Math 122 2013 HW 1

Solutions

Problem 1

$IM \neq \emptyset$, since $0 \in IM$. Let $r \in R$ and $x, y \in IM$. Write

$$x = \sum_{i=1}^{n} a_i m_i$$

and

$$y = \sum_{i=1}^{k} b_i n_i,$$

for some $k, n \in \mathbb{N}$, $a_i, b_j \in I$ and $m_i, n_i \in M$. Then

$$x + ry = \sum_{i=1}^{n} a_i m_i + \sum_{i=1}^{k} (rb_i) n_i,$$

and $rb_i \in I$ sice I is a left ideal. Further, this sum is finite, so $x + ry \in IM$, so IM is a submodule using the submodule criterion. (Note that some of the m_i and n_i can be equal, which doesn't change the finiteness of the sum)

Problem 2

(a) First note that $\mathrm{Tor}(M) \neq \emptyset$, since $0 \in \mathrm{Tor}(M)$.

Let $x, y \in \mathrm{Tor}(M)$ and $r \in R$. Since x, y are torsion, let $r_1, r_2 \in R$ nonzero be such that $r_1 x = r_2 y = 0$. Since R is an integral domain, $r_1 r_2 \neq 0$ and $r_1 r_2 = r_2 r_1$ and $r_1 r_2 r = r_1 r r_2$. We have

$$\begin{aligned} r_1 r_2 (x + ry) &= r_1 r_2 x + r_1 r_2 ry \\ &= r_2 (r_1 x) + r_1 r (r_2 y) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

So $x + ry \in \mathrm{Tor}(M)$, and using the submodule criterion, we conclude that $\mathrm{Tor}(M)$ is a submodule of M.

(b) Take $R = \mathbb{Z}_6$, and view $M = R$ as a module over itself. Then $\mathrm{Tor}(M) = \{0, 2, 3, 4\}$, which is not a submodule, since $2 + 3 = 5 \notin \mathrm{Tor}(M)$.

(c) Since R has zero divisors, let $a, b \in R$ be such that a, b are nonzero and $ab = 0$. Take any $m \in M$ nonzero.

If $bm = 0$, then $m \in \mathrm{Tor}(M)$, so M has nonzero torsion elements. If $bm \neq 0$, then $a(bm) = (ab)m = 0 \cdot m = 0$, and $a \neq 0$. Hence $bm \in \mathrm{Tor}(M)$, and again $\mathrm{Tor}(M)$ has nonzero elements.

Problem 3

Note that $0 \in \mathrm{Ann}(I)$, so $\mathrm{Ann}(I) \neq \emptyset$.

Now let $m_1, m_2 \in \mathrm{Ann}(I)$, and let $r \in R$. For any $a \in I$, we have $am_1 = 0$, since $m_1 \in \mathrm{Ann}(I)$. Since I is a right ideal of R, we have $ar \in I$, and $m_2 \in \mathrm{Ann}(I)$, so $(ar)m_2 = 0$. Then we get that

$$\begin{aligned} a(m_1 + rm_2) &= am_1 + (ar)m_2 \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

and since this holds for all $a \in I$, $m_1 + rm_2 \in \mathrm{Ann}(I)$. By the submodule criterion, $\mathrm{Ann}(I)$ is a submodule of M.

Problem 4

Since there is a bijection between $F[x]$-submodules and subspaces of V which are T-stable, it is enough to find all the T-stable subspaces. It is easy to see that U_k is T-stable, for all $k \leq n$.

Now let U be a T-stable subspace. Let $0 \leq k \leq n$ be maximum such that U contains a vector $v = (v_1, \ldots, v_k, 0, \ldots, 0)$ with $v_k \neq 0$. Then

$$v_k^{-1} T^{k-1}(v) = e_1 \in U,$$

since U is T-stable. Now since $e_1 \in U$ and U is T-stable, we get that

$$e_2 = v_k^{-1}(T^{k-2}(v) - v_{k-1}e_1) \in U.$$

Repeating this argument gives that $e_1, \ldots, e_k \in U$. Since $\{e_1, \ldots, e_k\}$ is a basis for U_k, it follows that $U_k \subset U$. Now if $U \neq U_k$, then U must contain an element w with the j^{th} entry nonzero, for $j > k$, contradiction with the maximality of k.

Hence $U = U_k$.

Problem 5

Note that the $F[x]$-submodules are the T-stable subspaces of $\mathbb{R}^2$, so it is enough to determine the T-stable subspaces. Let W be a T-stable subspace. Notice that $T(x, y) = (0, y)$, for all $x, y \in \mathbb{R}$.

