Math 215A Final Exam

Fall 2016

The exam will be due by midnight on Thursday, December 15. The exam should be sent electronically to gunnar@math.stanford.edu. Please do not consult any books or internet resources other than Hatcher.

1. Let S^n denote the subspace of $\mathbb{R}^{n+1}$ consisting of vectors $(x_0, x_1, \ldots, x_n)$ so that $\sum_i x_1^2 = 1$. Let $T \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ be the subspace of pairs (v, w) so that (a) $v, w \in S^n$, and (b) $v \cdot w = 0$. Compute the homology and cohomology (with cup products) of T with integer and mod 2 coefficients. (Hint: If we write $(v_0, \ldots, v_n, w_0, \ldots, w_n)$ for a point in T, consider the subsets T_+ and T_-, defined by

   $$T_+ = \{(v, w) | v_0 \geq 0\} \text{ and } T_- = \{(v, w) | v_0 \leq 0\}$$

   Prove that

   $$T_\pm \cong D_\pm \times S^{n-1}$$

   where $D_\pm$ is the set of points in S^n so that $(\pm 1)v_0 \geq 0$.)

2. Let $f : X \to X$ be a continuous map. By the *mapping torus* of f, we will mean the space

   $$X \times [0, 1]/ \sim$$

   where $\sim$ is the equivalence relation given by $(x, 0) \sim (f(x), 1)$.

   (a) Let X be the circle S^1, regarded as the complex numbers z of unit length, and let $f : S^1 \to S^1$ be the map defined by $f(z) = z^n$. Determine the homology and fundamental group of the mapping torus of f.

   (b) Let X be the space $\mathbb{R}P^2 \vee \mathbb{R}P^2$, where $\vee$ denotes one point union (see below). We assume that we are given a choice of base point in $\mathbb{R}P^2$, and that the same choice is made for both copies. Let $f : X \to X$ denote the map that flips the two copies of $\mathbb{R}P^2$. Determine the homology and fundamental group of the mapping torus of f. (Given spaces X and Y, equipped with base points x_0 and y_0, we define $X \vee Y$ to be the quotient space

   $$X \coprod Y/ \sim$$

   where $\sim$ denotes the equivalence relation defined by $x_0 \sim y_0$.)

3. Let X denote the space of all quadratic polynomials

   $$f = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$$

   for which

   $$\int\int_D f = 0 \text{ and } \int\int_D f^2 = 1$$

   where D denotes the unit disc.

(a) Compute the integer homology of X

(b) What familiar space is it homeomorphic to?

(c) Consider the subspace $Y \subseteq X$ consisting of the quadratic functions having the form $\varphi(l(x,y))$, where φ is a single variable quadratic function and $l(x,y) = ax + by$ is a linear function in x and y. What familiar space is it homeomorphic to?

4. Let M° denote the open Möbius band, give by $[0,1] \times (0,1)$, with the identifications given by $(0,y) \sim (1, 1-y)$. Compute $H^*_c(M^\circ)$. with integer coefficients.

5. Consider the self map θ_p of $\mathbb{C}P^2$ given in homogeneous coordinates by $\theta_p(z_0, z_1, z_2) = (z_1^p, z_0^p, z_2^p)$, where p is an integer. Compute the induced map on homology of θ_p.

6. Let X be a finite metric space. By the *Vietoris-Rips complex* of X with threshold r, denoted by $V(X,r)$, we will mean the abstract simplicial complex whose vertex set is the underlying set of X, and where $\{x_0, x_1, \ldots, x_k\}$ spans a k-simplex if and only if $d(x_i, x_j) \leq r$ for all $0 \leq i, j \leq k$. Let X be the four point metric space whose points are the corners of a square of side length 1. Determine the homology, with integer coefficients, of $V(X,r)$ for all values of r.