})^{Iw}\langle G, \rho\rangle\}_i$ are Mittag-Leffler, and therefore their lim^1 terms vanish. The comparison of inverse limits is straightforward, using the standard fact that the progroups $\{\pi_i T_l^k(A)\}_k$ and $\{\pi_i A/l^k\pi_i A\}_k$ are pro-isomorphic for any spectrum A (see [6]). □

4.3 Group rings and rings of endomorphisms

Let A be any S-algebra, and suppose that a group G acts on A via an action ρ. Suppose that $B \subseteq A$ is a sub S-algebra on which the group action is trivial.

Then we have a canonical homomorphism $G \to End_B(A)$, since each $g \in G$ constitutes B-linear automorphism of A. One the other hand, there is a canonical homomorphism $A \to End_B(A)$ given by right multiplication by elements of A. ($End_B(A)$ consists of the endomorphisms preserving the *left* B-module structure.) Using an obvious universality property of the twisted group ring construction, we obtain an S-algebra homomorphism $A\langle G, \rho\rangle \to End_B(A)$. More generally, let B be a S-algebra, and suppose that we are given a directed system of B-algebras $\underset{\sim}{A} = \{A_\alpha\}_{\alpha \in A}$ equipped with an action by a profinite group G as automorphisms of B-algebras, so that for each α, the G action factors through a finite quotient G/K of G. Let $A = \underset{\rightarrow}{lim}\ A_\alpha$. It is now clear that for each α, we obtain a map of S-algebras

$$\underset{\sim}{A}^{Iw}\langle G, \underset{\sim}{\rho}\rangle \longrightarrow End_B(A)$$

We now consider the case of $End^{\wedge}_{KF}(KE)$, where $G = G_{\overline{F}}$ is the absolute Galois group of F. Recall that in the $(-)^{\wedge}$-notation, the choice of the prime l is implicit. We first note that

$$End^{\wedge}_{KF}(KE) \cong \underset{\longleftarrow}{holim}\ Tot_k Hom_{KF}(KF_\alpha, T_l^{\cdot} KE)$$

where KF_α varies over a cofinal subset of Galois extensions of F, and where k varies over the positive integers. Let N_α denote the absolute Galois group of K_α. Then there is an evident map

$$G/N_\alpha \longrightarrow Hom_{KF}(KF_\alpha, KE) \longrightarrow Tot_k Hom_{KF}(KF_\alpha, T_l^{\cdot} KE)$$

induced by the inclusion $F_\alpha \hookrightarrow E$ followed by the action of an element of G. Let $\mathcal{O}_\alpha$ denote any ring of S-integers in K_α, for a finite set of primes S. We now have evident composite maps

$$T_l^k K\mathcal{O}_\alpha \to Hom_{T_l^k KF}(T_l^k KE, T_l^k KE) \to Hom_{KF}(KE, T_l^k KE) \to$$

$$\to Hom_{KF}(KF_\alpha, T_l^k KE)$$

Passing to homotopy inverse limits, we obtain maps

$$(T_l^k K\mathcal{O}_\alpha)^{Iw}\langle G, \rho\rangle \longrightarrow Hom_{KF}(KE, T_l^k KE)$$

We now pass to a colimit over the rings of S-integers $\mathcal{O}_\alpha$, to obtain a map

$$(T_l^k \mathcal{K})^{Iw}\langle G, \rho\rangle \to Hom_{KF}(KE, T_l^k KE)$$

Here, $\mathcal{K}$ denotes the Ind-ring spectra $\mathcal{K}\overline{\mathbb{Q}}$ and $\mathcal{K}\overline{\mathbb{F}}_p$ discussed in the previous section. Finally, we take total spectra to obtain a map

$$\mathcal{K}^{Iw}\langle G, \rho, l\rangle \to End^{\wedge}_{KF}(KE)$$

The conlusions are as follows.

Proposition 4.3.1 *There is a map of spectra* $\mathcal{K}^{Iw}\langle G,\rho,l\rangle \to End^{\wedge}_{KF}(KE)$ *making the diagram*

$$\begin{array}{ccc} K\overline{\mathbb{Q}}\langle G,\rho\rangle \text{ or } K\overline{\mathbb{F}}_p\langle G,\rho\rangle & \longrightarrow & KE\langle G,\rho\rangle \\ \downarrow & & \downarrow \\ \mathcal{K}^{Iw}\langle G,\rho,l\rangle & & End_{KF}KE \\ & \searrow & \downarrow \\ & & End^{\wedge}_{KF}(KE) \end{array}$$

commute.

Corollary 4.3.2 *There is a homomorphism of graded rings* $\mathbb{Z}_l[x]^{Iw}\langle G,\rho,l\rangle \to \pi_* End^{\wedge}_{KF}(KE)$ *making the diagram*

$$\begin{array}{ccc} \mathbb{Z}_l[x]\langle G,\rho\rangle & \longrightarrow & \pi_* End_{KF}(KE) \\ \downarrow & & \downarrow \\ \mathbb{Z}_l[x]^{Iw}\langle G,\rho,l\rangle & \longrightarrow & \pi_* End^{\wedge}_{KF}(KE) \end{array}$$

commute.

Proof This follows directly from Proposition 4.3.1, once we know that

$$\pi_*\mathcal{K}^{Iw}\langle G,\rho,l\rangle \cong \mathbb{Z}_l[x]^{Iw}\langle G,\rho,l\rangle$$

But this is the content of Proposition 4.2.5. □

4.4 A conjecture

Let $F \subseteq \overline{F}$ be the inclusion of a field F in its algebraic closure, and let G_F denote the absolute Galois group of F. In Proposition 4.3.1, we constructed a map of spectra

$$\mathcal{K}^{Iw}\langle G,\rho,l\rangle \to End^{\wedge}_{KF}(K\overline{F})$$

and consequently an induced homomorphism on homotopy groups

$$\mathbb{Z}_l[x]^{Iw}\langle G, \pi_*\rho\rangle \longrightarrow \pi_* End^{\wedge}_{KF}(K\overline{F})$$

It is interesting to ask about the nature of this map, whether it is injective or surjective. We will first give a heuristic argument explaining why the homomorphism should not be surjective.

Consider F, $\overline{F}$, and G as above. Let $g \in G$, and consider the map $g-1 : KF \to K\overline{F}$. Let $\mathbb{L}$ denote the Eilenberg-MacLane spectrum for the integers, and let $K\overline{F} \to \mathbb{L}$ denote the obvious map, which induces an isomorphism on π_0. The composite $K\overline{F} \stackrel{g-1}{\longrightarrow} K\overline{F} \longrightarrow \mathbb{L}$ is clearly null homotopic, since both g and the identity induce the identity on π_0. This means that we have the following diagram.

$$\begin{array}{ccccc} & & K\overline{F}[2,3,\ldots)^{\wedge}_l & & \\ & & \downarrow & & \\ K\overline{F}^{\wedge}_l & \xrightarrow{g-1} & K\overline{F}^{\wedge}_l & \longrightarrow & \mathbb{L}^{\wedge}_l \end{array}$$

where $K\overline{F}[2,3,\ldots)^{\wedge}_l$ denotes the 1-connected Postnikov cover of $K\overline{F}^{\wedge}_l$, with the horizontal composite null homotopic. It follows from Suslin's theorem and Bott periodicity that on the spectrum level, $K\overline{F}[2,3,\ldots)^{\wedge}_l \cong \Sigma^2 K\overline{F}_l$, and that under this identification, the vertical map in the diagram above may be identified with left multiplication by the Bott map β. Further, since the horizontal composite is null homotopic, there is a lift of $g-1$ to $K\overline{F}[2,3,\ldots)^{\wedge}_l$, giving us the following diagram.

$$\begin{array}{ccccc} & & \Sigma^2 K\overline{F}^{\wedge}_l & & \\ & \nearrow & \downarrow{\scriptstyle \times\beta} & & \\ K\overline{F}^{\wedge}_l & \xrightarrow{g-1} & K\overline{F}^{\wedge}_l & \longrightarrow & \mathbb{L}^{\wedge}_l \end{array}$$

This tells us that the map $g-1$is left divisible by the Bott map β, which isn't the case in the ring $\mathbb{Z}_l[x]^{Iw}\langle G, \pi_*\rho\rangle$. This certainly suggests that the above homomorphism is not surjective, and that we look for a suitably "universal construction " in which all elements $g-1$ become left divisible by β. A reasonable candidate for such an algebra is given by the ring $\hat{R}\langle G, \rho\rangle$ defined in 4.1.

Conjecture 4.4.1 *There is a commutative diagram*

$$
\begin{array}{ccc}
\mathbb{Z}_l[x]^{Iw}\langle G, \pi_*\rho\rangle & & \\
\downarrow & \searrow & \\
\hat{R}\langle G, \rho\rangle & \longrightarrow & \pi_* End^{\wedge}_{KF}(K\overline{F})
\end{array}
$$

and the lower horizontal arrow is an isomorphism of pro-rings.

4.5 Examples where F contains an algebraically closed subfield

We will now examine how this works in some specific examples. As usual, l will denote a prime at which we will be completing our spectra, and p will be denoting the characteristic of our field when we deal with a finite characteristic situation. All homology groups will be with $\mathbb{F}_l$-coefficients.

Example: Let k denote an algebraically closed field of characteristic zero, and let $F = k((x))$, field of quotients of power series with coefficients in k. The field F contains the subring $A = k[[x]]$ of actual power series, and is obtained from it by inverting x. We begin by analyzing the spectrum KF. It follows from the localization sequence (see [44]) that we have a fibration sequence of spectra

$$Kk \longrightarrow KA \longrightarrow KF$$

Further, the ring A is a *Henselian local ring* with residue class field k. It follows from a theorem of O. Gabber (see [19]) that the map of spectra $KA \to Kk$ induces an isomorphism on homotopy groups with finite coefficients, and consequently an equivalence on l-adic completions $KA^{\wedge}_l \to Kk^{\wedge}_l$. The fiber sequence above now becomes, up to homotopy equivalence, a fiber sequence

$$Kk^{\wedge}_l \longrightarrow Kk^{\wedge}_l \longrightarrow KF^{\wedge}_l$$

All three spectra in this sequence become module spectra over the commutative S-algebra $Kk^{\wedge}_l$, and the sequence consists of maps which are $Kk^{\wedge}_l$-module maps. Since we have $\pi_0 Kk^{\wedge}_l \cong \pi_0 KF^{\wedge}_l \cong \mathbb{Z}_l$, we find that the inclusion map $\pi_0 Kk^{\wedge}_l \to \pi_0 KA^{\wedge}_l$ is the zero map. Since the module $\pi_* Kk^{\wedge}_l$ is cyclic over the ring $\pi_* Kk^{\wedge}_l$, this means that the inclusion induces the zero map on all homotopy groups. The conclusion is that as a $\pi_* Kk^{\wedge}_l$-module,

$$\pi_* KF^{\wedge}_l \cong \pi_* Kk^{\wedge}_l \oplus \pi_* Kk^{\wedge}_l[1]$$

where the second summand is topologicaly generated by the unit x viewed as an element of $K_1(F)$. In fact, the algebra structure is also determined, since the square of the one dimensional generator is zero. The conclusion is that

$$\pi_* KF_l^\wedge \cong \Lambda_{\mathbb{Z}_l[x]}(\xi)$$

where Λ denotes the Grassmann algebra functor, where the polynomial generator x is in dimension 2, and where the exterior generator ξ is in dimension 1.

In order to analyze the homotopy groups of the pro-ring $\mathcal{E} = End^\wedge_{KF}(K\overline{F})$, we first describe the E_2-term of the universal coefficient spectral sequence converging to $\pi_*\mathcal{E}$. This spectral sequence has E_2-term given by

$$Ext^{cont}_{\Lambda_{\mathbb{Z}_l[x]}(\xi)}(\mathbb{Z}_l[x], \mathbb{Z}_l[x])$$

It is well-known that the *Ext*-algebra over a Grassmann algebra is equal to the polynomial algebra on same vector space, so in this case we find that the E_2-term is the bigraded algebra

$$\mathbb{Z}_l[x][y]$$

where x has bigrading (0,2), and where y has bigrading (1,-1). Terms in bigrading (1,-1), should they survive to E_∞, represent elements in $\pi_{-2}\mathcal{E}$. The spectral sequence collapses for dimensional regions, and it is easy to see that the structure of $\pi_*\mathcal{E}$ as a pro-ring is equivalent to the pro-ring

$$\{\mathbb{Z}_l[x][y]/(l,y)^i\}_{i\geq 0}$$

The homotopy groups of the inverse limit are given by the power series ring over $\mathbb{Z}_p[x]$ on a single generator in dimension -2.

Example: In this case, let F denote the field of functions in one variable x over an algebraically closed field k of characteristic 0 . We again analyze the K-theory of F directly. F contains the polynomial subring $A = k[x]$, and F is obtained from A by inverting all non-zero polynomials. Let $\underline{T}$ denote the category of all A-modules which are finitely generated as k-modules. The category of F-vector spaces can be obtained as the quotient category of the category $Mod(A)$ of all finitely generated A-modules by the subcategory $\underline{T}$, so we have a fibration sequence of spectra

$$K\underline{T} \longrightarrow KA \longrightarrow KF$$

by the localization theorem of Quillen.

We are able to analyze the structure of the category $\underline{T}$ as well. For any $\alpha \in k$, we say an object M in $\underline{T}$ is α-*primary* if the element $x - \alpha$ acts nilpotently on M. The structure theorem for modules over a P.I.D. shows that every object M in $\underline{T}$ decomposes canonically as

$$M \cong \coprod_{\alpha \in k} M_\alpha$$

where M_α is α-primary, and where the coproduct sign indicates that all but finitely many summands are trivial. This decomposition is canonical, so we obtain a decomposition of the category $\underline{T}$ as

$$\underline{T} \cong \prod_{\alpha \in k}^{res} \underline{T_\alpha}$$

where $\underline{T_\alpha}$ denotes the full subcategory of α-primary modules, and where $\prod^{res}$ indicates the restricted product consisting of objects where all but finitely many factors are $= 0$. An application of the "devissage theorem" as in [44] shows that the $K\underline{T_\alpha}$ is equivalent to the K-theory spectrum attached to the full subcategory of modules in which $x - \alpha$ acts *trivially*. The latter category is clearly equivalent to the category of finite dimensional k-vector spaces, and we obtain an equivalence of spectra

$$K\underline{T} \cong \prod_{\alpha \in k}^{res} Kk$$

On the other hand, we have the homotopy property for K-theory, as proved by Quillen (see [44]), which asserts that for any regular commutative ring B, the canonical map $KB \to KB[x]$ is an equivalence of spectra. The conclusion is that up to homotopy, we have a fibration sequence

$$\prod_{\alpha \in k}^{res} Kk \longrightarrow Kk \longrightarrow KF$$

As in the case of the power series ring, one easily checks that the homomorphism $\pi_0 Kk \to \pi_0 KF$ is an isomorphism, and therefore that we have an isomorphism of $\pi_* Kk$-modules

$$\pi_* KF \cong \pi_* Kk \oplus \bigoplus_{\alpha \in k} \pi_* Kk[1]$$

In dimension one, the generator for the summand corresponding to α is the unit $x-\alpha$, regarded as an element of $K_1(F)$. As in the case of the power series ring, the product is easy to describe. The cup products of all elements in degree 1 are trivial. Consequently, we obtain the following description of the ring $\pi_* KF_l^{\wedge}$.

$$\pi_* KF_l^{\wedge} \cong \mathcal{D}_{\mathbb{Z}_l[x]}(V(k,1))$$

where $\mathcal{D}_{\mathbb{Z}_l[x]}$ denotes the "ring of dual numbers"functor from graded $\mathbb{Z}_l$-modules to graded $\mathbb{Z}_l[x]$-algebras, and $V(k,1)$ denotes the $\mathbb{Z}_l$-module $\underleftarrow{lim} \bigoplus_{\alpha \in k} \mathbb{Z}/l^i\mathbb{Z}$, concentrated in dimension 1.

We now wish to describe the universal coefficient spectral sequence for computing $\pi_*\mathcal{E}$. Let V^* denote the topological $\mathbb{Z}/l\mathbb{Z}$ module $Hom(V(k,1)/lV(k,1), \mathbb{F}_l)$, where the topology is the inverse limit topology coming from the inverse limit of $\mathbb{F}_l$-duals of finite dimensional subspaces of $V(k,1)$. The E_2-term for the universal coefficient spectral sequence for computing $\pi_*\mathcal{E}$ is the free associative graded algebra over Kk_* on the topological module V^*. This means the "topological" tensor algebra, in the sense that it is computed using completed tensor products rather than ordinary tensor products. In this case, the elements in V^* are given homological degree 1 and geometric dimension -2. The spectral sequence degenerates for dimensional reasons, and one easily concludes that $\pi_*\mathcal{E}$ is isomorphic as a topological graded ring to the non-commutative power series ring

$$\mathbb{Z}_l[\beta]\langle\langle V^* \rangle\rangle$$

and where the topology is specified by the powers of the ideal $(l) + (V^*)$.

4.6 Finite fields

Example: We consider the case of a finite field $\mathbb{F}_q$, where, $q = p^r$ for some prime p. We will be studying the spectrum $K\mathbb{F}_q$ completed at a prime l distinct from p. We first recall the work of Quillen on the algbraic K-theory of $\mathbb{F}_q$

Theorem 4.6.1 *There is a cofibration sequence of spectra*

$$K\mathbb{F}_{q}{}^{\wedge}_{l} \longrightarrow K\overline{\mathbb{F}}_{q}{}^{\wedge}_{l} \xrightarrow{\varphi - 1} K\mathbb{F}_{q}{}^{\wedge}_{l}[2,3,\ldots) \cong \Sigma^2 K\mathbb{F}_{q}{}^{\wedge}_{l}$$

where φ denotes the r-th power of the Frobenius operator, and where $X[k, k+1,\ldots)$ denotes the $(k-1)$-connected Postnikov cover of the spectrum X.

Remark: Quillen's result was only stated on the level of zeroth spaces of spectra, but it is a straightforward argument to obtain the spectrum level statement.

We will now develop do some homological calculations in order to analyze the ring spectrum $End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$.

Throughout this section, all homology groups will be taken with $\mathbb{Z}/l\mathbb{Z}$-coefficients. Recall from the previous section that we have the Adams-Hom spectral sequence with E_2-term isomorphic to

$$Ext^{**}_{\mathcal{A}_*}(\mathbb{F}_l, \pi_* Hom^{\wedge}_{\mathbb{F}_q}(K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q))$$

converging to $\pi_* End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$. We will analyze this E_2-term. Recall that there is a standard adjunction

$$Hom_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q) \cong$$

$$Hom_{\mathbb{H} \underset{S^0}{\wedge} K\mathbb{F}_q}(\mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q)$$

The universal coefficient spectral sequence for computing homotopy groups of Hom-spectra in this case has the form

$$Ext_{H_* K\mathbb{F}_q}(H_* K\overline{\mathbb{F}}_q, H_* K\overline{\mathbb{F}}_q)$$

Recall that the mod-l homology of the mod-l Eilenberg MacLane spectrum is isomorphic to an graded polynomial algebra tensored with a graded exterior algebra over the field $\mathbb{F}_l$. The polynomial generators are called $\xi_i, i \geq 1$ and the exterior generators are named $\tau_i, i \geq 0$, and their dimensions are given by

$$|\xi_i| = 2p^i - 1 \quad \text{and} \quad |\tau_i| = 2p^i - 2$$

We also have the following result describing the decomposition of the l-completed ku-spectrum. See [26].

Proposition 4.6.2 *Let $C = \mathbb{Z}/(l-1)\mathbb{Z}$, and identify C with the group of $(l-1)$-roots of unity in $\mathbb{Z}_l$. There is an action of C on the spectrum $ku^{\wedge}_l$, with the property that the induced action of C on $\pi_2 ku^{\wedge}_l \cong \mathbb{Z}_l$ is multiplication by the corresponding group of roots of unity. Let $\chi = \chi_1$ denote the defining character via roots of unity, and let χ_s denote the character $\chi^{\otimes s}$, and conventionally set $\chi_0 = \chi^{\otimes(l-1)}$. For each $0 \leq i < l-1$, there is a summand $K(i)$ obtained by taking the "isotypical component" corresponding to the character χ_i. We have a decomposition*

$$ku_l^\wedge \cong K(0) \vee K(1) \vee \ldots \vee K(p-2)$$

$K(0)$ is a subring spectrum of $ku_l^\wedge$, and the decomposition is as module spectra. The summands are mutually isomorphic up to suspensions, i.e. $K(j) \cong \Sigma^{2j} K(0)$.

It follows from Suslin's theorem ([57]) that $K(\overline{\mathbb{F}}_q)_l^\wedge \cong ku_l^\wedge$. It is also well-known (see [26]) that the mod-l homology of $K(0)$ is isomorphic to the subalgebra of

$$H_*\mathbb{H} \cong \mathbb{F}_l[\xi_i; i \geq 1] \otimes \Lambda_{\mathbb{F}_l}(\tau_i; i \geq 0)$$

generated by all the ξ_i's and by the τ_i's for $i \geq 2$, as a module over the Steenrod algebra.

For simplicity, we now restrict ourselves to the case where $l|(q-1)$. Recall the cofibration sequence from 4.6.1. In this case, the map $\varphi^r - 1$ induces the zero map on homology, and we find that

$$H_*K\mathbb{F}_q \cong H_*K\overline{\mathbb{F}}_q \oplus H_{*-1}K\overline{\mathbb{F}}_q$$

To compute the multiplicative structure, it suffices to observe that $H_*K\mathbb{F}_q$ is a commutative algebra, and that any one-dimensional element has trivial square. The conclusion is that

$$H_*K\mathbb{F}_q \cong H_*K\overline{\mathbb{F}}_q[\theta]/(\theta^2)$$

i.e. the graded commutative exterior algebra on a single 1-dimensional generator over $H_*K\overline{\mathbb{F}}_q$.

We now consider the Adams-Hom spectral sequence for computing $\pi_*End^\wedge_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$.

The input to this spectral sequence is

$$\pi_*Hom_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q) \cong$$

$$\pi_*Hom_{\mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q}(\mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q,)$$

The E_2-term for the universal coefficient spectral sequence converging to this group has the form

$$Ext^{**}_{H_*K\mathbb{F}_q}(H_*K\overline{\mathbb{F}}_q, H_*K\overline{\mathbb{F}}_q)$$

A standard change of rings result now shows that this E_2-term is isomorphic to

$$Ext^{**}_{\Lambda^{\mathbb{F}_l}_*[\theta]}(\mathbb{F}_l, H_*K\overline{\mathbb{F}}_q) \cong \mathbb{F}_l[\eta] \otimes H_*K\overline{\mathbb{F}}_q$$

where η is a polynomial generator of homological degree 1 and geometric dimension -2.

Proposition 4.6.3 *This spectral sequence collapses.*

Proof It is clear that this spectral sequence is a spectral sequence of $H_*K\overline{\mathbb{F}}_q$-modules, so it will suffice to show that the generator η in geometric dimension -2 is an infinite cycle. To see this, recall that we have a cofiber sequence of $H_*K\overline{\mathbb{F}}_q$-module spectra

$$K\mathbb{F}_q \longrightarrow K\overline{\mathbb{F}}_q \stackrel{\varphi^r-1}{\longrightarrow} \Sigma^2 K\overline{\mathbb{F}}_q$$

so $K\overline{\mathbb{F}}_q/K\mathbb{F}_q \cong \Sigma^2 K\overline{\mathbb{F}}_q$. By abuse of notation, $\varphi^r - 1$ denotes the lift of the operator $\varphi^r - 1$ to the 1-connected connective cover of $K\overline{\mathbb{F}}_q$. In order to prove the result, it is clear that it will suffice to show that there is a map of $K\mathbb{F}_q$-module spectra

$$\lambda : K\overline{\mathbb{F}}_q \longrightarrow \Sigma^2 K\overline{\mathbb{F}}_q$$

which satisfies

- λ restricted to $K\mathbb{F}_q$ is null homotopic.
- In the homology of the mapping cone of λ, $C(\lambda)$, multiplication by the element θ in $H_1K\mathbb{F}_q$ from $H_1C(\lambda)$ to $H_2C(\lambda)$ is non-trivial.

But both conditions are clearly satisfied by the map

$$K\overline{\mathbb{F}}_q \stackrel{\varphi^r-1}{\longrightarrow} \Sigma^2 K\overline{\mathbb{F}}_q$$

□

Corollary 4.6.4 *There is an $\mathbb{F}_l$-algebra isomorphism*

$$\pi_* Hom_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q) \cong H_*K\overline{\mathbb{F}}_q[[\eta]]$$

where η has degree -2.