If $\dim(W) = 0$, then $W = 0$, which is easily seen to be T-stable.

If $\dim(W) = 1$, let (a, b) be a generator of W. If $a = 0$, then since $b \neq 0$, W is the y-axis, which is easily seen to be T-stable (indeed, $T(0, y) = (0, y)$, so $T(W) = W$. If $b = 0$, then we must have $a \neq 0$ and hence W is the x-axis, which again is T-stable ($T(x, 0) = (0, 0)$, so $T(W) = \{0\} \subset W$).

If $a, b \neq 0$, then since $T(W) \subset W$, we must have $T(a, b) = \lambda(a, b)$, and since $T(a, b) = (0, b)$, we get that $\lambda b = b$. Since $b \neq 0$, we get $\lambda = 1$, so $\lambda a = a = 0$, which is a contradiction. So the only 1-dimensional subspaces which are T-stable are the x-axis and the y-axis.

If W is two-dimensional, $W = V$ which is clearly T-stable.

Problem 6

Since a $\mathbb{Z}$-module is the same as an abelian group, it is enough to find the group homomorphisms $\phi : \mathbb{Z}/30\mathbb{Z} \to \mathbb{Z}/21\mathbb{Z}$. Any such homomorphism is uniquely determined by the value of $\phi(x)$, where x is a generator of $\mathbb{Z}/30\mathbb{Z}$, since $\mathbb{Z}/30\mathbb{Z}$ is cyclic. Since $30x = 0$ in $\mathbb{Z}/30\mathbb{Z}$, it follows that $0 = \phi(0) = \phi(30x) = 30\phi(x)$. Since $30 = 9 \bmod 21$, we get that $9\phi(x) = 0$ in $\mathbb{Z}/21\mathbb{Z}$, so $\phi(x) \in \{0, 7, 14\}$. Clearly $\phi : x \mapsto ax$ is a homomorphism, with $a \in \{0, 7, 14\}$, so these are all the $\mathbb{Z}$-homomorphisms.

Problem 7

Let $\phi : M \to M$, $\phi(m) = zm$. Let $m_1, m_2 \in M$ and $r \in R$. Then

$$\begin{aligned}\phi(m_1 + rm_2) &= z(m_1 + rm_2) \\ &= zm_1 + zrm_2 \\ &= zm_1 + rzm_2 \\ &= \phi(m_1) + r\phi(m_2),\end{aligned}$$

where in the third line we used the fact that $zr = rz$, since z is in the center of R. So ϕ is an R-module homomorphism.

Now let $\alpha : R \to \mathrm{End}_R(M)$ be defined by $\alpha(r) = \alpha_r = rI$. First note that since R is commutative, $\alpha(r)$ is indeed an R-module homomorphism.

Now let $m \in M$ and $r_1, r_2 \in R$. Then

$$\begin{aligned}\alpha_{r_1+r_2}(m) = (r_1 + r_2) \cdot m &= (r_1 \cdot m) + (r_2 \cdot m) \\ &= \alpha(r_1)(m) + \alpha(r_2)(m)\end{aligned}$$

Since this holds for any $m \in M$, it follows that $\alpha_{r_1+r_2} = \alpha_{r_1} + \alpha_{r_2}$. Now

$$\begin{aligned}\alpha(r_1r_2)(m) &= (r_1r_2)m = (r_2r_1)m = r_2(r_1m)\\ &= \alpha_{r_2}(\alpha_{r_1}(m))\\ &= (\alpha_{r_1} \circ \alpha_{r_2})(m)\\ &= (\alpha(r_1) \circ \alpha(r_2))(m),\end{aligned}$$

where in the first line we used commutativity of R. Since this holds for any $m \in M$, it follows that $\alpha(r_1r_2) = \alpha(r_1) \circ \alpha(r_2)$.

Remark. Note that the operation in $\mathrm{End}_R(M)$ is composition, and not multiplication.

Problem 8

Since $\bar{\phi}$ is surjective, it follows that for any $n \in N$, there exists some $m \in M$ such that $n - \phi(m) \in IN$. So $N = \phi(M) + IN$. Applying this again for N gives

$$N = \phi(M) + I(\phi(M) + IN).$$

Since $\phi(M) + I\phi(M) = \phi(M)$, it follows that $N = \phi(M) + I^2N$. Repeating this argument gives that

$$N = \phi(M) + I^kN,$$

for all $k \in \mathbb{N}$. Since I is nilpotent, there exists some $k \in \mathbb{N}$ such that $I^k = 0$. Then we get that $N = \phi(M)$, so ϕ is surjective.

Problem 9

If $n = m$, then clearly R^n and R^m are isomorphic.

