1 What are vectors?

When we do calculations we write down vectors as n-tuples of real numbers. But aren't n-tuples of real numbers just points in $\mathbb{R}^n$? So what's the difference between vectors in $\mathbb{R}^n$ and points in $\mathbb{R}^n$?

First a little fable. Once upon a time in a land far away, there was a gentle moo-cow named Petunia who lived near the sea. Petunia was so small that she slept in a matchbox and her owners had to milk her with tweezers. Her brittle legs were so long that she grazed on clover growing on the top of stratus clouds, and before she went to sleep she had to spend two hours meticulously rolling her legs up into four spirals that would fit in her matchbox. The faintest ocean breeze would knock her over, and the whole village would have to spend the whole day arighting her (they had to make a huge human pyramid). This caused so much trouble that the oldest and cleverest of the village eventually dreamed up a horribly confusing system of ropes to tether her to the highest hills (but that is for another time). One day while Petunia was eating clover and swatting at the cloud-bugs with her tail, she noticed that the alphabet and the numbers $\{-1, 0, ..., 27\}$ could be associated to each other. 'a' $\leftrightarrow$ 1. 'b' $\leftrightarrow$ 2 etc. 0 $\leftrightarrow$ a blank space and -1 $\leftrightarrow$ ".". Naturally she decided to do humanity an unwanted favor by translating great literature. She soon became bored and stopped, but not before she managed to produce (9,20,0,23,1,19,0,12,1,20,5,0,5,22,5,14,9,14,7,0,23,8,5,14,0,11,-1,0,1,18,18,9,22,5,4,-1). Not exactly Shakespeare or even Esquire. Though technically all the information necessary to read this sentence is present, it is patently insane to try to appreciate literature by reading strings of numbers in lieu of the written word (unless you are a computer or had a really bizarre homeschooling). Ok, that story kind of blew.

Well, points are...points. Vectors in $\mathbb{R}^n$ ARE line segments in $\mathbb{R}^n$ with a preferred direction (aka an arrow) considered up to translation (meaning two vectors that are translates of each other count as the same vector).

The way we assign points/"n-tuples of numbers" to vectors is by taking a vector and translating it so that its tail is at the origin (still the same vector) and assigning to it the point at the head of the vector. This works backwards too. Given a $p \in \mathbb{R}$ we associate to it the vector whose tail is at the origin and whose head is at p. Numerically this boils down to what you already knew: if a vector has its head at $(p_1, ..., p_n)$ and its tail at $(q_1, ..., q_n)$ then the n-tuple we associate to the vector is $\begin{bmatrix} p_1 - q_1 \\ \vdots \\ p_n - q_n \end{bmatrix}$.

In summary: vectors are not point/"n-tuples of real numbers". However, there is a way to associate a point to a vector and vice-versa. Like in the fable though, sometimes thinking of vectors of n-tuples of numbers will actually hide some of their important features. When someone says "vector", I hope in your private coconut, there appears the image of an arrow instead of an n-tuple of numbers.

2 helpful facts about linear independence

You really need to know the definition of linear independence. One incarnation is:

$\{\overrightarrow{v_1}, ..., \overrightarrow{v_n}\}$ is linearly independent if the only solution to $c_1\overrightarrow{v_1} + \cdots + c_n\overrightarrow{v_n} = \overrightarrow{0}$ is $c_1 = \cdots = c_n = 0$.

Some facts that are nice to know and good exercises to verify (please try them and see me if you can't do them)

1) $\{\overrightarrow{v}\}$ is always linearly independent unless $\overrightarrow{v} = \overrightarrow{0}$.

2) $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ is linearly independent unless $\overrightarrow{v_1}$ is a non-zero multiple of $\overrightarrow{v_2}$.

WARNING: nothing like this is true for collections of three or more vectors. Consider for instance the collection $\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\}$. Notice that none of the vectors is a multiple of another one in the collection. However as $\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ the collection is linearly dependent.

3) Try to show: If $\{\overrightarrow{v_1}, ..., \overrightarrow{v_k}\}$ is linearly independent and v_{k+1} is NOT in the span of $\{\overrightarrow{v_1}, ..., \overrightarrow{v_k}\}$, then the larger collection $\{\overrightarrow{v_1}, ..., \overrightarrow{v_{k+1}}\}$ is also linearly independent.