Proposition 4.6.5 *The action of the Steenrod algebra $\mathcal{A}_*$ on*

$$\pi_* Hom_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q)$$

is determined by the fact that these groups form a graded $H_*K\overline{\mathbb{F}}_q$-module, and that all the elements $1, \eta, \eta^2, \ldots$ are acted on trivially by all elements of $\mathcal{A}_*$ of negative degree.

Proof It is clear that if these facts are true, then they determine the $\mathcal{A}_*$-action. It is also clear that 1 is acted on trivially, and that the Cartan formula shows that it is now sufficient to show that all negative degree elements of $\mathcal{A}_*$ act trivially on η. This result will now follow from the observation that for any ring spectrum R and left module spectra M and N, the image of the map

$$\pi_* Hom_R(M, N) \to \pi_* Hom_R(M, \mathbb{H} \underset{S^0}{\wedge} N)$$

contains only elements on which all elements of positive degree in $\mathcal{A}_*$ vanish, under the action of $\mathcal{A}_*$ on $\pi_* Hom_R(M, \mathbb{H} \underset{S^0}{\wedge} N)$ defined above. □

We are now in a position to describe $\Lambda_* \cong \pi_* End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$. Recall that we have the Adams-Hom spectral sequence converging to this graded algebra whose E_2-term is

$$Ext_{\mathcal{A}_*}(\mathbb{F}_l, \pi_* Hom_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q, \mathbb{H} \underset{S^0}{\wedge} K\overline{\mathbb{F}}_q))$$

Proposition 4.6.6 *The E_2-term of the above spectral sequence has the form*

$$Ext_{\mathcal{A}_*}(\mathbb{F}_l; H_*K\overline{\mathbb{F}}_q)[[\theta]]$$

where θ has homological degree 0 and geometric dimension -2. The spectral sequence collapses.

Proof The form of the E_2-term follows directly from 4.6.5. The collapse result follows from the observation that the image of the map

$$\pi_* Hom_R(M, N) \to \pi_* Hom_R(M, \mathbb{H} \underset{S^0}{\wedge} N)$$

contains only infinite cycles on the zero line in the corresponding Adams-Hom spectral sequence. □

From this result it is possible to deduce the following about the graded ring Λ_*.

- Λ_* is topologically generated over $K\overline{\mathbb{F}}_q$ by the an element $\overline{\theta}$ of degree -2.
- Λ_* is a torsion free $\pi_* K\overline{\mathbb{F}}_q$ -module
- Elements of Λ_* are detected by their effect on homotopy groups, so that any element of Λ_* which acts trivially on homotopy groups is null homotopic.

It now follows easily from these observations that the element $\varphi^r - 1$ is left divisible by the Bott element in Λ_*, and hence that we obtain a homomorphism $\hat{R}\langle \mathbb{Z}_l, \rho \rangle$ to $\pi_* End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$, and that this homomorphism induces an isomorphism on associated graded algebras from $gr^J_* \hat{R}\langle G, \rho \rangle$ to the associated graded algebra for $\pi_* End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$. It is now follows that the map is an isomorphism, which gives the structure of $\pi_* End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$.

We can give an explicit description of $\pi_* End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$. As before, et $q = p^r$, and suppose $l|(q-1)$. Then we have an action of the group $\hat{\mathbb{Z}}$ on the graded ring $\mathbb{Z}_l[x, x^{-1}]$ by allowing the generator T corresponding to $1 \in \mathbb{Z}$ to act by $Tx = qx$. We may now form the twisted group ring $\mathcal{O}_* = \mathbb{Z}_l[x, x^{-1}]\langle G, \rho \rangle$. We let I_n denote the two sided ideal generated by l^n and $T^{l^n} - 1$. It is clear that $I_{n+1} \subseteq I_n$, and we form the inverse limit

$$\hat{\mathcal{O}}_* = \lim_{\leftarrow} \mathcal{O}/I_n$$

We also have the twisted group ring $\mathcal{L}_* = \mathbb{Z}_l[x]\langle G, \rho \rangle$, and its corresponding completion at the family of ideals I_n, denoted by $\hat{\mathcal{L}}_*$. The graded ring $\pi_* End^{\wedge}_{K\mathbb{F}_q}(K\overline{\mathbb{F}}_q)$ is isomorphic to the subring of $\hat{\mathcal{O}}_*$ generated by the subring $\hat{\mathcal{L}}_*$ together with the element $x^{-1}(1-T)$.

4.7 The real case

We will consider the case of descent from $K\mathbb{C}^{\wedge}_2$ to $K\mathbb{R}^{\wedge}_2$. In this case, the absolute Galois group is $\mathbb{Z}/2\mathbb{Z}$, generated by complex conjugation. It is perhaps surprising that the same relations which define $\hat{R}_*\langle G, \rho \rangle$ work equally well in this case to define the graded ring $\pi_* End^{\wedge}_{K\mathbb{R}}(K\mathbb{C})$, which we will again denote by Λ_*. In this case, the work of Suslin and of Karoubi allows us to identify the completed K-theory spectra.

Theorem 4.7.1 (Suslin, Karoubi (see [57], [27])) *We have equivalences of commutative ring spectra*

$$K\mathbb{C}^{\wedge}_2 \simeq ku^{\wedge}_2 \text{ and } K\mathbb{R}^{\wedge}_2 \simeq ko^{\wedge}_2$$

where ko and ku denote the connective real and complex K-theory spectra, respectively.

The homology algebras of these two ring spectra are also simple to describe. Recall that in the case $l = 2$, the dual to the Steenrod algebra $\mathcal{A}_2$ is a polynomial algebra $\mathbb{F}_2[\xi_1, \xi_2, \ldots]$, on generators ξ_j of degree $2^j - 1$.

Proposition 4.7.2 (Stong; see [56]) *The homology rings of the spectra $ku_2^\wedge$ and $ko_2^\wedge$ are given by*

$$H_* ku_2^\wedge \cong \mathbb{F}_2[\xi_1^2, \xi_2^2, \xi_3, \ldots]$$

and

$$H_* ko_2^\wedge \cong \mathbb{F}_2[\xi_1^4, \xi_2^2, \xi_3, \ldots]$$

Notice that $H_* ku_2^\wedge$ is a free $H_* ko_2^\wedge$-module of rank two, with basis given by the elements 1 and ξ_1^2. This decomposition has a geometric counterpart. To see this, we observe that the zeroth space in the spectrum ku is the space $BU \times \mathbb{Z}$, and that we have an evident inclusion $\mathbb{C}P^\infty \to BU \times \{1\}$. It follows that we have a map of spectra $i : \Sigma^\infty \mathbb{C}P^\infty_+ \to ku$, where $(-)_+$ denotes the addition of a disjoint basepoint. Both $\Sigma^\infty \mathbb{C}P^\infty_+$ and ku admit maps to the integral Eilenberg-MacLane spectrum $\mathbb{L}$, and we obtain a commutative diagram

$$\begin{array}{ccccc}
\Sigma^\infty \mathbb{C}P^\infty & \longrightarrow & \Sigma^\infty \mathbb{C}P^\infty_+ & \longrightarrow & \mathbb{L} \\
\downarrow{\scriptstyle \hat{\imath}} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id} \\
\Sigma^2 ku \simeq ku[2,3,\ldots) & \longrightarrow & ku & \longrightarrow & \mathbb{L}
\end{array}$$

One readily checks that the map $\hat{\imath}$ induces a surjection on H_2. By applying Ω^2 and restricting to $\mathbb{C}P^1$, we obtain

Proposition 4.7.3 *There is a map $\hat{\hat{\imath}} : \Sigma^{-2}\mathbb{C}P^1 \to ku$ which induces isomorphisms on H_0, H_1, and H_2. Since ku is a ko-module, we obtain a natural equivalence*

$$ko \wedge \Sigma^{-2}\mathbb{C}P^1 \longrightarrow ku$$

In particular, there is a cofibration sequence

$$ko \longrightarrow ku \longrightarrow \Sigma^2 ko$$

Of course, we obtain the corresponding sequence of 2-completed spectra.

Proposition 4.7.4 *There is a split cofibration sequence of spectra*

$$\Sigma^{-2} ku_2^\wedge \longrightarrow End_{ko}^\wedge(ku) \longrightarrow ku_2^\wedge$$

Consequently, we have a direct sum decomposition

$$\pi_* End^{\wedge}_{ko}(ku) \cong \pi_* ku^{\wedge}_2 \oplus \pi_{*+2} ku^{\wedge}_2$$

Proof The only thing which needs to be verified is that the sequence is split. But this follows since there is always an inclusion of B in $End_A(B)$, when A and B are commutative ring spectra. □

Remark: This is just the well known fact that ku can be obtained as the cofiber of multiplication by the element η in $\pi_1 ko$ viewed as a self map of ko.

It remains to determine the multiplicative structure of $\Lambda_* = \pi_* End^{\wedge}_{ko}(ku)$. First, it follows easily from this decomposition that as in the finite field case, elements of Λ_* are determined by their effect on homotopy groups.

It is also clear that a generator θ for Λ_{-2} is given by the class of the map

$$ku \longrightarrow ku/ko \cong \Sigma^2 ko \longrightarrow \Sigma^2 ku$$

On can check algebraically that $1 - T$ becomes divisible in $\pi_* End^{\wedge}_{ko}(ku)$, and thereby produces an hmomorphism from $\hat{R}\langle G, \rho\rangle$ to $\pi_* End^{\wedge}_{ko}(ku)$. This map is easily checked to be an isomorphism. In this case, the graded ring $\pi_* End^{\wedge}_{ko}(ku)$ is isomorphic to the subring of the twisted group ring $\mathbb{Z}_2[x, x^{-1}]\langle \mathbb{Z}/2\mathbb{Z}, \rho\rangle$ generated by $\mathbb{Z}_2[x]\langle G, \rho\rangle$ together with the element $x^{-1}(1 - T)$. Here, the group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}_2[x, x^{-1}]$ via $Tx = -x$.

5 Endomorphism rings and the Positselskii-Vishik conjecture

In this section, we will show that our conjecture on the structure of the graded algebra $\pi_* End^{\wedge}_{KF}(K\overline{F})$ can be proved for fields for which a very strong form of the *Block-Kato conjecture* holds. This conjecture is known to play a central role in the motivic theory. It's strengthening due to Positselskii and Vishik plays a similar role here. We will first outline the theory of Koszul duality algebras, then discuss the Bloch-Kato conjecture and the work of Positselskii and Vishik [43], continue to describe what we need from motivic theory and the work of Beilinson and Lichtenbaum on the Bloch-Lichtenbaum spectral sequence, and finally describe our results. Throughout this section, l will denote a prime at which we will complete our spectra, p will denote the characteristic in cases of fields of positive characteristic. H_* and H^* will always denote homology and cohomology with $\mathbb{Z}/l\mathbb{Z}$-coefficients, and $\mathbb{H}$ and $\mathbb{L}$ will denote the Eilenberg-MacLane spectra for the groups $\mathbb{Z}/l\mathbb{Z}$ and $\mathbb{Z}_l$, respectively. $\mathcal{A}_*$ will denote the mod-l Steenrod algebra.

5.1 Koszul duality algebras

The following is standard material on Koszul duality. [43] is a useful reference. Let A_* denote an augmented graded algebra over a field k, with augmentation $\epsilon : A_* \to k$. . For any graded A_*-module M_*, we let $M(n)_*$ denote the graded module obtained by shifting the grading of M_* by n, i.e. $M(n)_k \cong M_{n-k}$. Recall also that given two graded A_*-modules M_* and N_*, we construct a graded A_*-module $Hom_{A_*}(M_*, N_*)$ by letting $Hom_{A_*}(M_*, N_*)_n$ be the k-vector space of homomorphisms of graded A_*-modules from $M(n)_*$ to N_*. We may now construct a bigraded ring $Ext^{**}_{A_*}(k,k)$, where the first grading is the homological degree and the second one is the internal degree. The homological grading is *non-positive*, so the group Ext^i is regarded as sitting in homological grading $-i$. The ring structure on $Ext^{**}_{A_*}(k,k)$ as usual comes from Yoneda product. We write $H^*(A_*)$ for the ring $Ext^{**}_{A_*}(k,k)$ regarded as a graded ring, using the homological degree as our grading.

Definition 5.1.1 *Suppose that $A_0 \cong k$, and that A_* is non-negatively graded. We say A_* is a graded* quadratic algebra *if*

- *A_* is generated by A_1*
- *The kernel of the evident homomorphism of graded rings $k\langle A_1\rangle_* \to A_*$ is generated as a two sided ideal by elements in $A_1 \otimes A_1 \cong k\langle A_1\rangle_2$. Here $k\langle A_1\rangle_*$ denotes the free associative graded k-algebra on the k-vector space A_1.*

Definition 5.1.2 *Let A_* denote a graded quadatic algebra. We say A_* is a* Koszul duality algebra *if $H^*(A_*)$ is generated by elements in $Ext^1_{A_*}(k,k)$.*

Example: $k\langle V\rangle_*$. In this case, H^* is isomorphic to a graded *ring of dual numbers* $\mathcal{D}[V^*]$ on V^*, where V^* is assigned grading -1. $\mathcal{D}[V]$ is obtained by setting all two-fold products of elements in V to zero.

Example: The graded symmetric algebra $k[V]_*$ on a vector space, with V assigned grading 1. In this case, H^* is a *Grassmann algebra* on V^*, with V^* assigned grading -1.

Example: The graded *ring of dual numbers* $\mathcal{D}[V]_*$ on a k-vector space. In this case, H^* is isomorphic to $k\langle V^*\rangle$, where V is assigned grading -1.

Example: The graded Grassmann algebra $\Lambda[V]_*$ on a k-vector space V, again with V assigned grading 1. $H^* \cong k[V^*]$, with V^* assigned grading -1.

It is also important to keep track of the internal gradings in the bigraded ring $Ext^{**}_{A_*}(k,k)$. The following is shown in [43].

Proposition 5.1.3 *Let A_* be a Koszul duality algebra. Then the bigraded ring $Ext^{**}_{A_*}(k,k)$ is generated by classes in $Ext^{1,1}_{A_*}(k,k)$. It follows that $Ext^{i,j}_{A_*}(k,k) = 0$ for $i \neq j$.*

5.2 Koszul duality groups

Definition 5.2.1 *We say a group* Γ *is a* Koszul duality group at a prime l *if the cohomology ring $H^*(\Gamma; \mathbb{F}_l)$ is a Koszul duality algebra. If G is a profinite group, we say G is Koszul duality at l if $H^*_{cont}(G; \mathbb{F}_l)$ is a Koszul duality algebra.*

Example: $\Gamma = \mathbb{Z}^n$ for any prime l.

Example: Γ is a free group on a finite set of generators, again for any prime l.

Example: The fundamental group of an orientable surface, for any prime.

Example: The free pro-p group on any set of generators, at the prime l.

Example: The free profinite group on any set, at any prime.

The Koszul duality property for groups is related to the behavior of the *Eilenberg-Moore spectral sequence* for the fibration

$$\Gamma \longrightarrow E\Gamma \longrightarrow B\Gamma$$

Recall (see [55])that for any fibration

$$F \longrightarrow E \longrightarrow B$$

there is a spectral sequence with E_2-term $Ext_{H^*(B)}(H^*(B), H^*(pt))$, which for simply connected base space and fiber converges to $H_*(F)$. It is useful to ask what the spectral sequence converges to in some non-simply connected situations, such as in the case of the fibration $\Gamma \longrightarrow E\Gamma \longrightarrow B\Gamma$ above.

In order to do this, we consider the augmentation homomorphism $\epsilon : \mathbb{F}_l[\Gamma] \to \mathbb{F}_l$. Rings may always be viewed as S-algebra by a functorial Eilenberg-MacLane construction, and so we are in the situation of Section III, so there is a derived completion $\mathbb{F}_l[\Gamma]^{\wedge}_{\epsilon}$. According to 3.8, in order to analyze the groups $\pi_*\mathbb{F}_l[\Gamma]^{\wedge}_{\epsilon}$, we may use a spectral sequence whose E_2-term is identified with the *Ext*-groups over the homotopy groups of the pro-S-algebra $End^{\wedge}_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l)$. It is now easy to verify the following.

Proposition 5.2.2 *There is a canonical isomorphism*

$$\pi_* End^{\wedge}_{\mathbb{F}_l[\Gamma]}(\mathbb{F}_l) \xrightarrow{\sim} H^{-*}(\Gamma, \mathbb{F}_l)$$

The descent spectral sequence 3.8 can be identified with the Eilenberg-Moore spectral sequence. Consequently, the Eilenberg-Moore spectral sequence converges to $\pi_*\mathbb{F}_l[\Gamma]^{\wedge}_{\epsilon}$. *In the case of a profinite group* G*, it converges to* $\varprojlim \pi_*\mathbb{F}_l[G/K]^{\wedge}_{\epsilon}$*, where the inverse limit ranges over the subgroups of finite index in* G.

Now suppose we are in the case of a Koszul duality group, either in the discrete or profinite case. It follows from the construction of the spectral sequence that elements in $Ext^{i,j}_{H^*(\Gamma)}(\mathbb{F}_l, \mathbb{F}_l)$ correspond (at the E_∞-level) to homotopy elements in dimension $j - i$. Since by 5.1.3 these groups vanish except when $i = j$, it follows that the spectral sequence collapses at the E_2-level. We now have the following.

Proposition 5.2.3 *Let* G *be a profinite Koszul duality group. Then the Eilenberg-Moore spectral sequence for* $\pi_*\mathbb{F}_l[G]^{\wedge}_{\epsilon}$ *collapses at* E_2. $\pi_*\mathbb{F}_l[G]^{\wedge}_{\epsilon} = 0$ *for* $* \neq 0$*, and by 3.4* $\pi_0\mathbb{F}_l[G]^{\wedge}_{\epsilon}$ *is isomorphic to the completion (in the usual algebraic sense) of the ring* $\mathbb{F}_l[G]$ *at the augmentation ideal. The filtration on* $\pi_0\mathbb{F}_l[G]^{\wedge}_{\epsilon}$ *is exactly the augmentation ideal filtration.*

Proof The first two statements are clear. The statement about the filtration can be deduced as follows. The filtration associated to the spectral sequence is associated to a nested sequence of ideals $J_n \subseteq \pi_0\mathbb{F}_l[G]^{\wedge}_{\epsilon}$ so that $J_{n+1} \subseteq J_n$ and $J_m \cdot J_n \subseteq J_{m+n}$, and further so that $J_1 = I$, the augmentation ideal in the completion of the group ring. There is a smallest integer k so that $J_k \neq I^k$, and $k \geq 2$. It follows from 5.1.3 that the associated graded algebra $\oplus_n J_n/J_{n+1}$ is generated in dimension 1. Since $I/I^2 \to J_1/J_2$ is surjective, the map on associated graded algebras

$$\oplus_n I^n/I^{n+1} \longrightarrow \oplus_n J_n/J_{n+1}$$

is surjective. Any element in $J_k - I^k$ represents a non-zero element in $J_k/I^k + J_{k+1}$, but this group is trivial in view of the above mentioned surjectivity. We conclude that $J_k = I^k$ for all k, which is the desired result. □

Remark: In the profinite case, this result must be interpreted in terms not of the group ring but of the ring $\varprojlim \mathbb{F}_l[G/K] = \mathbb{F}_l^{Iw}[G]$. The augmentation ideal is a well defined in this ring, as is the augmentation ideal filtration.

Since $E_2 = E_\infty$ in the spectral sequence, we obtain

Proposition 5.2.4 *The group* $Ext^k_{H^*(G)}(\mathbb{F}_l, \mathbb{F}_l)$ *is isomorphic to the group* I^k/I^{k+1}*, where* I *denotes the augmentation ideal. This result holds both in the discrete case and the profinite case.*

5.3 Milnor K-theory, the Bloch-Kato conjecture, and the Bloch-Lichtenbaum spectral sequence

In this section, we will discuss the necessary background on algebraic K-theory of fields. See [3], [39], or Carlsson3 for a more complete account. Recall first that $\pi_1 KF \cong F^*$. Further, KF is a commutative S-algebra, from which it follows that $\pi_* KF \cong K_*(F)$ is a graded commutative ring. Consequently, we obtain a ring homomorphism $S(F^*) \longrightarrow K_*(F)$, where $S(F^*)$ denotes the free graded $\mathbb{Z}$-algebra on the $\mathbb{Z}$-module F^*. There is a well-known relation, called the *Steinberg relation*, in $K_2(F)$, which has the form $a \cup (1-a) = 0$, where $\cup$ is the cup product in algebraic K-theory, and $a \in F^*$. We now define the *Milnor K-theory* of F, $K_*^M(F)$, to be the quotient of $S(F^*)/I$ where I is ideal generated by all elements of the form $a \otimes (1-a)$. We have an evident homomorphism $K_*^M(F) \to K_*(F)$.

$K_*^M(F)$ is conjectured to be strongly related with Galois cohomology. Let G_F denote the absolute Galois group of F, and consider the short exact sequences of G_F-modules

$$0 \longrightarrow \mu_l \longrightarrow \overline{F}^* \xrightarrow{\times l} \overline{F}^* \longrightarrow 0$$

In the associated long exact sequence of cohomology groups, one portion has the form

$$H^0(G_F, \overline{F}^*) \xrightarrow{\times l} H^0(G_F, \overline{F}^*) \longrightarrow H^1(G_F, \mu_l) \longrightarrow H^1(G_F, \overline{F}^*)$$

From the isomorphisms $H^0(G_F, \overline{F}^*) \cong F^*$ and $H^1(G_F, \overline{F}^*) \cong 0$ (Hilbert's theorem 90), we conclude that $H^1(G_F, \mu_l) \cong F^*/lF^* \cong K_1^M(F)/lK_1^M(F)$. Now consider the graded ring

$$\mathcal{H}_* \cong \bigoplus_{i=0}^{\infty} H^i(G_F, \mu_l^{\otimes i})$$

We now have a natural isomorphism $\theta : K_*^M(F)/lK_*^M(F) \to \mathcal{H}_1$. Tate (see [61]) has shown that all elements of the form $\theta(a) \cup \theta(1-a)$ are zero in $H^2(G_F, \mu_l^{\otimes 2})$, so we obtain a homomorphism of graded rings $\theta_* : K_*^M(F)/lK_*^M(F) \to \mathcal{H}_*$, which is an isomorphism in degree 1. We now have

Conjecture 5.3.1 (Bloch, Kato [3], [28]) *The homomorphism θ_* is an isomorphism in all degrees.*

This conjecture has been proved by Voevodsky for all fields, when $p = 2$, and is known in various cases at other primes.

It turns out that Milnor K-theory plays a very central role in the algebraic K-theory of fields. The following results combine the work of Beilinson, Lichtenbaum, Suslin-Voevodsky, Geisser-Levine, and Friedlander-Suslin.

Theorem 5.3.2 *Let F be a field. There is a second quadrant spectral sequence called the* Bloch-Lichtenbaum spectral sequence *deriving from a filtration of KF whose subquotients are generalized Eilenberg-MacLane spectra, and whose E_1-term can be inetrpreted as motivic cohomology groups $H^i(Spec(F);\mathbb{N})$ for certain complexes of sheaves on $Spec(F)$. One may smash the filtration on KF to obtain a filtration and a spectral sequence for the mod-l homotopy $\pi_*KF \wedge M_l$, where M_l denotes the mod-l Moore spectrum. For a field F containing the l-th roots of unity, there is a second quadrant spectral sequence (called the* mod l Bloch-Lichtenbaum spectral sequence*) which converges to $\pi_*(KF, \mathbb{F}_l) = \pi_*KF \wedge M_l$, (where M_l denotes the mod l Moore space), and whose E_2-term is described in terms of Bloch's so-called higher Chow groups. The spectral sequence is induced by a tower of fibrations of spectra, so that the layers in the tower of fibrations are generalized mod l Eilenberg-MacLane spectra. If the Bloch-Kato conjecture holds for all fields of characteristic $p \neq l$, then the E_1-term of the mod l Bloch-Lichtenbaum spectral sequence has the form*

$$E_1^{s,3t} \cong H^{-s-t}(G_F, \mu^{\otimes t})$$

In particular, $E_1^{-2s,3s} \cong H^s(G_F, \mu^{\otimes s}) \cong K_s^M(F, \mathbb{F}_l)$.