Conversely, suppose that $\phi : R^n \to R^m$ is an isomorphism. Let I be a maximal ideal of R. First we show that $R^n/IR^n \simeq R^M/IR^m$. Consider

$$\bar{\phi} : R^n \to R^m/IR^m,$$

which is the composition of ϕ and the projection map $\pi : R^m \to R^m/IR^m$. Since $\phi(IR^n) \subset IR^m$, we have that $\bar{\phi}(IR^n) \subset \ker(\bar{\phi})$. Conversely, let $x \in \ker(\bar{\phi})$. Then $\bar{\phi}(x) = 0$, so $\phi(x) \in IR^m$. So there exist finitely many $a_i \in I$ such that $\phi(x) = \sum_i a_i x_i$, with $a_i \in I$ and $x_i \in R^m$. Since ϕ is surjective, there exist $y_i \in R^n$ such that $\phi(x_i) = \phi(y_i)$. Then we have

$$\phi(x) = \sum_i a_i\phi(y_i) = \phi(\sum a_iy_i).$$

Since ϕ is injective, $x = \sum_i a_i y_i$, so $x \in IR^n$. So $\ker(\bar{\phi}) = IR^n$, so $R^n/IR^n \simeq R^m/IR^m$.

Now $R^n/IR^n \simeq (R/IR)^n$ and $R^m/IR^m \simeq (R/IR)^m$. Since I is a maximal ideal of R, we have that $R/IR \simeq F$, with F a field. So $F^n \simeq F^m$, so we must have $m = n$.

Problem 10

Suppose that S is finitely generated as a module over itself. Let $a_1, \ldots, a_n$ be a set of generators and for each a_i, let k_i be the maximum position such that a_i has a nonzero k_i^{th}-entry and for all $N > k_i$, $a_{iN} = 0$ (this k_i is finite, since a_i has finitely many nonzero entries). Let $k = \max k_i + 1$. Then k is finite, since we have a finite number of generators.

Look at $e_k = (0, \ldots, 0, 1, 0, \ldots)$, which has 1 in the k^{th}-entry and 0 everywhere else. Then $e_k \notin \text{Span}(a_1, \ldots, a_n)$, so S is not finitely generated.

Problem 11

If M is irreducible, then let $m \in M$ be a nonzero element. Then Rm is a submodule of M, and since M is irreducible, we must have $M = Rm$. Since this holds for any nonzero $m \in M$, M is cyclic with any nonzero element as a generator.

Conversely, if $M \neq 0$ and M is cyclic with any nonzero element as a generator, suppose N is a submodule of M. Assume $N \neq 0$. Pick $n \in N$ nonzero. Then $Rn = M$ by hypothesis, and since $Rn \subset N$, we get that $N = M$, so M is irreducible.

Since $\mathbb{Z}$-modules are abelian groups, we have to find the cyclic groups with any nonzero element as a generator. Note that the group has to be finite (otherwise if a is a generator of the infinite cyclic group G, $2a$ cannot be a generator). The only finite cyclic groups are those of the form $\mathbb{Z}/n\mathbb{Z}$, for some $n \in \mathbb{N}$. The number of generators of $\mathbb{Z}/n\mathbb{Z}$ is $\phi(n)$, so if any nonzero element is a generator of $\mathbb{Z}/n\mathbb{Z}$, we must have $\phi(n) = n - 1$, so n must be prime. Hence the irreducible $\mathbb{Z}$ modules are $\mathbb{Z}/p\mathbb{Z}$, for p prime.

Problem 12

We have to show that $M = M_1 + \ldots M_k$ and that $M_j \cap (M_1 + \ldots M_{j-1} + M_{j+1} + \ldots M_k) = 0$ for all j.

Let $q_j = \prod_{i \neq j} p_i^{\alpha_i}$. Let I be the ideal generated by the q_j, for $j \in \{1, \ldots, k\}$. Since R is a PID, there exists some $b \in R$ such that $I = (b)$. Then $(q_i) \subset (b)$, so $b | q_i$. But the q_i have no common prime, so b must be a unit, and $I = R$. So $1 \in I$, and there exist $r_i \in R$ such that $\sum_i r_i q_i = 1$. Then we have

$$m = 1 \cdot m = \sum_i (r_i q_i m).$$

But $p_i^{\alpha_i}(r_i q_i m) = r_i a m = 0$, so $r_i q_i m \in M_i$. This shows that $M = M_1 + \ldots + M_k$.