When l is odd, the mod l Moore space admits a pairing $M_l \wedge M_l \to M_l$ which is homotopy associative and commutative. Consequently, the spectral sequence above is a spectral sequence of algebras, with differentials being derivations. We will describe the E_1-term of the Bloch-Lichtenbaum spectral sequence as a bigraded algebra. The notion of a tensor product of two bigraded algebras is defined in the obvious way. For a field F, let $\mathcal{K}^F_{**}$ denote the bigraded algebra defined by $\mathcal{K}_{-2s,3s} = K^M_s(F, \mathbb{F}_l)$ for $s \geq 0$, and $\mathcal{K}^F_{s,t} = 0$ for all other values of s and t. The multiplication is given by the multiplication in the Milnor K-theory in the obvious way. Also, let $S_{**}[\beta]$ be the bigraded polynomial algebra on a single generator β in bigrading $(-1, 3)$. The result is now the following

Proposition 5.3.3 *Suppose the Bloch-Kato conjecture holds. Then the E_1-term for the mod l Bloch-Lichtenbaum spectral sequence is isomorphic to the bigraded algebra*

$$\mathcal{K}_{**} \otimes S_{**}[\beta]$$

In particular, it is generated by the Milnor K-theory together with a single generator β in bidimension $(-1, 3)$.

Corollary 5.3.4 *If F contains a l-th root of unity, then the mod-l Bloch-Lichtenbaum spectral sequence collapses.*

Proof The existence of the mod-l root of unity permits the construction of a Bott element in $K_2(F, \mathbb{F}_l)$. The image of this element in the $E_1^{-1,3}$ term is non-zero, which shows that the element $\beta \in E_1^{-1,3}$ is an infinite cycle. There is also a map from the mod-l Milnor K-theory into $K_*(F, \mathbb{F}_l)$. Since β and the Milnor K-theory generate the E_1-term, all elements must be infinite cycles, and the spectral sequence collapses. □

5.4 The spectrum homology of $K_*(F, \mathbb{F}_l)$ and $K_*(F)$

5.4.1 The homology Bloch-Lichtenbaum spectral sequence

The mod-l Bloch Lichtenbaum spectral sequence as described in the previous paragraph is induced by a tower of fibrations of spectra, so that the layers are products of mod-l Eilenberg-MacLane spectra. It follows that for any spectrum X, we may smash this tower of fibrations with X and obtain a spectral sequence for computing $\pi_*(KF \wedge M_l \wedge X)$. If we let $X = \mathbb{H}$, , we obtain a spectral sequence for the spectrum homology $H_*(KF \wedge M_l, \mathbb{F}_l)$, which we'll call the mod-l homology Bloch Lichtenbaum Moore space (mod-l **hBLM**) spectral sequence. Similarly, if we smash with the spectrum $\mathbb{L}$, we obtain a spectral sequence. Using the equivalence $\mathbb{L} \wedge M_l \cong \mathbb{H}$, we see that this spectral sequence converges to the mod l spectrum homology of KF, and we refer to it as the mod l homology Bloch-Lichtenbaum (mod l **hBL**) spectral sequence. We will first focus attention on the **hBLM** spectral sequence. Let $\mathcal{H}_{**}$ denote the bigraded algebra with $\mathcal{H}_{0,t} \cong H_t(\mathbb{H}_p, \mathbb{F}_p)$, and $\mathcal{H}_{s,t} = 0$ for $t \neq 0$. Of course, $\mathbb{H}_p$ is a S-algebra, so $H_*(\mathbb{H}, \mathbb{F}_l)$ is a graded algebra, which makes $\mathcal{H}_{**}$ into a bigraded algebra.

Proposition 5.4.1.1 *Suppose the Bloch-Kato conjecture holds. Let F be a field containing a primitive l-th root of unity. Then the E_1-term of the mod l-**hBLM** spectral sequence has the form*

$$E_1^{**} \cong \mathcal{K}_{**} \otimes S_{**}[\beta] \otimes \mathcal{H}_{**}$$

The spectral sequence is a spectral sequence of modules over the mod l Steenrod algebra $\mathcal{A}_*$.

Proof The E_1-term is obtained by computing the homotopy groups of the layers of the tower of fibrations. In this case, the layers are all bouquets of mod-l-Eilenberg spectra. For any spectrum X which is a bouquet of mod-l Eilenberg-Maclane spectra, one finds that there is a canonical isomorphism

$H_*(X, \mathbb{F}_l) \cong \pi_* X \otimes H_*(\mathbb{H}, \mathbb{F}_l)$. This easily gives the description of the E_1-term. The Steenrod algebra statement holds since the filtration is a filtration of modules over $\mathbb{H}_l$. □

5.4.2 The hBLM spectral sequence in the algebraically closed case

We consider the case of an algebraically closed field of characteristic $p \neq l$. The results of Kane 4.6.2 and Stong 4.7.2 concerning the homology of the connective K-theory spectrum of the connective K-theory spectrum ku. By Suslin's theorem [57], it suffices to analyze the mod-l **hBLM** spectral sequence for one field of each characteristic.

Lemma 5.4.2.1 *In the mod-l* **hBLM** *spectral sequence for any algebraically closed field of characteristic $p \neq l$, d_i vanishes identically for $i < l-1$*

Proof If $l = 2$, there is no content to this statement, so suppose l is odd. We consider $d_1 : E_1^{0,*} \to E_1^{-1,*-1}$ first. We have $E_1^{0,*} \cong H_*(\mathbb{H}, \mathbb{F}_l)$ and $E_1^{-1,*} \cong H_*(\mathbb{H}, \mathbb{F}_l)[2]$, where for a graded module M_* over $\mathcal{A}_*$, $M[k]$ denotes the shifted graded module $M[k]_l = M_{k-l}$. Taking $\mathbb{F}_l$-duals, we obtain the dual homomorphism $\mathcal{A}(p)[3]_* \to \mathcal{A}_*$. Such a homomorphism is determined by an element in $\mathcal{A}_3$. Recall that $\mathcal{A}_* = 0$ for all t with $0 \leq t \leq 2l-1$ except for $t = 0, 1, 2l-2,$ and $2l-1$. In particular, there is no element of degree 3, so d_1 vanishes on $E_1^{0,*}$. β is known to be an infinite cycle by Suslin's theorem, and since E_1 is generated by $E_1^{0,*}$ and β, it follows that d_1' vanishes identically. Proceeding inductively, suppose that d_i is trivial for all $1 \leq i \leq s$, with $s < l-2$. We prove that d_{s+1} vanishes identically. For, as in the case $s = 1$, d_{s+1} is determined by an element in $\mathcal{A}_{2(s+1)+1}$. But we have

$$2(s+1)+1 < 2(l-1)+1 = 2l-1$$

Since $2(s+1)+1$ is odd and less than $2l+1$, it is zero which shows that d_{s+1} vanishes on $E_{s+1}^{0,*}$. But by the inductive hypothesis, we find that E_{s+1}^{**} is multiplicatively generated by $E_{s+1}^{0,*}$ and β, which gives the vanishing of d_{s+1}. □

We are now able to analyze the homology spectral sequence completely. Recall that for odd primes l,

$$H_{2l-1}(\mathbb{H}, \mathbb{F}_l) \cong \mathbb{F}_l \tau_1 \oplus \mathbb{F}_l \xi_1 \tau_0$$

Lemma 5.4.2.2 *In the case of an odd prime, d_{l-1} is given by $d_{l-1}(\xi_j) = 0$ for all j, $d_{l-1}(\tau_j) = 0$ for all j except 1, $d_{l-1}(\tau_1) = \beta$, and $d_{l-1}(\beta) = 0$. Consequently,*

$$E_l^{**} \cong S_{**}[\beta]/(\beta^{l-1}) \otimes \mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_0, \tau_2, \tau_3, \ldots]$$

In the case $l = 2$, $d_1(\xi_2) = \beta$, and we find that $E_2^{**} \cong \mathbb{F}_2[\xi_1, \xi_2^2, \xi_3, \xi_4, \ldots]$.

Proof It is known from calculations of Stong and Kane ([26],[56]) that the image

$$H_*(ku \wedge M_l, \mathbb{F}_l) \longrightarrow H_*(\mathbb{H}, \mathbb{F}_l)$$

is precisely the algebra $\mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_0, \tau_2, \tau_3, \ldots]$ in the case of odd primes and $\mathbb{F}_2[\xi_1, \xi_2^2, \xi_3, \xi_4, \ldots]$ in the case $l = 2$. Consequently, that subalgebra in the spectral sequence must consist of infinite cycles, so d_{l-1} vanishes on that subalgebra. Thus it suffices to describe d_{l-1} on the element τ_1 in the case of odd primes and on ξ_2 for $l = 2$. If the differential vanished, then we see that all of $H_*(\mathbb{H}, \mathbb{F}_l)$ would survive to E_∞, which would contradict the results of Stong and Kane. Consequently the differential is non-zero, so its value is a non-zero multiple of β. After performing a basis change on β, we obtain the desired result. □

Proposition 5.4.2.3 *The* **hBLM** *spectral sequence for an algebraically closed field collapses at E_l, i.e. $E_l = E_\infty$*

Proof The computations of Stong and Kane show that the algebras $\mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_0, \tau_2, \tau_3, \ldots]$ in the case of odd primes and $\mathbb{F}_2[\xi_1, \xi_2^2, \xi_3, \xi_4, \ldots]$ in the case $l = 2$ consist of infinite cycles, and we have that β is an infinite cycle. This shows that E_r is multiplicatively generated by infinite cycles for $r \geq l$, which is the result. □

5.4.3 The hBLM spectral sequence for general fields

We now consider a general field F of characteristic $p \neq l$. The analysis of the spectral sequence through E_l follows the same lines as the algebraically closed case.

Proposition 5.4.3.1 *Suppose the Bloch-Kato conjecture holds. In the mod-l* **hBLM** *spectral sequence for $H_*(KF \wedge M_l, \mathbb{F}_l)$, d_i vanishes identically for $i < l - 1$. d_{l-1} is given by $d_{l-1}(\tau_1) = \beta$ for odd primes and $d_1(\xi_2) = \beta$ for $l = 2$. (Recall that in 5.4.1.1 we described the E_1-term of the spectral sequence as $\mathcal{K}_{**} \otimes S_{**}[\beta] \otimes \mathcal{H}_{**}$, where $\mathcal{K}_{**}$ was Milnor K-theory and $\mathcal{H}_{**}$ was the homology of the Eilenberg-MacLane spectrum.)*

Proof The case $i = 1$ is identical to the algebraically closed case, and we proceed by induction on i. Suppose $s < l-2$, and we know that $d_j = 0$ for $0 < j \leq s$. We study the differential

$$d_{s+1} : E_{s+1}^{0,*} \longrightarrow E_{s+1}^{-(s+1),*-1}$$

We consider $\mathbb{F}_l$-duals, and identify each of these terms as free modules over $\mathcal{A}_*$. We find

$$(E_{s+1}^{-(s+1),*-1})^* \cong \bigoplus_{\{a,b|2a+b=s+1\}} (K_a^M(F)/lK_a^M(F))^* \otimes \mathcal{A}_*[2b+a+1]$$

Consequently, the possible homomorphisms are given by selecting for each (a,b) such that $2a+b = s+1$ an element in $\mathcal{A}_{2b+a+1}$. Note first that since $s < l-2$, we have that $2a+b < l-1$, so $4a+2b < 2l-2$. It follows that $2b+a+1 < 2l-2$ for all a and b satisfying these conditions, and thus that the differential vanishes. For the d_{l-1}-differential, we have the same parametrization of the possibilities, now with $2a+b = l-1$. In this case, $2b+a+1 < 2l-2$ unless $a = 0$ and $b = l-1$. This means that the only possiblity for a differential is the one in the statement of the proposition, and that the differential is as stated follows by comparison with the spectral sequence for $\overline{F}$. □

5.4.4 Collapse of the hBLM spectral sequence for fields containing the l-th roots of unity

Let F denote either a finite field of characteristic $p \neq l$ containing a primitive l-th root of unity, $\mathbb{Q}(\zeta_l)$ for l an odd prime, or $\mathbb{Q}(i)$. We will prove that the mod-l homology Bloch-Lichtenbaum spectral sequence collapses for these fields. It will then be an easy step to draw the same conclusion for any fields containing the l-th roots of unity if l is odd, or containing $\sqrt{-1}$ if $l = 2$.

Proposition 5.4.4.1 *For any F containing the l-th roots of unity, where l is an odd prime, the element $\xi_1 \in E_l^{0,2(l-1)}$ is an infinite cycle in the mod-l* **hBLM** *spectral sequence. If $\sqrt{-1} \in F$, then the element $\xi_1^2 \in E_2^{0,2}$ is an infinite cycle in the mod-2 spectral sequence.*

Proof For any field, one can consider the fiber Φ given by

$$\Phi(F) \longrightarrow KF \wedge M_l \longrightarrow \mathbb{H}$$

where are usual $\mathbb{H}$ denotes the mod-l Eilenberg-MacLane spectrum. The filtration in the Bloch-Lichtenbaum spectral sequence restricts to a filtration on

$\Phi(F)$, and we obtain a spectral sequence $\{\mathcal{E}(F)_r^{**}\}$ converging to the mod-l homology of Φ. The E_1-term $\mathcal{E}(F)_2^{**}$ is obtained by removing the vertical line $E_1^{0,*}$ from the original homology spectral sequence.

For any field containing the l-th roots of unity, we have a map

$$\Sigma^\infty B\mathbb{Z}/l\mathbb{Z}_+ \longrightarrow K(F) \wedge M_l$$

induced by the inclusion of the l-roots into the units and then further into K-theory. By subtracting off the similar map induced by the trivial homomorphism from $\mathbb{Z}/l\mathbb{Z}$ to the units, we obtain a map which induces the zero map on π_0 and hence a map into $\Phi(F)$. We can then project to the first subquotient of the filtration on $\Phi(F)$, and obtain a homomorphism $H_*(B\mathbb{Z}/l\mathbb{Z}, \mathbb{F}_l) \rightarrow \mathcal{E}(F)_l^{1,*}$. This graded group is precisely $\beta \otimes \mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_0, \tau_2, \tau_3, \ldots]$ when l is odd, and $\beta \otimes \mathbb{F}_2[\xi_1, \xi_2^2, \xi_3, \xi_4, \ldots]$ when $l = 2$. When the field is algebraically closed, we may analyze this map in terms of complex conective theory. In this case, it is clear that the image of the generator in $H_2(B\mathbb{Z}/l\mathbb{Z}, \mathbb{F}_l)$ under this map is β. Since the Steenrod operation P^1 (Sq^2 if $l = 2$) is non-trivial from $H_{2l}(B\mathbb{Z}/l\mathbb{Z}, \mathbb{F}_l)$ to $H_2(B\mathbb{Z}/l\mathbb{Z}, \mathbb{F}_l)$, it now readily follows that the generator in $H_{2l}(B\mathbb{Z}/l\mathbb{Z}, \mathbb{F}_l)$ maps to $\beta \otimes \xi_1$ ($\beta \otimes \xi_1^2$ if $l = 2$). It now follows that the elements $\beta \otimes \xi_1$ and $\beta \otimes \xi_1^2$ are infinite cycles in the homology Bloch-Lichtenbaum spectral sequence for F. But now since β is an infinite cycle, and since $d_r(\beta \otimes \xi_1) = \pm\beta d_r(\xi_1)$, the fact that multiplication by β is injective as a map from $E_r^{0,2(l-1)}$ to $E_r^{-1,2l+1}$ implies that $d_r(\xi_1) = 0$. □

Proposition 5.4.4.2 *Suppose the Bloch-Kato conjecture holds. The mod-l* **hBLM** *spectral sequence collapses at E_l for F as above.*

Proof Let $\mathcal{S}_*$ denote algebra $E_l^{0,*}$, i.e. $\mathcal{S}_* \cong \mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_0, \tau_2, \tau_3 \ldots]$ if l is odd and $\mathcal{S}_* \cong \mathbb{F}_2[\xi_1, \xi_2^2, \ldots]$ when $l = 2$. Let M_* denote a free graded $\mathcal{S}_*$-module which is equipped with an $\mathcal{A}_*$-module action which is compatible with the action of $\mathcal{A}_*$ on $\mathcal{S}_*$. Then we observe that Steenrod algebra module homomorphisms from $\mathcal{S}_*$ to M_* are in bijective correspondence with homomorphisms of graded $\mathbb{F}_l$-vector spaces from $\mathcal{S}_*$ to $\mathbb{F}_l \underset{\mathcal{S}_*}{\otimes} M_*$. This statement is dual to the fact that a homomorphism of $\mathcal{A}_*$-modules is determined by its values on a set of generators.

We must analyze the differentials d_r for $r \geq l$. We begin with d_l. We first note that

$$\mathbb{F}_l \underset{\mathcal{S}_*}{\otimes} E_r \cong \mathcal{K}_{**} \otimes S_{**}[\beta]$$

so that the d_l-differential is completely determined by a map of graded vector spaces of degree -1 from $\mathcal{S}_*$ to V_*, where V_* denotes that part of $\mathcal{K}_{**} \otimes S_{**}[\beta]$

whose s-bigrading is $-l$. We note that for the fields in question, Galois cohomology is non-vanishing only in degrees 1 and possibly 2. This means that the non-zero groups occur only for $0 \le k \le 2l$. On the other hand, we have that in the range $0 \le l \le 2l+1$, $\mathcal{S}_l$ is non-zero only in dimensions $l = 0, 1, 2l-2,$ and $2l-1$, where the basis element are $1, \tau_0, \xi_1,$ and $\xi_1\tau_0$. τ_0 and ξ_1 are now known to be infinite cycles, so multiplicativity shows that d_l vanishes. This analysis works equally well for any $r \ge l$, so the spectral sequence collapses at E_l. □

Corollary 5.4.4.3 *Suppose the Bloch-Kato conjecture holds at l. Suppose also that E is any field containing the l-th roots of unity if l is odd, or containing $\sqrt{-1}$ if $l = 2$. Then the mod-l* **hBLM** *spectral sequence for E collapses at E_l.*

Proof The hypothesis guarantees that E contains one of the model fields F above as a subfield. There is a map of spectral sequences from the spectral sequence for F to that for E. This map of spectral sequences may be extended to maps

$$\mathcal{K}_{**}(E) \underset{\mathcal{K}_{**}(F)}{\otimes} E_r^{**}(F) \longrightarrow E_r^{**}(E)$$

which commute with all differentials. From the above discusson, this map is an isomorphism, and it follows immediately that the spectral sequence for E collapses as well. □

5.4.5 The hBL spectral sequence

The **HBLM** spectral sequence was obtained by smashing the spectrum $KF \wedge M_l$ with the mod-l Eilenberg-MacLane spectrum, and converges to $H_*(KF \wedge M_l, \mathbb{F}_l)$. As we observed in 5.4.1, we may instead smash with the integral Eilenberg-MacLane spectrum and obtain a spectral sequence which converges to $H_*(KF, \mathbb{F}_l)$. We will describe the E_1-term of this spectral sequence, and prove that it also collapses at E_l. We first recall

Proposition 5.4.5.1 *We have isomorphisms of $\mathcal{A}_*$-modules*

$$H_*(\mathbb{L}, \mathbb{F}_l) \cong \mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_1, \tau_2, \ldots]$$

when l is odd and

$$H_*(\mathbb{L}, \mathbb{F}_2) \cong \mathbb{F}_2[\xi_1^2, \xi_2, \xi_3, \ldots]$$

when $l = 2$. We'll write $L_*(l)$ for $H_*(\mathbb{L}, \mathbb{F}_l)$. The map of spectra $\mathbb{L} \to \mathbb{H}$ induces the evident inclusions of algebras $L_*(l) \to \mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_0, \tau_1, \ldots]$ and $L_*(2) \to \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$

Proposition 5.4.5.2 *Suppose that the Bloch-Kato conjecture holds at l. Suppose the field F contains the l-th roots of unity, and $\sqrt{-1}$ if $l = 2$. Then the E_1-term of the mod p* **hBL** *spectral sequence is of the form*

$$\mathcal{K}_{**} \otimes S_{**}[\beta] \otimes \mathcal{L}_{**}$$

where $\mathcal{L}_{0*} \cong L_*(l)$ and $\mathcal{L}_{s*} \cong 0$. Moreover, the map of spectral sequences from the mod-l **hBL** *spectral sequence to the mod-l* **hBLM** *spectral sequence is induced by the evident inclusion $\mathcal{L}_{**} \to \mathcal{H}_{**}$.*

Proof Easily checked. □

Corollary 5.4.5.3 *For F as in 5.4.5.2, and assuming that the Bloch-Kato conjecure holds at l, the mod-l* **hBL** *spectral sequence collapses at E_l. The E_∞ term is of the form*

$$\mathcal{K}_{**} \otimes S_{**}[\beta] \otimes \mathcal{L}_{**}$$

Corollary 5.4.5.4 *Assume the Bloch-Kato conjecture holds at l. For F as in 5.4.5.2, the algebra $H_*(KF, \mathbb{F}_l)$ is a free graded module over $K_*^M(F)/lK_*^M(F)$.*

5.4.6 Multiplicative structure

The previous analysis easily permits the description of the graded ring $H_* = H_*(KF, \mathbb{F}_l)$ when F is a field of characteristic $\neq l$ and containing the l-th roots of unity, and $\sqrt{-1}$ when $l = 2$. From the previous section, we find that in the case of one of the model fields $\mathbb{F}_p(\zeta_l)$, $\mathbb{Q}(\zeta_l)$ for l odd, or $\mathbb{Q}(i)$, there is a family of ideals in H_* whose associated graded algebra is $K_*^M(F) \otimes H_*(ku, \mathbb{F}_l)$. Recall that $H_*(ku, \mathbb{F}_l) \cong \mathbb{F}_l[\beta]/(\beta^{l-1}) \otimes \mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_2, \tau_3, \ldots]$ when l is odd and $\cong \mathbb{F}_2[\xi_1^2, \xi_2^2, \xi_3, \ldots]$ when $l = 2$.

Proposition 5.4.6.1 *Assume the Bloch-Kato conjecture holds at l. For the model fields F, we have*

$$H_*(KF, \mathbb{F}_l) \cong K_*^M(F)/lK_*^M(F) \otimes H_*(ku, \mathbb{F}_l)$$

as algebras.

Proof When $l = 2$, $H_*(ku, \mathbb{F}_2)$ is a free commutative algebra. Since $H_*(KF, \mathbb{F}_2)$ is a commutative algebra, there is a section of the map $H_*(KF, \mathbb{F}_2) \to H_*(ku, \mathbb{F}_2)$. The result follows easily. For l odd, the subalgebra of $H_*(ku, \mathbb{F}_l)$ generated by the ξ_i's and τ_i's is a free graded commutative algebra. Again, since $H_*(KF, \mathbb{F}_l)$ is a graded commutative algebra, there is a homomorphism $\mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_2, \tau_3, \ldots]$ so that the composite

$$\mathbb{F}_l[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_l}[\tau_2, \tau_3, \ldots] \longrightarrow H_*(KF, \mathbb{F}_l) \longrightarrow H_*(ku, \mathbb{F}_l)$$

is the inclusion. It remains to extend this map to β in a multiplicative way, i.e. to see that there is an element in $\hat{\beta} \in H_2(KF, \mathbb{F}_l)$ whose image in $H_2(ku, \mathbb{F}_l)$ is non-zero and so that $\hat{\beta}^{l-1} = 0$. To see that this is possible, select any choice β^* projecting to β in $H_2(ku, \mathbb{F}_l)$. $\hat{\beta}^{l-1}$ is an element in $H_{2(l-1)}(KF, \mathbb{F}_l)$. From dimensional considerations, since the $K_*^M(F)/lK_*^M(F) = 0$ for $* > 2$, it is easy to see that the element is either zero or equal to a nonzero element x of the form $\kappa\beta^{l-2}$. Note that $\kappa^2 \in K_2^M(F)/lK_2^M(F) = 0$, so $x^2 = 0$. Now set $\hat{\beta} = \beta^* - \frac{1}{l-1}x$. $\hat{\beta}$ is now clearly the required element. □

Corollary 5.4.6.2 *Assume the Bloch-Kato conjecture holds at l. For any field E of characteristic $\neq l$ and containing the l-th roots of unity and $\sqrt{-1}$ if $l = 2$, we have an isomorphism of algebras*

$$H_*(KE, \mathbb{F}_l) \cong K_*^M(E)/lK_*^M(E) \otimes H_*(ku, \mathbb{F}_l)$$

Proof The result follows from the isomorphism

$$H_*(KE, \mathbb{F}_l) \cong K_*^M(E, \mathbb{F}_l) \underset{K_*^M(F,\mathbb{F}_l)}{\otimes} H_*(KF, \mathbb{F}_l)$$

□

5.4.7 Spectrum homology of KF and the endomorphism algebra $End^{\wedge}_{KF}(K\overline{F})$

The results of the preceding paragraphs will now permit us to describe the E_2-term of the Adams-Hom spectral sequence 3.10 converging to the graded algebra. $\pi_* End^{\wedge}_{KF}(K\overline{F})$. Throughout, let E denote the algebraic closure of F, and let G denote the absolute Galois group of F. We recall from 3.10 that the key input to this spectral sequence is the $\mathcal{A}$-module

$$M_*(F) = \pi_* Hom_{\mathbb{H}\wedge KF}(\mathbb{H} \wedge KE, \mathbb{H} \wedge KE) \cong \pi_* Hom_{KF}(KE, \mathbb{H} \wedge KE)$$

In order to analyze $M_*(F)$, note first that since $\pi_*\mathbb{H} \wedge KF \cong H_*KF$, we have a spectral sequence with E_2-term

$$Ext_{H_*KF}(H_*KE, H_*KE)$$

In 5.4.6.2, we showed that as an algebra, H_*KF is isomorphic to the graded algebra $K^M_*(F, \mathbb{F}_l) \otimes H_*KE$, and where the inclusion of $F \hookrightarrow E$ induces the evident augmentation $K^M_*(F, \mathbb{F}_l) \otimes H_*KE \to H_*KE$. Consequently, using a standard change of rings result in homological algebra [12], we find that

$$Ext_{H_*KF}(H_*KE, H_*KE) \cong Ext_{K^M_*(F,\mathbb{F}_l)}(\mathbb{F}_l, H_*KE)$$

Let $gr^s\mathbb{F}_l[G]$ denote the subquotient I^s/I^{s+1}, where I denotes the augmentation algebra in $\mathbb{F}_l[G]$.

Proposition 5.4.7.1 *Suppose that G is a Koszul duality group. In particular, the Bloch-Kato conjecture holds at l. There is a spectral sequence converging to the algebra $M_*(F)$ whose E_2^{st}-term is of the form*

$$\Lambda^F_{**} \hat{\otimes} H_*KE$$

where $\Lambda^F_{ss} \cong Ext^{-s,-s}_{K^M_*(F,\mathbb{F}_l)}(\mathbb{F}_l, \mathbb{F}_l) \cong gr^{-s}\mathbb{F}_l[G]$ and $\Lambda^F_{st} = 0$ for $s \neq t$, where H_*KE denotes H_*KE regarded as a bigraded algebra concentrated on the non-negative y-axis, and where $\hat{\otimes}$ denotes completed tensor product.

Proof The existence of the spectral sequence and the form of the E_2-term follows directly from the above analysis, together with the 5.2.4. □

We will refer to this spectral sequence as the M_* *spectral sequence.*

5.4.8 Collapse of the M_* spectral sequence

The next step will be to show that the M_* spectral sequence collapses. Throughout this section, we assume that the Positselskii-Vishik conjecture holds, i.e. that the absolute Galois group G of F is a Koszul duality group. We must first interpret the spectral sequence for low homological degree, i.e. $s = 0, 1$. The $s = 0$ line of the spectral sequence is given by K

$$Ext^{0*}_{H_*KF}(H_*E, H_*KE) \cong Hom^*_{H_*KF}(H_*KE, H_*KE)$$

We also have a homotopy M_* spectral sequence with E_2-term given by

$$Ext_{\pi_*KF}(\pi_*KE, \pi_*KE)$$

Moreover, the natural change of scalars map

$$id_{\mathbb{H}} \wedge - : Hom_{KF}(KE, KE) \longrightarrow Hom_{\mathbb{H}\wedge KF}(\mathbb{H} \wedge KE, \mathbb{H} \wedge KE)$$

induces a map of spectral sequences.

Lemma 5.4.8.1 *Given an element $\varphi \in Hom^t_{H_*KF}(H_*KE, H_*KE)$, suppose that there is a map of KF-modules $f : \Sigma^t KE \to KE$ which induces the homomorphism φ. Then φ is an infinite cycle in the M_* spectral sequence.*

Proof Clear from the map of spectral sequences, since $\pi_*(f)$ is an infinite cycle in the homotopy M_* spectral sequence. □

We now consider KF-linear maps which induce the zero map on mod-p homology. Such maps will map to filtration one in the M_* spectral sequence, and we wish to describe the image of such a map in $Ext^{1*}(H_*KE, H_*KE)$. Given a map $f : \Sigma^t KE \to KE$ of KF-modules, which induces the trivial map on homology, we obtain a cofibration sequence

$$\Sigma^t KE \xrightarrow{f} KE \longrightarrow C(f)$$

where $C(f)$ denotes the mapping cone of f. Since f induces the zero map on homology, we have a short exact sequence

$$0 \to H_*KE \longrightarrow H_*C(f) \longrightarrow H_{*-(t+1)}KE \to 0$$

of graded H_*KF-modules. Such an exact sequence represents an element $\chi(f)$ in $Ext^{1,t+1}_{H_*KF}(H_*KE, H_*KE)$ under the long exact sequence interpretation of the functor Ext.

Lemma 5.4.8.2 *Given an element $\xi \in Ext^{1,t+1}_{H_*KF}(H_*KE, H_*KE)$, if there is a map $f : \Sigma^t KE \to KE$ of KF-modules so that $\chi(f) = \xi$, then ξ is an infinite cycle in the M_* spectral sequence.*

Proof Clear. □

Since th M_* spectral sequence is generated multiplicatively by H_*KE and elements in $Ext^{1,-1}$, we will only need to show that all elements in $Ext^{1,-1}$ are infinite cycles. We first evaluate the group $Ext^{1,-1}_{H_*KF}(H_*KE, H_*KE)$.

Proposition 5.4.8.3 *There is an isomorphism*

$$Ext^{1,-1}_{H_*KF}(H_*KE, H_*KE) \cong Hom(K^M_1(F,\mathbb{F}_l),\mathbb{F}_l)$$

The isomorphism is natural with respect to inclusions of fields.

Proof First we have the change of rings isomorphism

$$Ext^{1,-1}_{H_*KF}(H_*KE, H_*KE) \cong Ext^{1,-1}_{K^M_*(F,\mathbb{F}_l)}(\mathbb{F}_l, H_*KE)$$

Elements in this *Ext*-group correspond to short exact sequences of graded $K^M_*(F,\mathbb{F}_l)$-modules

$$H_*KE \longrightarrow X_* \longrightarrow \mathbb{F}_l[-1]$$

Let α denote the generator of $\mathbb{F}_l[-1]$ (so α lies in dimension -1), and let $\beta \in H_0KE$ also be a generator. Attached to such an exact sequence ξ, we have a homomorphism $\theta(\xi) : K^M_1(F,\mathbb{F}_l) \rightarrow \mathbb{F}_l$ defined as follows. Lift α to an element $\hat{\alpha} \in X_{-1}$. For any element $u \in K^M_1(F,\mathbb{F}_l)$, we see that $u\hat{\alpha} \in H_0KE$, and therefore can be written uniquely as $x_u\beta$, where $x_u \in \mathbb{F}_l$. We define $\theta(\xi)(u) = x_u$, and this gives the required homomorphism $\theta : Ext^{1,-1}_{K^M_*(F,\mathbb{F}_l)}(\mathbb{F}_l, H_*KE) \rightarrow Hom(K^M_1(F,\mathbb{F}_l),\mathbb{F}_l)$. It is easy to check that it is an isomorphism. The naturality statement is also readily checked. □

Corollary 5.4.8.4 *Suppose l is odd, or if $l = 2$ let F be a field whose absolute Galois group is torsion free. For instance, this holds if F contains $\sqrt{-1}$. Then $Ext^{1,-1}_{H_*KF}(H_*KE, H_*KE)$ consists only of infinite cycles if and only if the same is true for all extensions of F whose absolute Galois groups are topologically cyclic.*

Proof This is an immediate consequence of the naturality statement in 5.4.8.3. □

We will now focus our attention on the case of fields F whose absolute Galois group is $\mathbb{Z}_l$. In this case, $K^M_1(F,\mathbb{F}_l) \cong \mathbb{F}_l$. Our next step will be to construct maps of KF-modules $KE \rightarrow \Sigma^2 KE$ whose corresponding *Ext*-classes generate $Ext^{1,-1}$. Let $\mathbb{L}$ denote the Eilenberg-MacLane spectrum for the group $\mathbb{Z}_l$. The "component map" gives a natural homomorphism $KF \rightarrow \mathbb{L}$, and these maps are compatible with maps induced by inclusions of fields. As before, let E denote the algebraic closure of F, and let G_F denote the absolute Galois group of F. Let $\gamma \in G_F$. We may now consider the composite

$$p_\gamma : KE \xrightarrow{\gamma - 1} KE \longrightarrow \mathbb{L}$$

p_γ is a map of KF-module spectra, and on π_0 it induces the zero map. We make the following stronger statement.

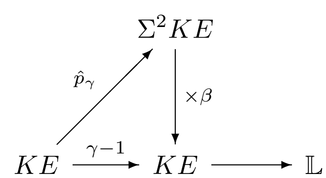
Proposition 5.4.8.5 *The map p_γ is null homotopic as a map of KF-module spectra. Moreover, there is a unique choice of null homotopy (up to homotopies of homotopies) as maps of KF-module spectra.*

Proof We consider the homotopy spectral sequence for computing the homotopy classes of maps of KF-module spectra from KE to $\mathbb{L}$. Its E_2-term is of the form

$$E_2^{s,t} \cong Ext_{\pi_* KF}^{-s,t}(\pi_* KE, \mathbb{Z}_l)$$

Since $\pi_* KF$ and $\pi_* KE$ are concentrated in non-negative degrees, and since $\pi_0 KF \to \pi_0 KE$ is an isomorphism, this spectral sequence is a third quadrant spectral sequence. This means that $E_2^{s,t} = 0$ for $s + t > 0$, and that the only non-vanishing group for which $s + t = 0$ is $E_2^{0,0} \cong \mathbb{Z}_l$. The conclusion from this is that the induced map on π_0 is the only invariant of KF-module maps from KE to $\mathbb{L}$, and we conclude that the map p_γ is null homotopic as a map of KF-module spectra. The spectral sequence for computing null homotopies is just the spectral sequence for maps of KF-module spectra from ΣKE to $\mathbb{L}$. In this spectral sequence, $E_2^{s,t} = 0$ for $s + t \geq 0$, and we conclude that there is an essentially unique nullhomotopy. □

Corollary 5.4.8.6 *There is a unique (up to homotopy) map of KF-module spectra $\hat{p} : KE \to \Sigma^2 KE$ making the following diagram commute.*



where $\times\beta$ denotes multiplication by the Bott element.

Proof Let $KE[2, 3, \ldots)$ denote the homotopy fiber of the map $KE \to \mathbb{L}$. It is the 1-connected Postnikov cover of KE. 5.4.8.5 shows that there is up to homotopy a unique map $\overline{p}_\gamma : KE \to KE[2, 3, \ldots)$ making the diagram

$$
\begin{array}{ccccc}
 & & KE[2,3,\ldots) & & \\
 & \overset{\overline{p}_\gamma}{\nearrow} & \downarrow & & \\
KE & \xrightarrow{\gamma-1} & KE & \longrightarrow & \mathbb{L}
\end{array}
$$

commute. But multiplication by the Bott element can be identified with the inclusion $KE[2,3,\ldots) \to KE$, which gives the result. □

Corollary 5.4.8.7 *There is a unique homomorphism η from I, the augmentation ideal in $\mathbb{Z}_l^{Iw}[G]$, into $\pi_{-2}End^{\wedge}_{KF}(KE)$, so that $\beta\eta(i) = i$*

Proof Direct consequence of 4.1.7 and 5.4.8. □

Lemma 5.4.8.8 *The map $\hat{p}_\gamma$ induces the zero map on homology.*

Proof The map $H_*(\hat{p}_\gamma) : H_*KE \to H_*\Sigma^2 KE$ is a map of H_*KF-modules. But H_*KE is a cyclic H_*KF-module, so it suffices to show that $H_0(\hat{p}_\gamma) = 0$. But this is clear since $H_0\Sigma^2 KE = 0$. □

It follows from this Lemma that we can attach the invariant

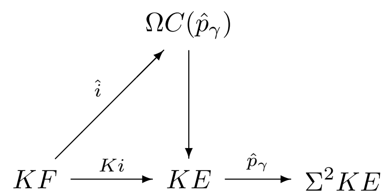
$$\chi(\hat{p}_\gamma) \in Ext^{1,-1}_{H_*KF}(H_*KE, H_*KE)$$

to $\hat{p}_\gamma$. Our goal is to show that this invariant is non-trivial when γ is chosen to be a topological generator of G_F.

The element $\chi(\hat{p}_\gamma)$ is represented by the mapping cone of $\hat{p}_\gamma$. This mapping cone is the suspension of the homotopy fiber of $\hat{p}_\gamma$. Now consider the diagram

$$
\begin{array}{ccccc}
 & & \Omega C(\hat{p}_\gamma) & & \\
 & & \downarrow & & \\
KF & \xrightarrow{Ki} & KE & \xrightarrow{\hat{p}_\gamma} & \Sigma^2 KE
\end{array}
$$

We claim that the composite $\hat{p}_\gamma \circ Ki$ is null homotopic as a map of KF-module spectra. For, KF is a free KF-module on the sphere spectrum S^0, and hence the homotopy classes of KF-module maps from KF to $\Sigma^2 KE$ are in one to one correspondence with the homotopy classes of maps of S^0 to $\Sigma^2 KE$, which are all trivial since $\Sigma^2 KE$ is 1-connected. This argument also shows that there is a unique (up to homotopy of homotopies) null homotopy, so there is a unique choice of map $\hat{i} \to \Omega(\hat{p}_\gamma)$ making the diagram



commute.

Proposition 5.4.8.9 *The element $\chi(\hat{p}_\gamma)$ is non-trivial if and only if the map $\hat{i}$ is injective on $H_1(-, \mathbb{F}_l)$.*

Proof If $\chi(\hat{p}_\gamma) = 0$, then $\mathbb{H} \wedge C(\hat{p}_\gamma) \simeq (\mathbb{H} \wedge \Sigma KE) \vee (\mathbb{H} \wedge \Sigma^2 KE)$ as $\mathbb{H} \wedge KF$-module spectra. Let $\alpha \in K_1^M(F, \mathbb{F}_l)$. Then α acts trivially on both factors in the wedge decomposition, and so α, regarded also as an element in $H_1 KF$, must map trivially into $H_1 \Omega C(\hat{p}_\gamma)$. This shows that if $\chi(\hat{p}_\gamma) = 0$, then the induced map on H_1 is not injective. On the other hand, if $\chi(\hat{p}_\gamma) \neq 0$, then multiplication by α from $H_0(\Omega C\hat{p}_\gamma)$ to $H_1(\Omega C\hat{p}_\gamma)$ is non-trivial, from 5.4.8.3. Since the map $\hat{i}$ clearly induces an isomorphism on H_0, the result follows. $\square$

We will now verify injectivity of the induced map on H_1. In order to do this, we will study the behavior of a the associated construction on units, rather than K-theory. The classifying space on units will map into the K-theory, and will control the behavios of the map. Recall that we are considering the case where our field F has a topologically cyclic absolute Galois group. This means (by Kummer theory) that $K_1^M(F, \mathbb{F}_l) \simeq \mathbb{F}_l$. Select any unit $u \in F^*$ which generates $K_1^M(F, \mathbb{F}_l)$. Then the extension E/F can be obtained by adjoining the l-th power roots of u. Let U denote the subgroup of E^* generated by the l-th power roots of u and the l-th power roots of 1. It is easy to see that this is a subgroup of E^* closed under the action of G_F. Now select any topological generator t for G_F. t specifies an automorphism of U, and thus gives an action by the group $\mathbb{Z}$ on U. One sees that there are three cases describing this group as an abelian group with action by t. To describe these cases, we let C_{l^∞} denote the group $\bigcup_n \mathbb{Z}/l^n\mathbb{Z}$. $Aut(C_{l^\infty})$ is in this case isomorphic to the product $\mathbb{Z}/(l-1)\mathbb{Z} \times \mathbb{Z}_l$. The factor $\mathbb{Z}_l$ is characterized as the subgroup which acts by the identity on the subgroup $\mathbb{Z}/l\mathbb{Z} \subseteq C_{l^\infty}$.

1. u is itself a l-th power root of unity. In this case $U \simeq C_{l^\infty}$, and the action of t is given by the choice of an element in $\mathbb{Z}_l \subseteq \mathbb{Z}/(l-1)\mathbb{Z} \times \mathbb{Z}_l \cong Aut(C_{l^\infty})$.

$F^* \cap U$ is in this case the finite cyclic subgroup of U fixed by t.

2. F contains all the l-th power roots of unity. In this case $U \cong C_{l^\infty} \times \mathbb{Z}[\frac{1}{l}]$, and the action of t is given by

$$t(z, w) = (z + \overline{w}, w)$$

where $w \to \overline{w}$ is the projection map $\mathbb{Z}[\frac{1}{l}] \to C_{l^\infty}$. $F^* \cap U$ is in this case $C_{l^\infty} \times \mathbb{Z} \subseteq C_{l^\infty} \times \mathbb{Z}[\frac{1}{l}]$.

3. In the general case, we have an action both on the l-th power roots of unity and on a separate unit. U is again isomorphic to $C_{l^\infty} \times \mathbb{Z}[\frac{1}{l}]$. In this case, the action of t is again specified by a choice of element $\phi \in \mathbb{Z}_l \subseteq \mathbb{Z}/(l-1)\mathbb{Z} \times \mathbb{Z}_l \cong Aut(C_{l^\infty})$. The action is specified by

$$t(z, w) = (\phi(z) + \overline{w}, w)$$

$F^* \cap U$ fits in a short exact sequence

$$\{e\} \longrightarrow (C_{p^\infty})^\phi \longrightarrow F^* \cap U \longrightarrow \mathbb{Z}[\frac{1}{l}] \longrightarrow \{e\}$$

In each case, we consider the classifying space BU. It is equipped with a $\mathbb{Z}$ action via Bt, which we also denote by t. We consider the spectra $\mathbb{H} \wedge BU$ and $\mathbb{H} \wedge BU_+$. Note that $\mathbb{H} \wedge BU$ can we viewed as the fiber of the map $\mathbb{H} \wedge BU_+ \to Be_+ \simeq S^0$. We have the self-map $t-1$ of $\mathbb{H} \wedge BU_+$, which lifts canonically to a map $\overline{t-1} : \mathbb{H} \wedge BU_+ \to \mathbb{H} \wedge BU$ since the composite $Bu_+ \to BU_+ \to S^0$ is canonically null homotopic. Let Φ denote the homotopy fiber of the map $\overline{t-1}$. There is an obvious map $j : \mathbb{H} \wedge B(F^* \cap U)_+ \to \Phi$, since $F^* \cap U$ is fixed by t.

Proposition 5.4.8.10 *The map j induces an injection on π_1.*

Proof We first evaluate $\pi_*\Phi$ for the relevant values of $*$. Note that $\pi_i \mathbb{H} \wedge BU_+ \simeq H_i BU_+$. We therefore have a long exact sequence

$$\cdots \to H_{i+1}BU_+ \stackrel{H_{i+1}(t-1)}{\longrightarrow} H_{i+1}BU \to \pi_i\Phi \to H_iBU_+ \to \cdots$$

We claim that in all cases above, $H_1BU_+ \cong 0$ and $H_2B_+ \cong H_2BU \cong \mathbb{F}_l$. In the first case, U is the group C_{l^∞}, for which it is well known that the inclusion $C_{l^\infty} \subseteq S^1$ induces an isomorphism on mod-l homology which gives the result. In Cases 2 and 3, U is a product of C_{l^∞} with the group $\mathbb{Z}[\frac{1}{l}]$, which is uniquely l-divisible, and hence has trivial homology, from which the result follows by the Künneth formula. We also observe that t acts trivially on H_2 in all cases. It follows that in all cases, $H_1\Phi \simeq \mathbb{F}_l$. Next, we observe that $H_1B(F^* \cap U)_+ \simeq \mathbb{F}_l$

in all cases as well. In Case 1, $F^* \cap U \cong C_{l^n}$ for some n, and $H_1 C_{l^n} \simeq \mathbb{F}_l$. In Case 2, we have $H_1 C_{l^\infty} = 0$ and $H_1 \mathbb{Z} \simeq \mathbb{F}_l$, so $H_1 B(F^* \cap U)_+ \simeq \mathbb{F}_l$. In Case 3, the exact sequence

$$\{e\} \longrightarrow (C_{l^\infty})^\phi \longrightarrow F^* \cap U \longrightarrow \mathbb{Z}[\frac{1}{l}] \longrightarrow \{e\}$$

shows that $(C_{l^\infty})^\phi \to F^* \cap U$ induces an isomorphism on homology, since $\mathbb{Z}[\frac{1}{l}]$ is uniquely l-divisible. Since $(C_{l^\infty})^\phi$ is a finite cyclic l-group, we again have that $H_1 B(F^* \cap U)_+ \simeq \mathbb{F}_l$. It remains to show that j induces an isomorphism.

In order to do this, we consider the Eilenberg-MacLane spaces of U and $F^* \cap U$ themselves, rather than the spectra obtained by smashing them with $\mathbb{H}$. Let Ψ denote the homotopy fiber of the map $t - 1 : BU \to BU$. There is an evident map $B(F^* \cap U) \to \Psi$. In Case 1, we have an exact sequence of abelian groups

$$0 \to F^* \cap U \to U \xrightarrow{t-1} U \to 0$$

The surjectivity of the map $t - 1$ is obtained from Hilbert's Theorem 90 in an obvious way. In Case 2 and Case 3, we find that the image of $t - 1$ in U is equal to C_{l^∞}, and the quotient U/C_{l^∞} is uniquely l-divisible and hence has trivial mod-l homology. It follows in all cases that the sequence

$$B(F^* \cap U) \to BU \xrightarrow{t-1} BU \tag{5–1}$$

is a fibration after l-adic completion, and it follows easily that the map

$$B(F^* \cap U) \to \Psi \tag{5–2}$$

indices an isomorphism on homotopy after l-adic completion.

For any abelian group A, let $\mathbb{H}A$ denote the Eilenberg-MacLane spectrum attached to A. Let $\mathbb{H}_l A$ denote the delooping of the fiber of the map $\times l : \mathbb{H}A \to \mathbb{H}A$, so we have a homotopy fiber sequence

$$\mathbb{H}A \xrightarrow{\times l} \mathbb{H}A \to \mathbb{H}_l A$$

Then we have a natural map $\rho : \mathbb{H} \wedge BA_+ \to \mathbb{H}_l A$, since $\mathbb{H}_l A$ is canonically a module spectrum over $\mathbb{H}$. It follows from the Hurewicz theorem that

1. ρ induces an isomorphism on π_1.
2. When A is a l-divisible group, ρ induces an isomorphism on π_2.

ρ is natural with respect to group homomorphisms, and it follows that we may construct a map $t - 1 : \mathbb{H}_l U \to \mathbb{H}_l U$. We will denote its homotopy fiber by $\overline{\Psi}$.

We also have an evident map $\overline{j} : \mathbb{H}_l(F^* \cap U) \to \overline{\Psi}$. It follows from 5–1 and 5–2 above that the map $\overline{j}$ induces an isomorphism on π_1. From 1 and 2 above, it follows that we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{H} \wedge B(F^* \cap U) & \xrightarrow{j} & \Phi \\
\downarrow & & \downarrow \\
\mathbb{H}_l(F^* \cap U) & \xrightarrow{\overline{j}} & \overline{\Psi}
\end{array}
$$

in which both vertical arrows induce isomorphisms on π_1 and in which $\overline{j}$ induces an isomorphism on π_1. It follows that j induces an isomorphism on π_1. $\square$

We can now demonstrate the non-triviality of the invariant $\chi(\hat{p}_\gamma)$.

Proposition 5.4.8.11 *The map $\hat{i} : KF \to C(\hat{p}_\gamma)$ induces in injection on H_1. Consequently, by 5.4.8.9, $\chi(\hat{p}_\gamma) \neq 0$.*

Proof For any field, the inclusion $BF^* \to BGL(F)$ induces a map of spectra

$$\nu : \Sigma^\infty BF^*_+ \to KF$$

This map induces an isomorphism on π_1. Passing to the 0-connected covering $K(F)[1, 2, \ldots)$, we obtain a map

$$\overline{\nu} : \Sigma^\infty BF^* \longrightarrow KF[1, 2, \ldots)$$

which also induces an isomorphism on π_1 and consequently on H_1 by the Hurewicz theorem. It now follows from the constructions of $C(\hat{p}_\gamma)$ and Φ that we have a commutative diagram

$$
\begin{array}{ccccc}
\Phi & \longrightarrow & \mathbb{H} \wedge BU_+ & \xrightarrow{\overline{t-1}} & \mathbb{H} \wedge BU \\
\downarrow & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \overline{\nu}} \\
\mathbb{H} \wedge \Omega C(\hat{p}_\gamma) & \longrightarrow & \mathbb{H} \wedge KE & \xrightarrow{\hat{p}_\gamma} & \mathbb{H} \wedge \Sigma^2 KE
\end{array}
$$

From this diagram, we obtain a diagram

$$
\begin{array}{ccccccc}
H_2BU & \xrightarrow{0} & H_2BU & \xrightarrow{\partial} & \pi_1\Phi & \longrightarrow & H_1BU \cong 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow \\
H_2KE & \xrightarrow{0} & \Sigma^2 KE & \xrightarrow{\partial} & H_1\Omega C(\hat{p}_\gamma) & \longrightarrow & H_1KE \cong 0
\end{array}
$$

It follows that the connecting homomorphisms are isomorphisms, and hence that the map $\pi_1\Phi \to H_1\Omega C(\hat{p}_\gamma)$ is an isomorphism.