Now let $j \in \{1, \ldots, k\}$ and let $m \in M_j \cap (M_1 + \ldots M_{j-1} + M_{j+1} + \ldots M_k)$. Then $p_j^{\alpha_j} m = 0$, and $q_j m = 0$. By the same reason as above, $R = (q_j, p_j^{\alpha_j})$, so there exist $b, c \in R$ with $bq_j + cp_j^{\alpha_j} = 1$. Then $bq_j m = 0$, and $cp_j^{\alpha_j} m = 0$. Adding these up, we get $m = 0$, so $M_j \cap (M_1 + \ldots M_{j-1} + M_{j+1} + \ldots M_k) = 0$.

Problem 13

(a) It is straightforward to check the fact that $\oplus_i M_i$ is an R-module.

To check that it is an R-submodule of $\prod_i M_i$, first note that $\oplus_i M_i$ is nonempty. Now let $x, y \in \oplus_i M_i$ and $r \in R$. Since finitely many entries of x, y are nonzero, we can pick some $N \in \mathbb{N}$, such that for all $n \geq N$, we have $x_n = y_n = 0$. Then $x_n + r y_n = 0$ for all $n \geq N$, so $x + ry$ has finitely many nonzero entries. This shows that $x + ry \in \oplus_i M_i$.

(b) Note that $\oplus_i M_i$ is torsion. Indeed if $x \in \oplus_i M_i$, let N be such that $x_n = 0$, for all $n > N$. Then $N! x = 0$ in $\oplus_i M_i$, so x is a torsion element. Hence every element is torsion in $\oplus_i M_i$.

On the other hand, there are elements in $\prod_i M_i$ which are not torsion. Take $x = (1, 1, \ldots)$ with 1 in every entry. Suppose there exists some $n \in \mathbb{Z}$ such that $nx = 0$ i.e: $n = 0$ in $\mathbb{Z}/k\mathbb{Z}$ for all k. In particular, $n = 0$ in $\mathbb{Z}/(n+1)\mathbb{Z}$, which is impossible.

Since any R-isomorphism between $\oplus M_i$ and $\prod M_i$ would preserve the torsion elements, it follows that $\oplus M_i$ is not isomorphic to $\prod_i M_i$.

Problem 14

(a) Let M_p denote the p-primary component. First note that $0 \in M_p$, so M_p is nonempty.

Now if $x, y \in M_p$ and $r \in R$, let $a, b \in \mathbb{N}$ such that $p^a x = p^b y = 0$. Let $k = \max a, b$. Then $p^k(x + ry) = 0$, so $x + ry \in M_p$. Using the submodule criterion, we conclude that M_p is a submodule.

(b) Let $I = (a) = \mathrm{Ann}(M)$ and write $a = \prod_{i=1}^k p_i^{\alpha_i}$. Let M'_{p_i} denote the p_i-primary component as defined in problem 18. It is clear that $M'_{p_i} \subset M_{p_i}$, so we have to show the reverse inclusion. Let $m \in M_{p_i}$ with $p_i^{b_i} m = 0$. Since also $am = 0$, let $(c) = (a, p_i^{b_i})$ (this is because R is a PID). Then $c|a$ and $c|p_i^{b_i}$, so $(c) = (p_i)^{\min\{b_i, \alpha_i\}}$. Since $am = p_i^{b_i} m = 0$, we have $cm = 0$, so $p_i^{\alpha_i} m = 0$.

(c) We will prove that for every element $m \in M$ of the submodule generated by the M_p, there are unique elements $m_i \in M_{p_i}$ sucht that $m = \sum_i m_i$ and all but finitely many of the m_i are zero.

Since M is torsion, the annihilator $\mathrm{Ann}(M)$ is a nontrivial ideal of R and since R is a PID, $\mathrm{Ann}(M) = (a)$, for some $a \in R$. Let $a = \prod_{i=1}^k p_i^{\alpha_i}$, up to multiplication by a unit. Using problem 18, we can write m uniquely as $m = \sum_i m_i$, with $m_i \in M'_{p_i}$ (and the sum is finite). Since $M'_{p_i} \subset M_{p_i}$, we have $m = \sum_i m_i$ with $m_i \in M_{p_i}$ and the sum is finite.

If there exists some different way of writing $m = \sum_j n_j$ with all but finitely many n_j zero, suppose without loss of generality that $y_1 \in M_p$ and p is a prime different from all the p_i. Then $y_1 = m - \sum_{j=2}^{r} y_j$. y_1 is annihilated by some power of p, and the right hand side is annihilated by some $q \in R$, which doesn't have p as a factor. Hence by the same reason we used in problem 18, we get that $y_1 = 0$.

Using the result in problem 21 (since $\sum_p M_p = M$), we find that $M = \oplus_p M_p$.