Now choose a unit u which generates $K_1^M(F, \mathbb{F}_l)$. As before, let U be the subgroup generated by all the l-th power roots of u and the l-th power roots of unity. We have the map $\rho\colon \Sigma^\infty B(F^* \cap U)_+ \to KF$, and its induced map

$$Id_{\mathbb{H}} \wedge \rho : \mathbb{H} \wedge B(F^* \cap U)_+ \to \mathbb{H} \wedge KF$$

The naturality of ρ gives us a commutative diagram

$$
\begin{array}{ccc}
\mathbb{H} \wedge B(F^* \cap U)_+ & \xrightarrow{j} & \Phi \\
\downarrow{\scriptstyle \rho} & & \downarrow \\
\mathbb{H} \wedge KF & \xrightarrow{id_{\mathbb{H}} \wedge i} & \mathbb{H} \wedge \Omega C(\hat{p}_\gamma)
\end{array}
$$

The corresponding diagram of fundamental groups is now

$$
\begin{array}{ccc}
\mathbb{F}_l \cong H_1B(F^* \cap U) & \xrightarrow{\cong} & \pi_1\Phi \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathbb{F}_l \cong H_1KF & \xrightarrow{H_1 i} & H_1\Omega C(\hat{p}_\gamma)
\end{array}
$$

The left hand vertical map carries the element u to a generator for $K_1^M(F, \mathbb{F}_l)$, which is a generator for H_1KF. □

Corollary 5.4.8.12 *Suppose that G_F is a Koszul duality group. Then the M_* spectral sequence collapses for odd primes or at the prime 2 if F contains $\sqrt{-1}$.*

Proof The E_2-term is generated multiplicatively over H_*KE by generators in $Ext^{1,-1}$, by 5.4.7.1. By the preceding analysis, together with 5.4.8.2, the result follows. □

Corollary 5.4.8.13 *Assume that G_F is a Koszul duality group. As before, let $\Gamma_*(G)$ denote the graded $\mathbb{F}_l$ vector space given by $\Gamma_i(G) = 0$ for i positive or odd, and $\Gamma_{-2k}(G) = gr_k \mathbb{F}_p^{Iw}[G]$ for k non-negative. Then $M_*(F) \cong \Gamma_*(G) \otimes H_*KE$ as graded H_*KE-modules.*

5.4.9 Steenrod algebra structure of $M_*(F)$ and the Adams spectral sequence for $\pi_* End^{\wedge}_{KF}(KE)$

We will need the $\mathcal{A}_*$-module structure of $M_*(F)$ as well. In order to obtain the $\mathcal{A}$-modules structure, we will need to show that the above spectral sequence collapses. We assume throughout this section that the Positselskii-Vishik conjecture holds for G_F, the absolute Galois group of a field F. We will need the following general result.

Proposition 5.4.9.1 *Let A be an S-algebra, with M and N left A-modules. Let μ_* denote the $\mathcal{A}_*$-module*

$$\pi_* Hom_{\mathbb{H}\wedge A}(\mathbb{H} \wedge M, \mathbb{H} \wedge N) \cong \pi_* Hom_A(M, \mathbb{H} \wedge N)$$

We have the evident inclusion $Hom_A(M, N) \hookrightarrow Hom_{\mathbb{H}\wedge A}(\mathbb{H} \wedge M, \mathbb{H} \wedge N)$ given by $f \to id_{\mathbb{H}} \wedge f$, and the corresponding Hurewicz homomorphism

$$h : \pi_* Hom_A(M, N) \to \mu_*$$

Then all elements of $\mathcal{A}_{*>0}$ vanish on the image of h. Also, all elements in the image of h are infinite cycles in the Adams spectral sequence with E_2-term given by

$$Ext_{\mathcal{A}_*}(\mathbb{F}_l, \pi_* Hom_A(M, \mathbb{H} \wedge N))$$

converging to $\pi_* Hom^{\wedge}_A(M, N)$

Proof Left to the reader. □

Corollary 5.4.9.2 *Suppose that G_F is a Koszul duality group. Then the isomorphism $M_*(F) \cong \Gamma_*(G) \otimes H_*KE$ of 5.4.8.13 is an isomorphism of $\mathcal{A}_*$-modules, when $\Gamma_*(G)$ is equipped with the trivial $\mathcal{A}_*$ action.*

Recall that we have an Adams spectral sequence with $E_2 \cong Ext^{**}_{\mathcal{A}_*}(\mathbb{F}_l, M_*(F))$ converging to $\pi_* End^{\wedge}_{KF}(KE)$. In order to describe its E_2-term, we let $\Gamma_{**}(G)$ denote the bigraded vector space given by $\gamma_{0q}(G) \cong \Gamma_q(G)$, and $\Gamma_{pq}(G) = 0$ for $p > 0$. Also, let $\mathcal{U}_{**}$ denote the E_2-term of the Adams spectral sequence for $\pi_* KE$.

Corollary 5.4.9.3 *Suppose that G_F is a Koszul duality group. Then the E_2-term for the Adams spectral sequence for*

$$\pi_* End^{\wedge}_{KF}(KE)$$

has the form

$$E_2^{**} \cong \Gamma_{**}(G) \otimes \mathcal{U}_{**}$$

Moreover, the entire Adams spectral sequence is a tensor product of a collapsing spectral sequence with $E_2^{**} \cong \Gamma_{**}(G)$ with the Adams spectral sequence for $\pi_* KE \cong \pi_* ku$.

Corollary 5.4.9.4 *Suppose that G_F is a Koszul duality group. then $\pi_* End^{\wedge}_{KF}(KE)$ is torsion free. Moreover, as a left $K_*(E) \cong \mathbb{Z}_l[\beta]$-module, it is torsion free.*

Proof The E_2-term is always a module over the algebra $Ext^{**}_{\mathcal{A}_*}(\mathbb{F}_l, \mathbb{F}_l)$. It is well known that there is an element $h_0 \in Ext^{1,-1}_{\mathcal{A}_*}(\mathbb{F}_l, \mathbb{F}_l)$ which is an infinite cycle in the Adams spectral sequence for the sphere, and so that multiplication by h_0 induces the associated graded version of multiplication by l. It is also well known that the E_∞-term of the Adams spectral sequence for $\pi_* ku$ has the form of a bigraded algebra $\mathbb{F}_l[a, b, \theta]/(b^{l-1})$, where a has bigrading $(-1, 1)$, b has bigrading $(0, 2)$, and θ has bigrading $(-1, 2l-1)$. Further, multiplication by θ induces the associated graded version of multiplication by β^{l-1}. Since the E_∞-term of the spectral sequence in question is free as a module over the elements h_0 and θ, the result follows easily. □

We are now in a position to describe $\pi_* End^{\wedge}_{KF}(KE)$. Recall that we are given a homomorphism $\alpha : \mathbb{Z}_l[x]^{Iw}\langle G, \rho, l\rangle \to \pi_* End^{\wedge}_{KF}(KE)$, and an inclusion $\mathbb{Z}_l[x]^{Iw}\langle G, \rho, l\rangle \hookrightarrow \hat{R}_*\langle G, \rho\rangle$.

Proposition 5.4.9.5 *There is a canonical extension of α to $\hat{\alpha}$ making the diagram*

$$\begin{array}{ccc} \mathbb{Z}_l[x]^{Iw}\langle G, \rho, l\rangle & \longrightarrow & \hat{R}\langle G, \rho\rangle \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \hat{\alpha}} \\ & & \pi_* End^{\wedge}_{KF}(KE) \end{array}$$

Proof It is shown in 5.4.8.7 that for every element i in the augmentation ideal I in $\mathbb{Z}_l^{Iw}[G]$, there is a unique element $\hat{i}$ in $\pi_{-2} End^{\wedge}_{KF}(KE)$ so that $\beta \hat{i} = i$. It now follows from the universal property 4.1.7 (using 5.4.9.4) that there exists an extension of α making the diagram commute, which sends the element θ_γ to $\hat{p}_\gamma$. □

We first observe that the Adams spectral sequence for $\pi_* End^{\wedge}_{KF}(KE)$ is a multiplicative spectral sequence, and that the $E_\infty-$term is the associated graded ring to a descending sequence of ideals $I_k \subseteq \pi_* End^{\wedge}_{KF}(KE)$ with the property that $I_m \cdot I_n \subseteq I_{m+n}$. We have the homomorphism $\hat{\alpha} : \hat{R}_*\langle G, \rho\rangle \to \pi_* End^{\wedge}_{KF}(KE)$, and we easily observe that the elements l and x^{l-1} in $\hat{R}_*\langle G, \rho\rangle$ have Adams filtration 1, and hence map to the ideal I_1. We conclude that $\hat{\alpha}$ induces a homomorphism of associated graded rings

$$gr^J_*(\hat{R}_*\langle G, \rho\rangle) \cong \bigoplus_k J^k/J^{k+1} \xrightarrow{gr(\hat{\alpha})} E_\infty$$

where E_∞ denotes the E_∞-term for the Adams spectral sequence for

$$\pi_* End^{\wedge}_{KF}(KE)$$

and J denotes the two-sided ideal (l, x^{l-1}). Note that $gr^J_*(\hat{R}_*\langle G, \rho\rangle)$ is actually bigraded, one grading being the filtration degree and the second being the grading from the grading on $\hat{R}_*\langle G, \rho\rangle)$ and on J. It is a well-known fact that if we can show that the map on associated graded rings is an isomorphism, then the original homomorphism $\hat{\alpha}$ is an isomorphism. This follows since $\hat{R}_*\langle G, \rho\rangle)$ is complete in the J-adic topology.

To prove that the $gr(\hat{\alpha})$ is an isomorphism, we first study the 0-th graded part, i.e. defined on $\hat{R}_*\langle G, \rho\rangle)/J$ with target $E^{0,*}_\infty$. We recall from 4.1.9 that $\hat{R}_*\langle G, \rho\rangle)/J \cong \Gamma_*(G, \mathbb{F}_l) \underset{\mathbb{F}_l}{\otimes} \mathbb{F}_l[x]/(x^{l-1})$. As for $E^{0,*}_\infty$, it follows from Corollaries 5.4.9.2 and 5.4.9.3 that $E^{0,*}_\infty \cong \Gamma_*(G, \mathbb{F}_l) \underset{\mathbb{F}_l}{\otimes} \mathbb{F}_l[x]/(x^{l-1})$. Consequently, the two groups in filtration zero are abstractly isomorphic, and we will need to show that the isomorphism is induced by $gr(\hat{\alpha})$.

A preliminary observation in this direction is

Lemma 5.4.9.6 *In order to prove that $gr_0(\hat{\alpha})$ is an isomorphism, it will suffice to prove that the induced homomorphism*

$$gr^J_{0,*}(\hat{R}\langle G, \rho\rangle)/(x) \to E^{0,*}_\infty / x E^{0,*}_\infty$$

is an isomorphism.

Proof Both are free modules over the subalgebra $\mathbb{F}_l[x]$. A homomorphism between free modules over connected graded rings which induces an isomorphism after factoring out the ideal of positive dimensional elements is an isomorphism. □

Note that $gr^J_{0,*}(\hat{R}\langle G, \rho \rangle)/(x) \cong \Gamma_{**}(G, \mathbb{F}_l)$, and that $E^{0,*}_\infty / x E^{0,*}_\infty \cong \Gamma_{**}(G, \mathbb{F}_l)$ as well. We now have

Proposition 5.4.9.7 *Suppose that G_F is a Koszul duality group. Then $gr_0(\hat{\alpha})$ is an isomorphism.*

Proof We wish to prove that the homomorphism $\Gamma_{0*}(G, \mathbb{F}_l) \to E^{0,*}_\infty / x E^{0,*}_\infty$ is an isomorphism. Since both sides are multiplicatively generated in dimension -2, it will clearly suffice to prove that the map is an isomorphism in degree -2. We identify the groups in question. $\Gamma_{-2}(G, \mathbb{F}_l) \cong \mathbb{F}_l \otimes I/I^2$, where I denotes the augmentation ideal in the Iwasawa algebra $\mathbb{Z}_l^{Iw}[G]$. The "Iwasawa algebra version" of the standard isomorphism $I(G)/I^2(G) \cong G^{ab}$ in group rings shows that $\Gamma_{-2}(G, \mathbb{F}_l) \cong G^{ab}/lG^{ab}$ as profinite abelian l-groups. On the other hand, we must analyze $E^{0,*}_\infty / x E^{0,*}_\infty$. We first note that $E^{0,*}_2 \cong M_*(F)$, where $M_*(F)$ is defined above. It follows from the collapse result 5.4.8.12 that $M_*(F)$ is a free module over H_*ku, and that $(M_*(F) \underset{H_*ku}{\otimes} \mathbb{F}_l)_{-2k} \cong Ext^{k,-k}_{K^M_*(F,\mathbb{F}_l)}(\mathbb{F}_l, \mathbb{F}_l)$. Similarly, $E^{0,*}_\infty$ is a free module over the ring $A_* = \mathbb{F}_l[x]/(x^{l-1})$, and we have that $(E^{0,*}_\infty \underset{A_*}{\otimes} \mathbb{F}_l)_{-2k} \cong Ext^{k,-k}_{K^M(F;\mathbb{F}_l)}(\mathbb{F}_l, \mathbb{F}_l)$. Consequently, $(E^{0,*}_\infty / x E^{0,*}_\infty)_{-2k} \cong Ext^{k,-k}_{K^M(F,\mathbb{F}_l)}(\mathbb{F}_l, \mathbb{F}_l)$. We now examine the case $k = 1$. It was shown in Proposition 5.4.8.3 that

$$Ext^{1,-1}_{K^M(F;\mathbb{F}_l)}(\mathbb{F}_l, \mathbb{F}_l) \cong Hom(K^M_1(F;\mathbb{F}_l), \mathbb{F}_l) \cong Hom(F^*/lF^*, \mathbb{F}_l)$$

and that this isomorphism is natural with respect to inclusions of fields. $\hat{\alpha}$ now has given us a homomorphism $\hat{\alpha}_F : G^{ab}/lG^{ab} \to Hom(F^*/lF^*, \mathbb{F}_l)$. One can check that $\hat{\alpha}_F$ is natural with respect to inclusions of fields in the sense that if F' is any extension of F, with Galois group G', then the diagram

$$\begin{array}{ccc}
(G')^{ab}/l(G')^{ab} & \xrightarrow{\hat{\alpha}_{F'}} & Hom(F'^*/lF'^*, \mathbb{F}_l) \\
\downarrow & & \downarrow \\
G^{ab}/lG^{ab} & \xrightarrow{\hat{\alpha}_F} & Hom(F^*/lF^*, \mathbb{F}_l)
\end{array}$$

commutes, where the right hand vertical arrow is induced by the homomorphism $F^*/lF^* \to F'^*/lF'^*$. We also showed in Proposition 5.4.8.11 that the homomorphism is non-trivial whenever the Galois group is $\mathbb{Z}_l$, i.e. is topologically cyclic. To prove that $\hat{\alpha}_F$ is injective, consider an element $\gamma \in ker(\hat{\alpha}_F)$. Let $g \in G$ be an element which projects to γ in G^{ab}/lG^{ab}, and let F^g denote its fixed field. It follows from Kummer theory that $(F^g)^*/l(F^g)^* \cong \mathbb{F}_l$, and that the natural homomorphism $F^*/lF^* \to (F^g)^*/l(F^g)^*$ is surjective. Consequently, the dual map $Hom((F^g)^*/l(F^g)^*, \mathbb{F}_l) \to Hom(F^*/lF^*, \mathbb{F}_l)$ is injective. We know that

$\hat{\alpha}_{F^g}$ is injective since the absolute Galois group of F^g is topologically cyclic. Consequently, the above diagram shows that the composite

$$\langle g \rangle / l \langle g \rangle \longrightarrow G^{ab}/lG^{ab} \longrightarrow Hom(F^*/lF^*, \mathbb{F}_l)$$

is injective. In particular, it contradicts the assumption that $\gamma \in ker(\hat{\alpha}_F)$, since the image of $\langle g \rangle$ is the subgroup generated by γ. For surjectivity, consider a homomorphism $\varphi : F^*/lF^* \to \mathbb{F}_l$. Kummer theory tells us that φ corresponds to an element $\gamma_\varphi \in G^{ab}/lG^{ab}$, which we lift to an element $g_\varphi \in G$. Kummer theory again tells us that if we set F_φ equal to the fixed field of g_φ, then the homomorphism φ is in the image of the group $Hom(F^*_\varphi/lF^*_\varphi, \mathbb{F}_l) \to Hom(F^*/lF^*, \mathbb{F}_l)$. But the homomorphism $\hat{\alpha}_{F_\varphi}$ is an isomorphism, and the result follows by applying the above diagram again. □

We have now shown that the filtration degree zero part of the homomorphism $gr(\hat{\alpha})$ is an isomorphism. In Corollary 5.4.9.3, we showed that E_∞ is the tensor product of the Γ_{**} with a bigraded algebra $\mathcal{U}_{**}$, which is the E_∞ term for the Adams spectral sequence for $\pi_* ku$. This spectral sequence has been analyzed by Kane [26].

Theorem 5.4.9.8 (Kane) *The bigraded algebra $\mathcal{U}_{**}$ is isomorphic to a bigraded algebra $\mathbb{F}_l[h_0, \beta(l-1), x]/(x^{l-1})$, where h_0 has bidegree $(1,1)$, x has bidegree $(0,2)$, and $\beta(l-1)$ has bidegree $(1, 2l-1)$.*

Corollary 5.4.9.9 *Let $E(0)_{**}$ denote the filtration degree 0 part of $(E_\infty)_{**}$. Then $(E_\infty)_{**}$ is isomorphic to a polynomial algebra $E(0)_{**}[h_0, x]$ where h_0 has bidegree $(1,1)$ and where x has bidegree $(1, 2l-1)$.*

Theorem 5.4.9.10 *Suppose that the Positselskii-Vishik conjecture holds at l . Then the homomorphism $\hat{\alpha}$ is an isomorphism of graded rings.*

Proof Let S_{**} and $E(0)_{**}$ denote the bidegree $(0,*)$ parts of $gr^J(\hat{R}\langle G, \rho \rangle)$ and of E_∞ respectively. We have shown that $gr(\hat{\alpha})$ restricts to an isomorphism $S_{**} \to E(0)_{**}$. We know from Corollary 5.4.9.9 that $E_\infty \cong E(0)_{**}[h_0, x]$ with generators in bidegrees $(1,1)$ and $(1, 2l-1)$ respectively. on the other hand, we have shown in 4.1.11 that there is a surjective homomorphism $\lambda : S_{**}[h'_0, x'] \to gr^J_{**}(\hat{R}\langle G, \rho \rangle$, where h'_0 and x' are in the same bidegrees as the corresponding elements in E_∞. One now checks, using the behavior of the Adams spectral sequence for $\pi_* ku$, that $h'_0 \to h_0$ and $x' \to x$ under $gr(\hat{\alpha}) \circ \lambda$. Consequently $gr(\hat{\alpha}) \circ \lambda$ is an isomorphism. This shows that λ is in fact injective, hence an isomorphism also. This gives the result. □

5.5 Comparison of the descent spectral sequence with the Bloch-Lichtenbaum spectral sequence

Let F be a field. Assume throughout this section that the Positselskii-Vishik conjecture holds at l. We have homomorphisms of S-algebras

$$KF \xrightarrow{\varepsilon} K\overline{F} \xrightarrow{\nu} \mathbb{L} \xrightarrow{\eta} \mathbb{H}$$

Every composite in this diagram induces an isomorphism on π_0 and a surjection on π_1, so we obtain equivalences of spectra

$$KF_p^{\wedge} \cong KF_{\varepsilon}^{\wedge} \cong KF_{\nu\varepsilon}^{\wedge} \cong KF_{\eta\nu\varepsilon}^{\wedge}$$

Each of the maps ε, $\nu\varepsilon$, and $\eta\nu\varepsilon$ gives rise to a different filtration and consequently different spectral sequences converging to $\pi_* KF_l^{\wedge}$. We have some conjectures concerning the relationship of these spectral sequences with the Bloch-Lichtenbaum spectral sequence. We begin with the case of integral homology, i.e. the spectral sequence attached to the homomorphism $KF \to \mathbb{L}$.

Conjecture 5.5.1 *The homotopy spectral sequence for the cosimplicial spectrum constructed using $KF \to \mathbb{L}$ coincides with the l-completed Bloch-Lichtenbaum spectral sequence.*

We regard this conjecture as plausible because the subquotients of the Bloch-Lichtenbaum spectral sequence are Eilenberg-MacLane, and moreover they are obtained by studying π_0 applied to various K-theory spectra. This construction appears closely related to the functor $- \underset{KF}{\wedge}$, which suggests the possibility that the filtrations are identical.

We discuss a variant of the Bloch-Lichtenbaum spectral sequence. We consider the pro-spectrum $\underline{\mathcal{M}} = \{M_{l^n}\}_{n \geq 0}$, with the evident reduction maps $M_{l^{n+1}} \to M_{l^n}$ as the bonding maps in the inverse system. The homotopy inverse limit of this pro-spectrum is the l-completed sphere. In particular, $\underleftarrow{holim}\, KF \wedge M_{l^n} \cong KF^{\wedge}l$. Let $KF\langle n \rangle$ denote the n-th quotient of the Bloch-Lichtenbaum spectral sequence, so $KF \cong \underleftarrow{holim}\, KF\langle n \rangle$. We call the spectral sequence for K_*F based on the inverse system $KF\langle n \rangle \wedge M_{l^n}$ the *Bloch-Lichtenbaum spectral sequence with Bocksteins.*

Conjecture 5.5.2 *The homotopy spectral sequence for the cosimplicial spectrum constructed using $KF \to \mathbb{H}$ can be identified with the Bloch-Lichtenbaum spectral sequence with Bocksteins.*

The homomorphism of S-algebras $KF \to K\overline{F}$ of course induces the descent spectral sequence. We believe its relationship with the l-completed Bloch-Lichtenbaum spectral sequence is as follows.

Conjecture 5.5.3 *The descent spectral sequence is an accelerated form of the Bloch-Lichtenbaum spectral sequence. Specifically, the E_2-terms should be abstractly isomorphic, but with a shift in grading.*

We picture the first few stages of the E_2-terms of the two spectral sequences in diagram form. Here is a picture of the Bloch-Lichtenbaum spectral sequence. We consider the geometric case, when F contains the l-th power roots of unity for simplicity.

H^2	H^1	$\mathbb{Z}_l$	0	0		0	H^3	H^2	H^1	$\mathbb{Z}_l$
0	0	0	0	0		0	0	0	0	0
0	0	0	0	0		0	0	H^2	H^1	$\mathbb{Z}_l$
0	0	H^1	$\mathbb{Z}_l$	0		0	0	0	0	0
0	0	0	0	0		0	0	0	H^1	$\mathbb{Z}_l$
0	0	0	0	0		0	0	0	0	0
0	0	0	0	$\mathbb{Z}_l$		0	0	0	0	$\mathbb{Z}_l$

Here H^i denotes $H^i(G, \mathbb{Z}_l)$. The left hand diagram should be the Bloch-Lichtenbaum spectral sequence, and the right hand diagram is the descent E_2-term.

Finally, we make conjectures concerning the structure of $\pi_* End^{\wedge}_{KF}(\mathbb{L})$ and $\pi_* End^{\wedge}_{KF}(\mathbb{H})$. Recall that $\pi_* End^{\wedge}_{ku}(\mathbb{L}) \cong \Lambda_{\mathbb{Z}}(\theta)/(\theta^2)$, where θ has degree -3.

Let $\Gamma_*(G)$ denote the associated graded ring to the augmentation ideal of the Iwasawa algebra $\mathbb{Z}_l^{Iw}[G]$, with the renormalization of the grading given by $\Gamma_{-2s}(G) \cong I^s(G)/I^{s+1}(G)$. Note that $\Gamma_{-2}(G) \cong G^{ab}$. For an element $g \in G$, let $X_g = 1 - g$. Also, let u_g denote the l-adic integer defining the action of g on the inverse limit of the l-th power roots of unity. Consider the graded algebra $\Lambda^{twisted}_{\Gamma_*(G)}(\theta)$ defined to be the quotient of the free algebra on θ over $\Gamma_*(G)$ (non-commutatively, so θ does not commute with elements in $\Gamma_*(G)$ in general) by the two sided ideal generated by θ^2 and $X_g\theta - ((1 - u_g)\theta + u_g\theta X_g)$. θ has degree -3.

Conjecture 5.5.4 *Suppose that the field F contains a primitive l-th root of unity. Then $\pi_* End^{\wedge}_{KF}(\mathbb{L}) \cong \Lambda^{twisted}_{\Gamma_*(G)}(\theta)$.*

Conjecture 5.5.5 *Suppose again that F contains a primitive l-th root of unity. Then we have an isomorphism*

$$\pi_* End^{\wedge}_{KF}(\mathbb{H}) \cong \Lambda^{twisted}_{\Gamma_*(G)}(\theta) \otimes \Lambda_{\mathbb{F}_l}(Q)$$

where Q has degree -1.

6 Representations of Galois groups and descent in K-theory

Let F be a field, with E its algebraic closure. In the previous sections, we have discussed the pro-S-algebra $\mathcal{E} = End^{\wedge}_{KF}(KE)$ of KF-linear self-maps of KE. We have formulated a conjecture concerning $\pi_*\mathcal{E}$, and shown that its validity depends on a very strong form of the Bloch-Kato conjecture. We have also shown that complete knowledge of $\pi_*\mathcal{E}$ will allow the evaluation of the E_2-term of a descent spectral sequence which converges "on the nose" to algebraic K-theory of F.

We have not, though, yet suggested a model for the homotopy type of the spectrum KF itself. In this section, we will produce conjectural descriptions of KF, depending only on G_F, and verify them in the case of some examples. We will also discuss what would be required to prove the validity of this conjectural description in more generality.

6.1 Categories of descent data

Let F be any field, equipped with a group action by a profinite group G. We assume that the action of G is continuous, in the sense that the stabilizer of any element of F is an open and closed subgroup of finite index in G.

Definition 6.1.1 *By a* linear descent datum *for the pair (G, F), we will mean a finite dimensional F-vector space V, together with a continuous action of G on V so that $g(fv) = f^g g(v)$ for all $g \in G$, $f \in F$, and $v \in V$. We define two categories of linear descent data, $V^G(F)$ and $V(G, F)$. The objects of $V^G(F)$ are all linear descent data for the pair (G, F), and the morphisms are all equivariant F-linear morphisms. The objects of $V(G, F)$ are also all linear descent data for (G, F), but the morphisms are all F-linear morphisms (without any equivariance requirements). The group G acts on the category $V(G, F)$, by conjugation of maps (so the action is trivial on objects), and the fixed point subcategory is clearly $V^G(F)$. Note that both categories are symmetric monoidal categories under direct sum.*

We note that $\underset{F}{\otimes}$ provides a coherently associative and commutative monoidal structure on $V(G, F)$ and $V^G(F)$.

Definition 6.1.2 *We define the spectra $K^G(F)$ and $K(G,F)$ to be the spectra obtained by applying an infinite loop space machine ([35] or [49] to the symmetric monoidal categories of isomorphisms of $V^G(F)$ and $V(G,F)$, respectively. $K(G,F)$ is a spectrum with G-action, with fixed point spectrum $K^G(F)$. The tensor product described above makes each of these spectra into commutative S-algebras using the results on [36].*

There are various functors relating these categories (and therefore their K-theory spectra). We have the fixed point functor

$$(-)^G : V^G(F) \longrightarrow V^{\{e\}}(F^G) \cong Vect(F^G)$$

defined on objects by $V \to V^G$. We also have the induction functor

$$F \underset{F^G}{\otimes} - : VectF \cong V^{\{e\}}(F) \longrightarrow V^G(F)$$

given on objects by $V \to F \underset{F^G}{\otimes} V$. The following is a standard result in descent theory. See [23] for details.

Proposition 6.1.3 *Suppose that the action of G on F is faithful, so that $F^G \hookrightarrow F$ is a Galois extension with Galois group G. Then both $(-)^G$ and $F \underset{F^G}{\otimes} -$ are equivalences of categories.*

We also have

Proposition 6.1.4 *The category $V(G,F)$ is canonically equivalent to the category $Vect(F)$.*

Proof The definition of the morphisms has no dependence on the group action, and the result follows immediately. □

Remark: Note that when the group action is trivial, i.e. G acts by the identity, then the category $V^G(F)$ is just the category of finite dimensional continuous F-linear representations of G.

Definition 6.1.5 *In the case when the G-action is trivial, we will also write $Rep_F[G]$ for $V^G(F)$.*

Proposition 6.1.6 *The functor*

$$Rep_{F^G}[G] \cong V^G(F^G) \xrightarrow{F \underset{F^G}{\otimes} -} V^G(F) \cong V^{\{e\}}(F^G) \cong Vect(F^G)$$

respects the tensor product structure, and $K(F^G)$ becomes an algebra over the S-algebra $KRep_{F^G}[G]$.

6.2 An example

Let k denote an algebraically closed field of characteristic 0, let $k[[x]]$ denote its ring of formal power series, and let $k((x))$ denote the field of fractions of $k[[x]]$. We have earlier (in 4.5) analyzed the K-theory of the field $k((x))$. In this section, we wish to show that the representation theory of its absolute Galois group can be used to create a model for the spectrum $Kk((x))$.

The following result is proved in [66]. It is derived from the fact that the algebraic closure of $k((x))$ is the field of *Puiseux series*, i.e. the union of the fields $k((x^{\frac{1}{n}}))$.

Proposition 6.2.1 *The absolute Galois group G of $k((x))$ is the group $\hat{\mathbb{Z}}$, the profinite completion of the group of integers.*

Let F denote the field $k((x))$, and let E denote $\overline{k((x))}$. We have observed in 6.1.6 that the spectrum KF, which is equivalent to $K^G(E)$, becomes an algebra spectrum over the S-algebra $KRep_F[G] \cong K^G F$. We wish to explore the nature of this algebra structure, and to use derived completion to demonstrate that the algebraic K-theory of F can be constructed directly from the representation theory of G over an algebraically closed field, e.g. $\mathbb{C}$.

We have the map of S-algebras $KRep_F[G] \to KV^G(E) \cong KF$, induced by the functor

$$id_E \underset{F}{\otimes} - : V^G(F) \longrightarrow V^G(E)$$

We may compose this map with the canonical map $KRep_k[G] \to KRep_F[G]$ to obtain a map of S-algebras

$$\hat{\alpha} : KRep_k[G] \to K^G(E)$$

We note that as it stands this map doesn't seem to carry much structure.

Proposition 6.2.2 $\pi_* KRep_k[G] \cong R[G] \otimes ku_*$, *where $R[G]$ denotes the complex representation ring. (The complex representation ring of a profinite group is defined to be the direct limit of the representation rings of its finite quotients.)*

Proof Since k is algebraically closed of characteristic zero, the representation theory of G over k is identical to that over $\mathbb{C}$. This shows that $\pi_0 KRep_k[G] \cong R[G]$. In the category $Rep_k[G]$, every object has a unique decomposition into irreducibles, each of which has $\mathbb{C}$ as its endomorphism ring. The result follows directly. □

We also know from 4.5 that $K_*F \cong K_*k \oplus K_{*-1}k$. It is now easy to check that the map $KRep_k[G] \to K^G E \cong KF$ induces the composite

$$R[G] \otimes K_*k \xrightarrow{\varepsilon \otimes id} K_*k \hookrightarrow K_*k \oplus K_{*-1}k \cong K_*F$$

which does not appear to carry much information about KF. However, we may use derived completion as follows. As usual, let $\mathbb{H}$ denote the mod-l Eilenberg-MacLane spectrum. We now have a commutative diagram of S-algebras

$$\begin{array}{ccc} KRep_k[G] & \xrightarrow{\hat{\alpha}} & K^G E \\ \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathbb{H} & \xrightarrow{id} & \mathbb{H} \end{array}$$

where ε in both cases denotes the augmentation map which sends any vector space or representation to its dimension. The naturality properties of the derived completion construction yields a map

$$\alpha^{rep} : KRep_k[G]^\wedge_\varepsilon \to K^G E^\wedge_\varepsilon \xrightarrow{\sim} KF^\wedge_\varepsilon$$

which we refer to as the *representational assembly* for F. The goal of this section is to show that despite the fact that $\hat{\alpha}$ carries little information about K_*F, α^{rep} is an equivalence of spectra.

Proposition 6.2.3 *$KF^\wedge_\varepsilon$ is just the l-adic completion of KF.*

Proof This is just the definition of l-adic completion, as in section 3. □

It remains to determine the structure of $\pi_* KRep_k[G]^\wedge_\varepsilon$. To do this we will use the descent spectral sequence 3.8.

Proposition 6.2.4 *There is an equivalence of spectra*

$$KRep_k[G]^\wedge_\varepsilon \cong ku^\wedge_l \vee \Sigma ku^\wedge_l$$

In particular the homotopy groups are given by $\pi_i KRep_k[G]^\wedge_\varepsilon \cong \mathbb{Z}_l$ for all $i \geq 0$, and $\cong 0$ otherwise.

Proof The descent spectral sequence of 3.8 shows that in order to compute $\pi_* KRep_k[G]^\wedge_\varepsilon$, we must compute the derived completion of modules over the *ring* $R[G]$, regarded as an S-algebra via the Eilenberg-MacLane construction. In order to study this derived completion, we will analyze the S-algebra $End^{\wedge}_{\underline{R[G]}}(\mathbb{F}_l)$, using the Adams spectral sequence. We must first compute the input to this spectral sequence, which is $\pi_* Hom_{\mathbb{H}\wedge\underline{R[G]}}(\mathbb{H}\wedge\mathbb{H}, \mathbb{H}\wedge\mathbb{H})$, for which there is a spectral sequence with E_2-term given by $Ext_{H_*\underline{R[G]}}(H_*\mathbb{H}, H_*\mathbb{H})$. $H_*\underline{R[G]}$ is given by $R[G]/lR[G] \otimes H_*\mathbb{L}$, where $\mathbb{L}$ is the $\mathbb{Z}_l$ Eilenberg-MacLane spectrum. $H_*\mathbb{H}$ is the dual to the mod l Steenrod algebra. It now follows directly that

$$Ext_{H_*\underline{R[G]}}(H_*\mathbb{H}, H_*\mathbb{H}) \cong Ext^*_{R[G]/lR[G]}(\mathbb{F}_l, \mathbb{F}_l) \otimes Ext_{H_*\mathbb{L}}(H_*\mathbb{H}, H_*\mathbb{H})$$

We evaluate each of these tensor factors. For finite abelian groups A, it is known that $R[A] \cong \mathbb{Z}[A^*]$, where A^* denotes the character group $Hom(A, S^1)$. It follows (after a passage to direct limits over the finite quotients of $\hat{\mathbb{Z}}$ that $R[G]/lR[G] \cong \mathbb{F}_l[\mathbb{Q}/\mathbb{Z}]$, since $\hat{\mathbb{Z}}^* \cong \mathbb{Q}/\mathbb{Z}$. We therefore have that

$$Ext^*_{R[G]/lR[G]}(\mathbb{F}_l, \mathbb{F}_l) \cong H^*(\mathbb{Q}/\mathbb{Z}, \mathbb{F}_l)$$

It is now a standard calculation in the cohomology of cyclic finite groups, together with a comparison with the cohomology of $\mathbb{C}P^\infty$, that $H^*(\mathbb{Q}/\mathbb{Z}, \mathbb{F}_l) \cong \mathbb{F}_l[b]$, where b is a generator in degree 2.

It is also well known that $H_*\mathbb{H}$ is a free $H_*\mathbb{L}$-module on two generators, one the element 1 in degree 0, and the other the element τ_0 in degree 1 (Sq^1 in the case $l = 2$). It follows that the higher Ext-groups vanish in this case, and that

$$Hom_{H_*\mathbb{L}}(H_*\mathbb{H}, H_*\mathbb{H}) \cong \Lambda^{\mathbb{F}_l}_*(\theta) \otimes \mathbb{H}_*$$

where $\mathbb{H}_*$ denotes the dual Steenrod algebra. Consequently,

$$Ext_{H_*\underline{R[G]}}(H_*\mathbb{H}, H_*\mathbb{H}) \cong \mathbb{F}_l[b] \otimes \Lambda^{\mathbb{F}_l}_*(\theta) \otimes \mathbb{H}_*$$

where b is in homological degree 2 and total degree 0 and where θ is in homological degree 0 and total degree -1. We can now observe that b is in the image of the Hurewicz map

$$\pi_* Hom_{\underline{R[G]}}(\mathbb{H}, \mathbb{H}) \longrightarrow \pi_* Hom_{\mathbb{H}\wedge\underline{R[G]}}(\mathbb{H}\wedge\mathbb{H}, \mathbb{H}\wedge\mathbb{H})$$

as follows. The homotopy spectral sequence for $\pi_* Hom_{\underline{R[G]}}(\mathbb{H}, \mathbb{H})$ has E_2-term of the form $Ext_{R[G]}(\mathbb{F}_l, \mathbb{F}_l) \cong \mathbb{F}_l[B, Q]/(Q^2)$, where B and Q commute, and where B has degree 2 and Q has degree 1. One readily checks that B maps to the element b under the Hurewicz map. On the other hand, we have the ring homomorphism $\varepsilon : R[G] \to \mathbb{Z}$, and the corresponding map of S-algebras $\underline{R[G]} \to \mathbb{L}$, and we now have a map

$$Hom_{\mathbb{H}\wedge\mathbb{L}}(\mathbb{H}\wedge\mathbb{H},\mathbb{H}\wedge\mathbb{H}) \longrightarrow Hom_{\mathbb{H}\wedge\underline{R[G]}}(\mathbb{H}\wedge\mathbb{H},\mathbb{H}\wedge\mathbb{H})$$

The left hand spectral sequence collapses since it is concentrated in homological degree 0. On the other hand, it is clear that θ is in the image of the E_2-term of this spectral sequence. It follows that the spectral sequence for $\pi_*Hom_{\mathbb{H}\wedge\underline{R[G]}}(\mathbb{H}\wedge\mathbb{H},\mathbb{H}\wedge\mathbb{H})$ collapses, and further that all Steenrod operations vanish on B and θ. It follows that the E_2-term of the Adams spectral sequence for $\pi_*End^{\wedge}_{\underline{R[G]}}(\mathbb{F}_l)$ has form $\mathbb{F}_l[b]\otimes\Lambda^{\mathbb{F}_l}_*(\theta)\otimes Ext_{\mathcal{A}_*}(\mathbb{F}_l,\mathbb{H}_*)$. The above analysis also shows that b and θ are also infinite cycles in this spectral sequence, so its E_∞-term has the form $\mathbb{F}_l[b,\theta]/(\theta^2)$, where b and θ commute. It is readily checked that for dimensional regions, we have that the structure of $\pi_*End^{\wedge}_{\underline{R[G]}}(\mathbb{F}_l)$ is of the same form, i.e,

$$\pi_*End^{\wedge}_{\underline{R[G]}}(\mathbb{F}_l)\cong\mathbb{F}_l[b,\theta]/(\theta^2)$$

It now follows from 3.8 that for any $R[G]$-module M, there is a spectral sequence with E_2-term $Ext^{**}_{\mathbb{F}_l[b,\theta]}(\mathbb{F}_l,Tor_{R[G]}(M,\mathbb{F}_l))$ converging to $\pi_*M^{\wedge}_{\varepsilon}$. In particular, for $R[G]$ itself, we obtain a spectral sequence with E_2-term $Ext^{**}_{\mathbb{F}_l[b,\theta]}(\mathbb{F}_l,\mathbb{F}_l)$ converging to $\pi_*\underline{R[G]}^{\wedge}_{\varepsilon}$. This E_2-term has the form of a bigraded algebra over $\mathbb{F}_l$ on a polynomial generator h of bidegree (-1,1) and an exterior generator of bidegree (-1, 2). The polynomial algebra on h corresponds to the integral generator in the image of $R^{\wedge}_{\varepsilon}\to\mathbb{Z}_l$, and hence no element in this algebra is in the image of any differential. It follows that the spectral sequence collapses, and that $\pi_0R[G]^{\wedge}_{\varepsilon}\cong\pi_1R[G]^{\wedge}_{\varepsilon}\cong\mathbb{Z}_l$ and that $\pi_iR[G]^{\wedge}_{\varepsilon}=0$ for $i\geq 2$. Moreover, it is clear that the multiplication is such that $\pi_*R[G]^{\wedge}_{\varepsilon}\cong\Lambda_{\mathbb{Z}_l}(\rho)$, where ρ has dimension 1.

The spectral sequence 3.4 now has E_2-term isomorphic to $ku_*\otimes\Lambda_{\mathbb{Z}_l}(\rho)$. It is easily verified that the spectral sequence collapses, and that $\pi_*KRep_k[G]^{\wedge}_{\varepsilon}\cong\Lambda_{ku_*}(\rho)$ where ρ has dimension 1. This gives the result. □

We have now shown that the derived completion $KRep_k[G]^{\wedge}_{\varepsilon}$ has the same homotopy groups as the K-theory spectrum KF. We must now show that the representational assembly induces an isomorphism on homotopy groups. We now recall the localization sequence [44] which allows us to compute $K_*(F)$. Consider the category $\mathcal{M}=Mod(k[[x]])$ of finitely generated modules over $k[[x]]$. Let $\mathcal{T}\subseteq\mathcal{P}$ denote the full subcategory of x-torsion modules. Then the category $Vect(F)$ may be identified with the quotient abelian category $\mathcal{M}/\mathcal{T}$, and we have the usual localization sequence

$$K\mathcal{T}\to K\mathcal{M}\to K\mathcal{P}/\mathcal{T}$$

which is identified with the fiber sequence

$$Kk \to Kk[[x]] \to KF$$

We now observe that there is a twisted version of this construction. Let $\mathcal{O}_E$ denote the integral closure of $k[[x]]$ in E. $\mathcal{O}_E$ is closed under the action of the group G. We now define $\mathcal{E}^G$ to be the category whose objects are finitely generated $\mathcal{O}_E$ modules equipped with a G-action so that $g(em) = e^g g(m)$. Further, let $\mathcal{T}^G$ denote the full subcategory of torsion modules.

Proposition 6.2.5 *The quotient abelian category $\mathcal{E}^G/\mathcal{T}^G$ can be identified with the category of linear descent data $V^G(E)$.*

Proof Straightforward along the lines of Quillen's proof that $\mathcal{M}/\mathcal{T} \cong VectF$. □

Corollary 6.2.6 *There is up to homotopy a fiber sequence of spectra*

$$K\mathcal{T}^G \longrightarrow K\mathcal{E}^G \longrightarrow KV^G(E) \cong KF$$

We note that for each of the categories $\mathcal{T}^G$, $\mathcal{E}^G$, and $V^G(E)$, objects may be tensored with finite dimensional k-linear representations of G to give new objects in the category. It follows that each of the K-theory spectra are module spectra over $KRep_k[G]$.

Lemma 6.2.7 *The homotopy fiber sequence of 6.2.6 is a homotopy fiber sequence of $KRep_k[G]$-module spectra.*

It follows from its definition that derived completion at ε preserves fiber sequences of module spectra, so we have a homotopy fiber sequence

$$(K\mathcal{T}^G)^\wedge_\varepsilon \longrightarrow (K\mathcal{E}^G)^\wedge_\varepsilon \longrightarrow (KV^G(E))^\wedge_\varepsilon \cong KF$$

The last equivalence follows from 6.2.3. We will now show that

- $(K\mathcal{T}^G)^\wedge_\varepsilon \simeq *$
- $(K\mathcal{E}^G)^\wedge_\varepsilon \simeq KRep_k[G]^\wedge_\varepsilon$

The result will follow immediately.

Proposition 6.2.8 $(K\mathcal{T}^G)^\wedge_\varepsilon \simeq *$

Proof Let $\mathcal{T}_n^G$ denote the subcategory of finitely generated torsion $k[[t^{\frac{1}{n}}]]$-modules M equipped with G-action so that $g(fm) = f^g g(m)$ for $g \in G$, $f \in k[[t^{\frac{1}{n}}]]$, and $m \in M$. We have obvious functors

$$\tau_n^k = k[[t^{\frac{1}{k}}]] \underset{k[[t^{\frac{1}{n}}]]}{\otimes} - : \mathcal{T}_n^G \to \mathcal{T}_k^G$$

whenever n divides k, and an equivalence of categories

$$\underset{\rightarrow}{lim}\, \mathcal{T}_n^G \longrightarrow \mathcal{T}^G$$

A straightforward devissage argument now shows that the inclusion of the full subcategory of objects of $\mathcal{T}_n^G$ on which $x^{\frac{1}{n}}$ acts trivially induces an equivalence on K-theory spectra. It follows that $K_*\mathcal{T}_n^G \cong R[G] \otimes K_*k$, and moreover that $K\mathcal{T}_n^G \simeq KRep_k[G]$ as module spectra over $KRep_k[G]$. We will now need to analyze the map of K-theory spectra induced by the functors τ_n^k. Recall that $R[G] \cong \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. For all positive integeres n and s, let $\nu_{n,s}$ denote the element in $R[G]$ corresponding to the element $\sum_{i=0}^{s-1}[i/ns] \in \mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$. We claim that we have a commutative diagram

$$\begin{array}{ccc} K_*\mathcal{T}_n^G & \xrightarrow{\sim} & R[G] \otimes K_*k \\ \downarrow{\scriptstyle \tau_n^{ns}} & & \downarrow{\scriptstyle (\cdot \nu_{n,s}) \otimes id} \\ K_*\mathcal{T}_{ns}^G & \xrightarrow{\sim} & R[G] \otimes K_*k \end{array}$$

That we have such a diagram is reduced to a π_0 calculation, since the diagram is clearly a diagram of $R[G] \otimes K_*k$-modules. It therefore suffices to check the commuting of this diagram on the element $1 \in K_*\mathcal{T}_n^G$. But 1 under τ_n^{ns} clearly goes to the module $k[[x^{\frac{1}{ns}}]]/(x^{\frac{1}{n}})$. It is easily verified that as an element in the representation ring, this element is $\nu_{n,s}$.

We are now ready to describe $\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}^G$. Since smash products commute with filtering direct limits, we see that

$$\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}^G \simeq \underset{\rightarrow}{lim}\, \mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}_n^G$$

But since $K_*\mathcal{T}_n^G$ is a free $R[G] \otimes K_*k$-module of rank 1, $\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}_n^G \cong \mathbb{H}$. It is readily checked that the induced map

$$id_{\mathbb{H}} \underset{KRep[G]}{\wedge} (\cdot \nu_{ns}) \otimes id$$

is multiplication by s on $\pi_*\mathbb{H}$. It now clearly follows that $\pi_*\mathbb{H} \underset{KRep[G]}{\wedge} K\mathcal{T}^G = 0$. 3.8 and 3.10 now show that $(K\mathcal{T}^G)^\wedge_\varepsilon \simeq *$. □

Corollary 6.2.9 *The map* $(K\mathcal{E}^G)^\wedge_\varepsilon \to (KV^G(E))^\wedge_\varepsilon$ *is a weak equivalence of spectra.*

Proof Completion preserves fibration sequences of spectra. □

We next analyze $K_*\mathcal{E}^G$ in low degrees. Let $\mathbb{N}^\cdot$ be the partially ordered set of positive integers, where $m \le n$ if and only if $m|n$. We define a directed system of $R[G]$-modules $\{A_n\}_{n>0}$ parametrized by $\mathbb{N}^\cdot$ by setting $A_n = R[G]$, and where whenever $m|n$, we define the bonding map from A_m to A_n to be multiplication by the element $\nu_{m,\frac{n}{m}}$. We let $QR[G]$ denote the colimit of this module. It follows from the proof of 6.2.8 that $K_*\mathcal{T}^G \cong QR[G] \otimes K_*k$. In particular, $K_*\mathcal{T}^G = 0$ in odd degrees. On the other hand, we know that $K_*F \cong \mathbb{Z}_l$ in odd degrees, but that in these odd elements map non-trivially in the connecting homomorphism in the localization sequence. Consequently, we have

Proposition 6.2.10 $K_*\mathcal{E}^G = 0$ *in odd degrees. In particular,* $K_1\mathcal{E}^G = 0$.

Proposition 6.2.11 $K_0\mathcal{E}^G \cong R[G]$.

Proof Objects of $\mathcal{E}^G$ are the same thing as modules over the twisted group ring $\mathcal{O}_E\langle G\rangle$, and a devissage argument shows that the inclusion

$$Proj(\mathcal{O}_E\langle G\rangle) \hookrightarrow Mod(\mathcal{O}_E\langle G\rangle)$$

induces an isomorphism, so that we may prove the corresponding result for projective modules over $\mathcal{O}_E\langle G\rangle$. Note that we have a ring homomorphism

$$\pi : \mathcal{O}_E\langle G\rangle \longrightarrow k[G]$$

given by sending all the elements $x^{\frac{1}{n}}$ to zero. $\mathcal{O}_E\langle G\rangle$ is complete in the I-adic topology, where I is the kernel of π. As in [60], isomorphism classes of projective modules over $\mathcal{O}_E\langle G\rangle$ are now in bijective correspondence via π with the isomorphism classes of projective modules over $k[G]$, which gives the result. □

Corollary 6.2.12 *The functor* $\mathcal{O}_E \underset{k}{\otimes} - : Rep_k[G] \to \mathcal{E}^G$ *induces a weak equivalence on spectra. Consequently, the map* $KRep_k[G]^\wedge_\varepsilon \to (K\mathcal{E}^G)^\wedge_\varepsilon$ *is a weak equivalence.*

Proof 6.2.10 and 6.2.11 identify the homotopy groups of $K\mathcal{E}^G$ in degrees 0 and 1. From the localization sequence above, together with the fact that multiplication by the Bott element induces isomorphisms $K_i\mathcal{T}^G \to K_{i+2}\mathcal{T}^G$ and $K_iF \to K_{i+2}F$, it follows that it also induces an isomorphism $K_i\mathcal{E}^G \to K_{i+2}\mathcal{E}^G$. So, the map of the statement of the corollary induces an isomorphism on homotopy groups, which is the required result. □

Theorem 6.2.13 *The map $\alpha^{rep} : KRep_k[G]^\wedge_\varepsilon \to KV^G(E)^\wedge_\varepsilon \cong KF^\wedge_\varepsilon$ is a weak equivalence of spectra.*

Proof α^{rep} is the composite of the maps of Corollary 6.2.9 and Corollary 6.2.12, both of which we have shown are weak equivalences. □

We will now show how to extend this result to the case of geometric fields (i.e. containing and algebraically closed subfield) F of characteristic zero with G_F a free pro-l abelian group. Let F be geometric, and let $k \subseteq F$ denote an algebraically closed subfield. Let $A = k[t_1^{\pm 1}, t_2^{\pm 1}, \ldots t_n^{\pm 1}]$. We let B denote the ring obtained by adjoining all p-th power roots of unity of the variables t_i to A, so

$$B = \bigcup_l k[t_1^{\pm\frac{1}{p^l}}, t_2^{\pm\frac{1}{p^l}}, \ldots t_n^{\pm\frac{1}{p^zeta_n l}}]$$

We choose for a sequence of elements ζ_n in k so that ζ_n is a primitive p^n-th root of unity, and so that $\zeta_n^p = \zeta_{n-1}$. We define an action of the group $\mathbb{Z}_p^n$ on B by $\tau_i(t_i^{\frac{1}{p^n}}) = \zeta_n t_i^{\frac{1}{p^n}}$, and $\tau_j(t_i^{\frac{1}{p^n}}) = t_i^{\frac{1}{p^n}}$ for $i \neq j$, where $\{\tau_1, \tau_2, \ldots, \tau_n\}$ is a set of topological generators for $\mathbb{Z}_p^n$.

Proposition 6.2.14 *The p-completed K-theory groups of A and B are given by*

- $K_*A \cong \Lambda_{K_*k}(\theta_1, \theta_2, \ldots, \theta_n)$
- $K_*B \cong K_*k$

Proof The first result is a direct consequence of the formula for the K-groups of a Laurent polynomial ring. The second also follows from this fact, together with analysis of the behavior of the p-th power map on such a Laurent extension. □

We now may construct a representational assembly in this case as well, to obtain a representational assembly

$$\alpha^{rep}_{Laurent} : KRep_k[G]^\wedge_\varepsilon \longrightarrow KA^\wedge_\varepsilon \cong KA$$

Proposition 6.2.15 *$\alpha^{rep}_{Laurent}$ is an equivalence of spectra.*

Proof We first consider the case $n = 1$, and the field $F = k((x))$, where we have already done the analysis. It is immediate that the inclusion $A \to F$ defined by $^{t} \to X$ extends to an equivariant homomorphism $B \to E$ of k-algebras, and consequently that we get a commutative diagram

$$\begin{array}{ccc} KRep_k[G] & & \\ {\scriptstyle \alpha^{rep}_{Laurent}} \downarrow & \searrow {\scriptstyle \alpha^{rep}_F} & \\ KA & \xrightarrow{\cong} & KF \end{array}$$

It is readily checked that the inclusion $KA \to KF$ is a weak equivalence, and so that the horizontal arrow in the diagram is a weak equivalence. On the other hand, we have already shown that $\alpha^{rep}F$ is a weak equivalence of spectra, which shows that $\alpha^{rep}_{Laurent}$ is an equivalence. For larger valued of n, the result follows by smashing copies of this example together over the coefficient S-algebra K_*k.

□

Now consider any geometric field F of characteristic zero, with $G_F \cong \mathbb{Z}_l^n$, and let k be an algebraically closed subfield. It is a direct consequence of Kummer theory that there is a family of elements $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ so that $E = \overline{F}$ can be written as

$$E = \bigcup_n F(\sqrt[l^n]{\alpha_1}, \sqrt[l^n]{\alpha_2}, \ldots, \sqrt[l^n]{\alpha_n})$$

It is therefore further clear that we may define a k-algebra homomorphism $i : B \to E$ by setting $i(t_i^{\frac{1}{l^n}}) = \sqrt[l^n]{\alpha_i}$, where $\sqrt[l^n]{\alpha_i}$ is chosen as the choice of l^n-th root so that the action of τ_i is via multiplication by ζ_n. It follows that we obtain a commutative diagram

$$\begin{array}{ccc} KRep_k[G]^{\wedge}_{\varepsilon} & & \\ {\scriptstyle \alpha^{rep}_{Laurent}} \downarrow & \searrow {\scriptstyle \alpha^{rep}_F} & \\ KA & \xrightarrow{Ki} & KF \end{array}$$

Since we now know that $\alpha^{rep}_{Laurent}$ is a weak equivalence, it will now suffice to show that Ki is a weak equivalence of spectra. It follows from the l-completed Bloch-Lichtenbaum spectral sequence that $K_*F \cong \Lambda_{K_*k}(\eta_1, \eta_2, \ldots, \eta_n)$. since we already know that $K_*A \cong \Lambda_{K_*k}(\theta_1, \theta_2, \ldots, \theta_n)$, it will suffice to show that the map Ki induces an isomorphism on π_1. But this is clearly the case, as one sees from Kummer theory.

Proposition 6.2.16 *α_F^{rep} is an equivalence of spectra for geometric fields of characteristic zero with abelian absolute Galois group.*

Remark: We believe that the techniques used here will also extend to the finite characteristic case, for characteristic prime to l. The techniques of section 6.5 allows us to extend to the general case of a field containing all the l-th power roots of unity.

6.3 Representational assembly in the geometric case

Suppose now that we are in the case of a *geometric field*, i.e. a field F containing an algebraically closed subfield k. Let E denote the algebraic closure of F, and let G denote the absolute Galois group. As in the case of the example of the previous section, we have a composite functor

$$Rep_k[G] \longrightarrow Rep_F[G] \cong V^G(F) \longrightarrow V^G(E)$$

and the induced map of completed spectra

$$KRep_k[G]^\wedge_\varepsilon \longrightarrow KV^G(E)^\wedge_\varepsilon \cong KF^\wedge_\varepsilon$$

We may ask whether this map is an equivalence as occurred in the case of the example in the previous section. In order to discuss the plausibility of this kind of statement, let us examine what occurs in the example, when $G = \mathbb{Z}_l$. In this case, $\pi_0 KRep_k[G] \cong \mathbb{Z}_l[\mathbb{Q}/\mathbb{Z}]$. This is a very large, non-Noetherian ring. However, after derived completion, we find that $\pi_0 KRep_k[G]^\wedge_\varepsilon \cong \mathbb{Z}_l$. On the other hand, we have from 3.4 that $\pi_0 KRep_k[G]^\wedge_\varepsilon$ is isomorphic to the ordinary (not derived) completion of the ring $R[G]$ at the ideal $J = (l) + I$, where I denotes the augmentation ideal in $R[G]$. So the conclusion is that $\varprojlim R[G]/J^n \cong \mathbb{Z}_l \cong (R[G]/I)^\wedge_l$. The reason this happens is the following result about the ideal I.

Proposition 6.3.1 *We have that I/I^2 is a divisible group, and that $I^k = I^{k+1}$ for all k.*

Proof It is a standard result for any group G that in the group ring $\mathbb{Z}[G]$, the augmentation ideal $I[G]$ has the property that $I[G]/I[G]^2 \cong G^{ab}$. Since $\mathbb{Q}/\mathbb{Z}$ is abelian, we find that $I/I^2 \cong \mathbb{Q}/\mathbb{Z}$, which is divisible. For the second statement, we note that I^k/I^{k+1} is a surjective image of $\underbrace{I/I^2 \otimes \ldots \otimes I/I^2}_{k \text{ factors}}$. But it is clear that $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z} = 0$, so $I^k/I^{k+1} = 0$. This gives the result. $\square$

Corollary 6.3.2 *We have that* $((l) + I)^k = (l^k) + I$. *Equivalently, for any element* $\theta \in I$*, and any positive integers* s *and* t*, there is an* $\eta \in I$ *and* $\mu \in I^k$ *so that*

$$\theta = l^s \eta + \mu$$

Proof We have

$$((l) + I)^k = \sum_t l^t I^{k-l} = (l^k) + l^{k-1} I + I^2$$

But 6.3.1 implies that $l^{k-1} I + I^2 = I$, which gives the result. $\square$

It now follows that $R[G]/J^k \cong R[G]/(I + (l^k)) \cong \mathbb{Z}/l^k\mathbb{Z}$, and therefore that the completion of $R[G]$ at the ideal J is just $\mathbb{Z}_l$.

This result extends to torsion free abelian profinite groups. That is, any such group also has the property that if I is the augmentation ideal in the representation ring, then $I^k = I^2$ for $k \geq 2$, and that I/I^2 is divisible. However, for non-abelian Galois groups, there appears to be no obvious reason why such a result should hold, and indeed we believe that it does not. However, there is a modification of this statement which is true for absolute Galois groups and which suggests a conjecture which can be formulated for any geometric field.

Recall that the representation ring of a finite group is not just a ring, but is actually a part of a *Green functor*. This was discussed in section 2, where there is a discussion of Mackey and Green functors. See [5] for a thorough discussion of these objects. A Mackey functor for a group G is a functor from a category of orbits to abelian groups. In the case of the representation ring, there is a functor $\mathcal{R}$ given on orbits by $\mathcal{R}(G/K) = R[K]$. The maps induced by projections of orbits induce restriction maps on representation rings, and transfers induce inductions. This functor is actually a commutative *Green functor*, in the sense that there is a multiplication map $\mathcal{R} \square \mathcal{R} \to \mathcal{R}$, which is associative and commutative. Also for any finite group G, there is another Green functor $\mathcal{Z}$ given on objects by $\mathcal{Z}(G/K) = \mathbb{Z}$, and for which projections of orbits induce the identity map and where transfers associated to projections induce multiplication by the degree of the projection. We may also consider $\mathcal{Z}/l\mathcal{Z}$, which is obtained by composing the functor $\mathcal{Z}$ with the projection $\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z}$. The *augmentation* is the morphism of Green functors $\mathcal{R} \to \mathcal{Z}$ which is given on an object G/K by the augmentation of $R[K]$. The mod-l augmentation ε is the composite $\mathcal{R} \to \mathcal{Z} \to \mathcal{Z}/l\mathcal{Z}$. The theory of Green functors is entirely parallel with the theory of rings, modules, and ideals. We may therefore speak of the *augmentation ideal* $\mathcal{I}$ and the ideal $\mathcal{J} = (l) + \mathcal{I}$, as well as the powers of these ideals. We can therefore also speak of completion at a Green functor ideal. We also observe that the theory of Mackey and Green functors estends in an obvious way to profinite groups, by considering the catgory of finite G-orbits. We want to identify a class of profinite groups G for which

$$\mathcal{R}_l^{\wedge} = \lim_{\leftarrow} \mathcal{R}/\mathcal{J}^n$$

is isomorphic to $\mathcal{Z}_l^{\wedge} = \mathbb{Z}_l \otimes \mathcal{Z}$.

Definition 6.3.3 *Let G be a profinite group. We say G is* totally torsion free *if every subgroup of finite index has a torsion free abelianization.*

Example: Free profinite groups and free profinite l-groups are totally torsion free.

Example: Free profinite abelian and profinite l-abelian groups.

Example: let Γ denote the integral Heisenberg group, i.e. the group of upper triangular integer matrices with ones along the diagonal. Then the profinite and pro-l completion of Γ is not totally torsion free.

We have

Proposition 6.3.4 *Let G be the absolute Galois group of a geometric field F. Let G_l denote the maximal pro-l quotient of G. Then G_l is totally torsion free.*

Proof Consider any subgroup K of finite l-power index in G. Then let E denote the extension of F corresponding to K. Then the abelianization of K corresponds to the maximal pro-l Abelian extension of E. Let $\mathbb{N}^{\cdot}$ deonte the partially ordered set of positive integers, where $m \leq n$ if and only if $m|n$. Define a functor Φ from $\mathbb{N}^{\cdot}$ to abelian groups by $\Phi(n) = k^*/(k^*)^n$, and on morphisms by $\Phi(m \to mn) = k^*/(k^*)^m \stackrel{(-)^n}{\to} k^*/(k^*)^{mn}$. Let $\mathcal{K}$ denote the direct limit over $\mathbb{N}^{\cdot}$ of Φ. Kummer theory then asserts the existence of a perfect pairing

$$K^{ab} \times \mathcal{K} \to \mathbb{Q}/\mathbb{Z}$$

So $K^{ab} \cong Hom(\mathcal{K}, \mathbb{Q}/\mathbb{Z})$. The group $\mathcal{K}$ is clearly divisible, and it is easily verified that the $\mathbb{Q}/\mathbb{Z}$-dual of a divisible group is torsion free. □

We have the following result concerning totally torsion free profinite groups.

Proposition 6.3.5 *Let G be a totally torsion free l-profinite group. Then the natural map $\mathcal{R}_l^\wedge \to \mathcal{Z}_l^\wedge$ is an isomorphism. Consequently, for G the maximal pro-l quotient of the absolute Galois group of a geometric field, we have that $\mathcal{R}_l^\wedge \cong \mathcal{Z}_l^\wedge$.*

Proof We will verify that $\mathcal{R}_l^\wedge(G/G) \to \mathcal{Z}_l^\wedge(G/G)$ is an isomorphism. The result at any G/K for any finite index subgroup will follow by using that result for the totally torsion free group K. It will suffice to show that for every finite dimensional representation ρ of G/N, where N is a normal subgroup of finite index, and every choice of positive integer s and t, there are elements $x \in \mathcal{R}(G/G)$ and $y \in \mathcal{I}^t(G/G)$ so that $[dim\rho] - [\rho] = l^s x + y$. We recall *Blichfeldt's theorem* [51], which asserts that there is a subgroup L of G/N and a one-dimensional representation ρ_L of L so that ρ is isomorphic to the representation of G/N induced up from ρ_L. It follows that $[dim\rho] - [\rho] = i_L^{G/N}(1 - \rho_L)$. Let $\overline{L}$ denote the subgroup $\pi^{-1}L \subseteq G$, where $\pi : G \to G/N$ is the projection, and let $\rho_{\overline{L}} = \rho_L \circ \pi$. Then we clearly also have $[dim\rho] - [\rho] = i_{\overline{L}}^G(1 - \rho_{\overline{L}})$. Now, $1 - \rho_{\overline{L}}$ is in the image of $R[\overline{L}^{ab}] \to R[\overline{L}]$, and let the corresponding one-dimensional representation of $\overline{L}^{ab}$ be $\rho_{\overline{L}^{ab}}$. Since $\overline{L}^{ab}$ is abelian and torsion free (by the totally torsion free hypothesis), we may write $1 - \rho_{\overline{L}^{ab}} = l^s\xi + \eta$, where $\eta \in I^t(\overline{L}^{ab})$ and where $\xi \in R[\overline{L}^{ab}]$, by 6.3.2. This means that we may pull ξ and η back along the homomorphism $\overline{L} \to \overline{L}^{ab}$, to get elements $\overline{\xi} \in R[\overline{L}]$ and $\overline{\eta} \in I^t(\overline{L})$ so that $\rho_{\overline{L}} = l^s\overline{\xi} + \overline{\eta}$. Since $\mathcal{I}^t$ is closed under induction, we have that $i_{\overline{L}}^G(\overline{\eta}) \in \mathcal{I}^t(G/G)$. Now we have that $[dim\rho] - \rho = i_{\overline{L}}^G(1 - \rho_{\overline{L}}) = p^s i_{\overline{L}}^G\overline{\xi} + i_{\overline{L}}^G\eta$. The result follows. □

We recall the relationship between Mackey functor theory and equivariant stable homotopy theory. The natural analogue for homotopy groups in the world of equivariant spectra takes its values in the category of Mackey functors. The Adams spectral sequence and the algebraic-to-geometric spectral sequence have E_2-terms which are computed using derived functors of Hom and □-product, which we will denote by *Ext* and *Tor* as in the non-equivariant case. We will now observe that the constructions we have discussed are actually part of equivariant spectra.

Proposition 6.3.6 *For any profinite group G, there is a stable category of G-spectra, which has all the important properties which the stable homotopy theory of G-spectra for finite groups has. In particular, there are fixed point subspectra for every subgroup of finite index as well as transfers for finite G-coverings. The "tom Dieck" filtration holds as for finite groups. The homotopy groups of a G-spectrum form a Mackey functor. Moreover, the homotopy groups of a G S-algebra are a Green functor, and the homotopy groups of a G-module spectrum are a module over this Green functor.*

Proof For any homomorphism $f : G \to H$ of finite groups, there is a pullback functor f^* from the category of H-spectra to the category of G-spectra. Hence, for a profinite group g, we get a direct limit system of categories parametrized by the normal subgroups of finite index, with the value at N being the category of G/N-spectra. A G spectrum can be defined as a family of spectra $\mathcal{S}_N$ in the category of G/N-spectra, together with isomorphisms

$$(G/N_1 \to G/N_2)^*(\mathcal{S}_{N_2}) \xrightarrow{\sim} \mathcal{S}_{N_1}$$

This general construction can be made into a theory of G spectra with the desired properties. □

Proposition 6.3.7 *Let F be a field, with E its algebraic closure, and G the absolute Galois group. Then there is a G-S-algebra with total spectrum $KV(G,E)$, with fixed point spectra $KV(G,E)^H \cong KV^H(E)$. The attached Green functor is given by*

$$G/L \to \pi_* KV(G,E)^L \cong \pi_* KV^L(E) \cong K_*(E^L)$$

Similarly, there is a G-S-algebra with total spectrum $KV(G,F)$, and with fixed point spectra $KV(G,F)^L \cong KV^L(F) \cong KRep_F[L]$. (Note that the G-action on F is trivial.) In this case, the associated Green functor is given by

$$G/L \to \pi_* KV(G,F)^L \cong \pi_* KV^L(F) \cong K_* Rep_F[L]$$

In the case when F contains all the l-th power roots of unity, we find that the Mackey functor attached to $KV(G,F)$ is $ku_* \otimes \mathcal{R}$. The functor $E \underset{F}{\otimes} -$ induces a map of G-S-algebra $KV(G,F) \to KV(G,E)$, which induces the functors of Proposition 6.1.6 on fixed point spectra.

Proof This result is essentially a consequence of the equivariant infinite loop space recognition principle [54]. □

We also recall the results of [31], where it was shown that in the category of G-equivariant spectra, there is an Eilenberg-MacLane spectrum for every Mackey functor. We will let $\mathcal{H}$ denote the Eilenberg-MacLane spectrum attached to the Green functor $\mathcal{Z}/l\mathcal{Z}$. Derived completions of homomorphisms of S-algebras are defined as in the non-equivariant case.

Proposition 6.3.8 *Let F be a geometric field, with the algebraically closed subfield k. There is a commutative diagram of G-S-algebras*

$$\begin{array}{ccc} KV(G,k) & \xrightarrow{E\otimes_k} & KV(G,E) \\ & \searrow^{\varepsilon} & \downarrow{\scriptstyle \varepsilon} \\ & & \mathcal{H} \end{array}$$

where, the maps ε are given by "dimension mod l". Consequently, there is a map of derived completions.

$$\alpha^{rep} : KV(G,k)^{\wedge}_{\varepsilon} \longrightarrow KV(G,E)^{\wedge}_{\varepsilon}$$

It follows from the equivariant algebraic-to-geometric spectral sequence that $\pi_0 KV(G,k)^{\wedge}_{\varepsilon} \cong \mathcal{R}^{\wedge}_l$, and this is isomorphic by Proposition 6.3.5 to $\mathcal{Z}^{\wedge}_l$, which is in turn isomorphic to the Green functor $\pi_0 KV(G,E)$. This makes the plausible the conjecture the following conjecture.

Conjecture 6.3.9 *Let F be a geometric field. The representational assembly map*

$$\alpha^{rep} : KV(G,k)^{\wedge}_{\varepsilon} \to KV(G,E)^{\wedge}_{\varepsilon}$$

is an equivalence of G-S-algebras. On G-fixed point spectra, we have an equivalence of S-algebras

$$(\alpha^{rep})^G : (KV(G,k)^{\wedge}_{\varepsilon})^G \longrightarrow (KV(G,E)^{\wedge}_{\varepsilon})^G \cong K(E^G) \cong KF$$

6.4 Representational assembly in the twisted case

In the case of a field F which does not contain an algebraically closed subfield, it is not as easy to construct a model for the K-theory spectrum of the field. As usual, let F denote a field, E its algebraic closure, and G its absolute Galois group. We observe first that for any subfield of $F' \subseteq E$ which is closed under the action of G, we obtain a map

$$KV(G,F')^{\wedge}_{\varepsilon} \longrightarrow KV(G,E)^{\wedge}_{\varepsilon}$$

We may conjecture (as an extension of Conjecture 6.3.9 above) that this map is always an equivalence, and indeed we believe this to be true. In the case when F' is algebraically closed, we have believe that the domain of the map has what we would regard as a simple form, i.e. is built out of representation theory of G over an algebracially closed field. In general, though, the K-theory of the category $V^G(F')$ may not be a priori any simpler than the K-theory of F. However, when F' is a field whose K-theory we already understand well, then we expect to obtain information this way. In addition to algebraically closed fields, we have an understanding of the K-theory of finite fields from the work of Quillen [45]. So, consider the case where F has finite characteristic p, distinct from l. We may in this case let F' be the maximal finite subfield contained in F, which is of the form $\mathbb{F}_q$, where $q = p^n$ for some n. In this case we have, in parallel with Conjecture 6.3.9,

Conjecture 6.4.1 *For F of finite characteristic $p \neq l$, with $\mathbb{F}_q$ the maximal finite subfield of F, the map*

$$KV(G, \mathbb{F}_q)^{\wedge}_{\varepsilon} \longrightarrow KV(G, E)^{\wedge}_{\varepsilon}$$

is an equivalence of G-S-algebras. In particular, we have an equivalence

$$(KV(G, \mathbb{F}_q)^{\wedge}_{\varepsilon})^G \simeq KV(G, E)^{\wedge}_{\varepsilon} \cong KF$$

We now argue that this is a reasonable replacement for Conjecture 6.3.9 in this case. We note that the fixed point category of $V(G, \mathbb{F}_q)$ is the category $V^G(\mathbb{F}_q)$ of linear descent data. We have the straightforward observation

Proposition 6.4.2 *For any finite group G acting on $\mathbb{F}_q$ by automorphisms, the category of linear descent data $V^G(\mathbb{F}_q)$ is equivalent to the category of left modules over the twisted group ring $\mathbb{F}_q\langle G\rangle$. Similarly, for a profinite group G acting on $\mathbb{F}_q$, we find that the category of linear descent data is equivalent to the category of continuous modules over the Iwasawa algebra version of $\mathbb{F}_q\langle G\rangle$.*

Note that $\mathbb{F}_q\langle G\rangle$ is a finite dimensional semisimple algebra over $\mathbb{F}_p$, so it is a product of matrix rings over field extensions of $\mathbb{F}_p$. Consequently, we should view its K-theory as essentially understood, given Quillen's computations for finite fields. For this reason, we regard the above conjecture as a satisfactory replacement for Conjecture 6.3.9.

In the characteristic 0 case, though, we have more difficulties, since in this case we do not know the K-theory of the prime field $\mathbb{Q}$. Indeed, we would like to make conjectures about the K-theory of $\mathbb{Q}$ involving the representation theory of $G_{\mathbb{Q}}$. We do, however, thanks to the work of Suslin [58], understand the K-theory of the field $\mathbb{Q}_p$ and $\mathbb{Z}_p$ where $p \neq l$.

Theorem 6.4.3 (Suslin; see [58]) *Let K denote any finite extension of $\mathbb{Q}_p$, and let $\mathcal{O}_K$ denote its ring of integers. Let (π) denote its unique maximal ideal. The quotient homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ induces an isomorphism on l-completed K-theory.*

Now, suppose we have any field F containing $\mathbb{Q}_p$, containing the l-th roots of unity. and let E denote its algebraic closure. Let L be $\bigcup_n \mathbb{Q}_p(\zeta_{l^n}) \subseteq E$. L is of course closed under the action of the absolute Galois group $G = G_F$. By abuse of notation, we will write $V(G, \mathcal{O}_L)$ for the category of twisted $G-\mathcal{O}_L$-modules over $\mathcal{O}_L$, i.e. finitely generated $\mathcal{O}_L$-modules M equipped with a G-action so that $g(rm) = r^g g(m)$ for all $r \in \mathcal{O}_L$, $m \in M$, and $g \in G$. The following is an easy consequence of the Theorem 6.4.3 above.

Proposition 6.4.4 *The functor $V(G, \mathcal{O}_L) \to V(G, \mathcal{O}_L/\pi\mathcal{O}_L)$ induces an equivalence $K^G\mathcal{O}_L \to K^G\mathcal{O}_L/\pi\mathcal{O}_L$.*

Since $\mathcal{O}_L/\pi\mathcal{O}_L$ is a semisimple algebra over a finite field, we will regard it as an understood quantity. This means that in this case, we also have a version of the representational assembly via the diagram

$$(K^G\mathcal{O}_L/\pi\mathcal{O}_L)^\wedge_\varepsilon \xleftarrow{\sim} (K^G\mathcal{O}_L)^\wedge_\varepsilon \xrightarrow{E \underset{\mathcal{O}_L}{\otimes}} (K^G E)^\wedge_\varepsilon \cong KF$$

where as usual ε is the natural map defined to the Eilenberg-MacLane spectrum attached to the Green functor $\mathcal{Z}$. This kind of result extends to the case where F contains a Henselian local ring which residue class field algebraic over a finite field.

6.5 The ascent map and assembly for the case $\mu_{l^\infty} \subseteq F$

The constructions of the previous section provide explicit maps from derived completions of spectra attached to representation categories of the absolute Galois group G of a field F to the KF. However, they do not apply sufficiently generally. For instance, even in the non-twisted situation (i.e. with trivial action of G on the l-th power roots of unity), we do not obtain a representational assembly for a general field of characteristic zero containing the l-th power roots of unity. In this section, we will discuss a method, which we call the *ascent* method, which allows us the construct the assembly in the case where the l-th power roots of unity are contained in F. The method also gives an alternate criterion for verifying that the representational assembly is an equivalence. The terminology "ascent" refers to the fact that the method gives a description of KE coming from information about KF, rather than describing KF in terms of KE, as is done in the descent. In the interest of clarity, we will first describe the approach as it works in the abelian case, where it is not necessary to pass to the generality of Mackey and Green functors, and then indicate the changes necessary when we deal with the more general situation. Finally, we suppose that F contains all the l-th power roots of unity.

Consider the forgetful functor $V^G(E) \xrightarrow{\phi} V(G,E) \cong Vect(E)$, which simply forgets the G-action. It induces a map

$$KF \cong KV^G(E) \xrightarrow{K\phi} KV(G,E) \cong KE$$

We now have a commutative diagram

$$\begin{array}{ccc}
KRep_F[G] \wedge KV^G(E) & \longrightarrow & KV^G(E) \\
\Big\downarrow {\scriptstyle \varepsilon \wedge id} & & \Big\downarrow {\scriptstyle K\phi} \\
KF \wedge KV^G(E) & \longrightarrow & KE \\
\Big\downarrow {\scriptstyle id \wedge K\phi} & & \Big\downarrow {\scriptstyle id} \\
KF \wedge KE & \longrightarrow & KE
\end{array}$$

of spectra, where the horizontal map are multiplication maps defining algebra structures, and where the vertical maps are built out of the augmentation map ε and the forgetful functor ϕ. Note that the middle horizontal map exists because ε is induced by the forgetful functor $Rep_F[G] \to Vect(F)$ which forgets the G action. Considering the four corners of this diagram, we obtain a map of spectra

$$asc_F : KF \underset{KRep_F[G]}{\wedge} KV^G(E) \longrightarrow KV(G,E) \cong KE$$

It is readily verified that asc_F is a homomorphism of commutative S-algebras. This statement can be formulated as stating that the commutative diagram

$$
\begin{array}{ccccc}
 & & KV(G,E) \cong KE & & \\
 & \overset{i}{\nearrow} & & \nwarrow & \\
KV^G(E) & & & & KF \\
 & \nwarrow & & \overset{\varepsilon}{\nearrow} & \\
 & & KRep_F[G] & &
\end{array}
$$

is a pushout in the category of commutative S-algebras. We will now examine the consequences of the assumption that asc_F is a weak equivalence.

For any homomorphism of S-algebras $f : D \to E$, we let $T^{\cdot}(D \to E)$ denote the cosimplicial S-algebra given by

$$
T^k(D \to E) = \underbrace{E \underset{D}{\wedge} E \underset{D}{\wedge} \cdots \underset{D}{\wedge} E}_{k+1 \text{ factors}}
$$

so $Tot(T^{\cdot}(D \to E)) \cong D^{\wedge}_{f}$. Whenever we have a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & B'
\end{array}
$$

of S-algebras, we have an induced map

$$
T^{\cdot}(A \to B) \to T^{\cdot}(A' \to B')
$$

Now suppose we have a two S-algebra homomorphisms $A \to B$ and $A \to C$, together with an S-algebra homomorphism $k \to A$, so that each of the composites $k \to A \to B$ and $k \to A \to C$ are weak equivalences. We then obtain a commutative diagram of commutative S-algebras

$$
\begin{array}{ccc}
B & \longrightarrow & B \underset{A}{\wedge} C \\
\downarrow & & \downarrow \\
B \underset{A}{\wedge} C \underset{k}{\wedge} A & \longrightarrow & B \underset{A}{\wedge} C
\end{array}
$$

The upper horizontal arrow and the left hand vertical arrow are obvious inclusions, the right hand arrow is the identity, and the lower horizontal arrow is the multiplication map obtained by regarding $B \underset{A}{\wedge} C$ as a right module over the k-algebra A. We consequently obtain a map of cosimplicial S-algebras

$$T^{\cdot}(B \to B \underset{A}{\wedge} C) \longrightarrow T^{\cdot}(B \underset{A}{\wedge} C \underset{k}{\wedge} A \to B \underset{A}{\wedge} C)$$

Proposition 6.5.1 *Under the hypotheses above, i.e. that $k \to B$ and $k \to C$ are weak equivalences, this map of cosimplicial S-algebras is a levelwise weak equivalence, and hence induces an equivalence on taking total spectra.*

Proof There is a commutative diagram

$$\begin{array}{ccc}
B \underset{A}{\wedge} T^{\cdot}(A \to C) & \xrightarrow{\sim} & T^{\cdot}(B \to B \underset{A}{\wedge} C) \\
\downarrow & & \downarrow \\
B \underset{A}{\wedge} T^{\cdot}(C \underset{k}{\wedge} A \to C) & \xrightarrow{\sim} & T^{\cdot}(B \underset{A}{\wedge} C \underset{k}{\wedge} A \to B \underset{A}{\wedge} C)
\end{array}$$

where the horizontal arrows are levelwise weak equivalences. This follows from a standard isomorphism involving tensor products, which is verified to work in the context of commutative S-algebras. It therefore suffices to verify that the map $T^{\cdot}(A \to C) \to T^{\cdot}(C \underset{k}{\wedge} A \to C)$ is a levelwise weak equivalence. But this follows readily from the fact that the composite

$$A \cong k \underset{k}{\wedge} A \longrightarrow C \underset{k}{\wedge} A$$

is a weak equivalence. □

We now examine the consequences of this result in our case. The spectra will be $k = KF$, $A = KRep_F[G]$, $B = KV^G(E)$, and $C = KF$, viewed as an A-algebra via the augmentation which forgets the action. This data clearly satisfies the hypotheses above. If asc_F is a weak equivalence, then the arrow $B \to B \underset{A}{\wedge} C$ can be identified with the arrow $KV^G(E) \to KV(G, E)$, which in turn can be identified canonically with the arrow $KF \to KE$. It follows that

$$Tot(T^{\cdot}(B \to B \underset{A}{\wedge} C)) \cong Tot(T^{\cdot}(KF \to KE)) \cong KF_l^{\wedge}$$

On the other hand, the algebra $B \underset{A}{\wedge} C \underset{k}{\wedge} A$ is weakly equivalent to $KE \underset{KF}{\wedge} KRep_F[G]$, by the assumption that asc_F is a weak equivalence. We now have

Proposition 6.5.2 *When all l-th power roots of unity are in F, the evident map $KRep_F[G] \underset{KF}{\wedge} KE \to KRep_E[G]$ is a weak equivalence. Consequently, $B \underset{A}{\wedge} C \underset{k}{\wedge} A \cong KRep_E[G]$.*

Proof It is a standard fact in representation theory that for any field F which contain the l-th power roots of unity, the isomorphism classes of representations of any finite l-group are in bijective correspondence with the isomorphism classes of representations of the same group in $\mathbb{C}$. Moreover, the endomorphism ring of any irreducible representation is a copy of F. We therefore have isomorphisms

$$\pi_* KRep_F[G] \cong K_*F \otimes R[G]$$

and

$$\pi_* KRep_E[G] \cong K_*E \otimes R[G]$$

where $R[G]$ denotes the complex representation ring. The E_2-term of the Künneth spectral sequence for $\pi_* KRep_F[G] \underset{KF}{\wedge} KE$ now has the form

$$Tor_{K_*F}(K_*F \otimes R[G], K_*E) \cong K_*Rep_E[G]$$

which gives the result. □

The fact that asc_F is an equivalence shows that $B \underset{A}{\wedge} C \cong KE$, and it is clear that the arrow $B \underset{A}{\wedge} C \underset{k}{\wedge} A \to B \underset{A}{\wedge} C$ corresponds to the augmentation $KRep_E[G] \to KE$. We now have the following conclusion.

Theorem 6.5.3 *Suppose that F contains all the l-th power roots of unity. Suppose further thatn asc_F is a weak equivalence. Then we have a weak equivalence*

$$KF_l^\wedge \cong KRep_E[G]_\varepsilon^\wedge$$

where $\varepsilon : KRep_E[G] \to KE$ denotes the augmentation.

We observed in Section 6.3 that we did not expect to have an equivalence

$$KF \cong KRep_k[G]_\varepsilon^\wedge$$

as stated except in the abelian case, but that there is an analogous "fully equivariant version" involving equivariant spectra and Green functors which we expect to hold in all cases. This suggests that we should not expect the ascent map as formulated above to hold except in the abelian case. However, it is possible to modify the ascent map to extend it to the fully equivariant version of the completion, and we indicate how. Let $\mathbb{H}$ denote the equivariant Eilenberg-MacLane spectrum attached to the Mackey functor $\mathcal{Z}/l\mathcal{Z}$ of section 6.3. We have a commutative diagram of equivariant S-algebras

$$\begin{array}{ccc}
KV(G,F) & \xrightarrow{E \underset{F}{\otimes} -} & KV(G,E) \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\
\mathbb{H} & \xrightarrow{id} & \mathbb{H}
\end{array}$$

and a consequent map $KV(G,F)^{\wedge}_{\varepsilon} \to KV(G,E)^{\wedge}_{\varepsilon}$. We have the cosimplicial G-S-algebras $\mathcal{C}^{\cdot}$ and $\mathcal{D}^{\cdot}$ given in codimension k by

$$\mathcal{C}^k = \underbrace{\mathbb{H} \underset{KV(G,F)}{\wedge} \mathbb{H} \underset{KV(G,F)}{\wedge} \cdots \underset{KV(G,F)}{\wedge} \mathbb{H}}_{k+1 \text{ factors}}$$

and

$$\mathcal{D}^k = \underbrace{\mathbb{H} \underset{KV(G,E)}{\wedge} \mathbb{H} \underset{KV(G,E)}{\wedge} \cdots \underset{KV(G,E)}{\wedge} \mathbb{H}}_{k+1 \text{ factors}}$$

respectively. We have an evident map $\mathcal{C}^{\cdot} \to \mathcal{D}^{\cdot}$ which induces the map of completions given above. Because of the convergence theorem 3.10, we find that $Tot(\mathcal{D}^{\cdot})^G \cong KV(G,E)^G \cong KF$. Also on ambient spectra, we find that $Tot\mathcal{C}^{\cdot} \cong KF$ and $Tot\mathcal{D}^{\cdot} \cong KE$. We have a commutative diagram of S-algebras

$$\begin{array}{ccc} (\mathcal{C}^{\cdot})^G & \longrightarrow & (\mathcal{D}^{\cdot})^G \\ \downarrow & & \downarrow \\ \mathcal{C}^{\cdot} & \longrightarrow & \mathcal{D}^{\cdot} \end{array}$$

and consequently a map of cosimplicial S-algebras

$$asc_F : \mathcal{C}^{\cdot} \underset{(\mathcal{C}^{\cdot})^G}{\wedge} (\mathcal{D}^{\cdot})^G \longrightarrow \mathcal{D}^{\cdot}$$

The induced map of total spectra is the proper analogue of the ascent map from above, and it yields a map

$$(KV(G,F)^{\wedge}_{\varepsilon})^G \to KF^{\wedge}_l$$

for all fields F containing all the l-th power roots of unity.

6.6 Derived representation theory and Milnor K-theory

It is interesting to speculate what the consequences of these ideas would be for the standard motivic spectral sequence. In 5.5, we conjectured that the l-adic

Bloch-Lichtenbaum spectral sequence and the descent spectral sequence based on the ring homomorphism $KF \to \mathbb{L}$, where $\mathbb{L}$ denotes the l-adic Eilenberg-MacLane spectrum are identical. On the other hand, we have the homotopy spectral sequence for $\pi_* KRep_E[G]^\wedge_\varepsilon$, based on the cosimplicial spectrum attached to the map of equivariant S-algebras $KV(G, E^0) \to \mathbb{L}$, where E^0 denotes E equipped with trivial G-action and where $\mathbb{L}$ denotes the equivariant Eilenberg-MacLane spectrum attached to the Mackey functor $\mathcal{Z}$. We propose the following plausible conjectures about these spectral sequences.

Conjecture 6.6.1 *The homotopy spectral sequence for $\pi_* KRep_E[G]^\wedge_\varepsilon$ has the property that $E_2^{i,j} = 0$ for $3i + 2j < 0$ (recall that this spectral sequence is a second quadrant spectral sequence). We have isomorphisms*

$$E_2^{-2i,3i} \cong \pi_i \mathcal{R}^\wedge_\varepsilon(G/G)$$

where $\mathcal{R}$ denotes the representation ring Green functor, and $\mathcal{R}^\wedge_\varepsilon$ denotes its derived completion at the ring homomorphism $\mathcal{R} \to \mathcal{Z}/l\mathcal{Z}$.

Remark: We view this conjecture is plausible because we believe that the p-adic motivic spectral sequence is just an accleration of the descent spectral sequence attached to the ring map $KF \to KE$, which in turn can be identified with the algebraic-to-geometric spectral sequence.

Conjecture 6.6.2 *Suppose that F contains all the l-th power roots of unity. Then there is a canonical isomorphism of spectral sequences from the homotopy spectral sequence for $\pi_* KRep_E[G]^\wedge_\varepsilon$ and the l-adically completed Bloch-Lichtenbaum spectral sequence. Moreover, under this isomorphism, we obtain an isomorphism*

$$\pi_i \mathcal{R}^\wedge_\varepsilon(G/G) \simeq K_i^M(F)^\wedge_l$$

Remark: Assuming the validity of this conjecture, it would be particularly interesting to find a direct algebraic interpretation of this isomorphism from the derived representation theory of the absolute Galois group of F to the Milnor K-theory.

In the case when F only contains the l-th roots of unity, we first define a new Green functor $\mathcal{R}_\mu$ attached to G, by which we mean the functor $G/H \to K_0 V^H(F_0)$, where $F_0 \subseteq F$ is the algebraic closure of $\mathbb{F}_p$ in F. For the case of fields of finite characteristic p, we have the following parallel conjectures to Conjectures 6.6.1 and 6.6.2 above.

Conjecture 6.6.3 *Suppose that F contains the l-th roots of unity. The homotopy spectral sequence for $\pi_* KRep_E[G]^\wedge_\varepsilon$ has the property that $E_2^{i,j} = 0$ for $3i + 2j < 0$ (recall that this spectral sequence is a second quadrant spectral sequence). We have isomorphisms*

$$E_2^{-2i,3i} \cong \pi_i(\mathcal{R}_\mu)^\wedge_\varepsilon(G/G)$$

where $\mathcal{R}_\mu$ is defined above.

Conjecture 6.6.4 *Suppose that F contains the l-th roots of unity. Then there is a canonical isomorphism of spectral sequences from the homotopy spectral sequence for $\pi_* KV^G(E_\mu)^\wedge_\varepsilon$ and the l-adically completed Bloch-Lichtenbaum spectral sequence. Moreover, under this isomorphism, we obtain an isomorphism*

$$\pi_i(\mathcal{R}_\mu)^\wedge_\varepsilon(G/G) \simeq K_i^M(F)^\wedge_l$$

There should also be an isomorphism

$$\pi_i((\mathcal{R}_\mu)^\wedge_\varepsilon; \mathbb{Z}/l\mathbb{Z})(G/G) \simeq K_i^M(F; \mathbb{F}_l)$$

where $\pi_*(-;\mathbb{Z}/l\mathbb{Z})$ denotes mod l homotopy, defined as $\pi_*(X;\mathbb{Z}/l\mathbb{Z}) \cong \pi_*(X \wedge M_l)$, where M_l denotes the mod l Moore space.

6.7 Derived representation theory and deformation K-theory

So far, our understanding of the derived completion of representation rings is limited to the knowledge of spectral sequences for computing them, given information about *Tor* or *Ext* functors of these rings. In this (entirely speculative) section, we want to suggest that there should be a relationship between derived representation theory and deformations of representations.

We consider first the representation theory of finite l-groups G. In this case, the derived representation ring of G is just the l-adic completion of $R[G]$, as can be verified using the results of Atiyah [1]. In particular, the derived representation ring has no higher homotopy. However, when we pass to a profinite l-group, such as $\mathbb{Z}_l$, we find that the derived representation ring does have higher homotopy, a single copy of $\mathbb{Z}_l$ in dimension 1. This situation appears to be parallel to the following situation. Consider the infinite discrete group $\Gamma = \mathbb{Z}$. We may consider the category $Rep_{\mathbb{C}}[\Gamma]$ of finite dimensional representations of Γ, and its K-theory spectrum $KRep_{\mathbb{C}}[\Gamma]$. The homotopy of this spectrum is given by

$$\pi_* KRep_{\mathbb{C}}[\Gamma] \cong \mathbb{Z}[S^1] \otimes K_*\mathbb{C}$$

where S^1 is the circle regarded as a discrete group. This isomorphism arises from the existence of Jordan normal form, which (suitably interpreted) shows that every representation of Γ admits a filtration by subrepresentations so that the subquotients are one-dimensional, and are therefore given by multiplication by a uniquely defined non-zero complex number. This construction does not take into account the topology of $\mathbb{C}$ at all. We might take the topology into account as in the following definition.

Definition 6.7.1 *Consider any discrete group Γ. For each k, we consider the category $Rep^k_{\mathbb{C}}[\Gamma]$ whose objects are all possible continuous actions of Γ on $\Delta[k] \times V$ which preserve the projection $\pi : \Delta[k] \times V \to \Delta[k]$, and which are linear on each fiber of π. It is clear that the categories $Rep^k_{\mathbb{C}}[\Gamma]$ fit together into a simplicial symmetric monoidal category, and we define the* **deformation K-theory spectrum of** Γ*, $K^{def}[\Gamma]$, as the total spectrum of the simplicial spectrum*

$$k \to KRep^k_{\mathbb{C}}[\Gamma]$$

Remark: The terminology is justified by the observation that, for example, an object in the category of 1-simplices $Rep^1_{\mathbb{C}}[\Gamma]$ is exactly a path in the space of representations of Γ in $GL(V)$, or a deformation of the representation at $0 \times V$.

One can easily check that $\pi_0 K^{def}[\mathbb{Z}] \cong \mathbb{Z}$, with the isomorphism given by sending a represenatation to its dimension. This follows from the fact that any two representations of the same dimension of $\mathbb{Z}$ can be connected by a deformation. In fact, one can apply the functor π_0 levelwise to the simplicial spectrum $KRep^{\cdot}_{\mathbb{C}}[\mathbb{Z}]$, and attempt to compute the homotopy groups of the simplicial abelian group $k \to \pi_0 KRep^k_{\mathbb{C}}[\mathbb{Z}]$. As above, it is easy to see that π_0 of this simplicial abelian group is zero, and it appears likely that

$$\pi_*(\pi_0 KRep^{\cdot}_{\mathbb{C}}[\mathbb{Z}]) \cong \pi_*\mathbb{Z}[S^1] \cong H_*(S^1)$$

where S^1 is the circle regarded as a topological group. Note that S^1 is the character group of the original discrete group $\mathbb{Z}$.

Now consider the profinite group $\mathbb{Z}_l$. We may define the deformation K-theory of this group as the direct limit of its finite quotients, and in view of the rigidity of complex representations of finite groups we have that

$$KRep_{\mathbb{C}}[\mathbb{Z}_l] \cong K^{def}[\mathbb{Z}_l]$$

We have an inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_l$, and therefore a map of spectra $K^{def}[\mathbb{Z}_l] \to K^{def}[\mathbb{Z}]$. We now obtain a composite map of derived completions

$$KRep_{\mathbb{C}}[\mathbb{Z}_l]^{\wedge}_{\varepsilon} \simeq K^{def}[\mathbb{Z}_l]^{\wedge}_{\varepsilon} \longrightarrow K^{def}[\mathbb{Z}]^{\wedge}_{\varepsilon}$$

Since the homotopy groups on the two sides appear isomorphic, it appears likely that this composite is a homotopy equivalence. We now observe that for any discrete group Γ, we may consider its pro-l completion $\Gamma^{\wedge}_l$, and obtain a map

$KRep_{\mathbb{C}}[\Gamma_l^\wedge] \to K^{def}[\Gamma]$. This map induces a map j of derived completions, and we ask the following question.

Question: For what discrete groups does the map j induce a weak equivalence of spectra after derived completion at the augmentation map $\varepsilon : K^{def}[\Gamma] \to K\mathbb{C}$? Note that the completion may have to be taken in the category of $\Gamma_l^\wedge$-equivariant spectra.

We can further ask if the effect of derived completion can be computed directly on the pro-l group, rather than by permitting deformations on a discrete subgroup. This cannot be achieved over $\mathbb{C}$ since representations of finite groups are rigid, but perhaps over some other algebraically closed field.

Question: Can one construct a deformation K-theory of representations of a pro-l group G, using an algebraic deformation theory like the one discussed in [38], so that it coincides with the derived completion of $KRep_{\mathbb{C}}[G]$?

We conclude by pointing out an analogy between our theory of the representational assembly and other well known constructions. Consider a $K(\Gamma, 1)$ manifold X. The category $VB(X)$ of complex vector bundles over X is a symmetric monoidal category, and we can construct its K-theory. On the other hand, we have a functor from the category of complex representations of Γ to $VB(X)$, given by

$$\rho \to (\tilde{X} \underset{\Gamma}{\times} \rho \to X)$$

where $\tilde{X}$ denotes the universal covering space of X. This functor produces a map of K-theory spectra. It is also easy to define a "deformation version" of $KVB(X)$, and one obtains a map

$$K^{def}[\Gamma] \to K^{def} VB(X)$$

This map should be viewed as the analogue in this setting for the representational assembly we discussed in the case of K-theory of fields, using the point of view that fields are analogues of $K(G, 1)$ manifolds in the algebraic geometric context.

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