1 Tangent Planes To Graphs Of Functions

For this discussion f is a VECTOR valued differentiable function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

Let P be the plane tangent to the graph of f at the point $(\vec{p}, f(\vec{p}))$.

Remember!:

· This plane is an n-dimensional affine plane that is a subset of $\mathbb{R}^{n+m}$ (domain $\times$ codomain) containing the point $(\vec{p}, f(\vec{p}))$

· This plane P is exactly the same thing as the graph of the 1st order approximation to f at $\vec{p}$.

Quickly some notation: I want to identify n+m vectors as what I get when I stack an n-vector on top of an m-vector. For example when n=3 and m=2 I can stack $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \in \mathbb{R}^3$ and $\begin{bmatrix} 4 \\ -5 \end{bmatrix} \in \mathbb{R}^2$ to get $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \\ -5 \end{bmatrix} \in \mathbb{R}^5$. For two abstract vectors $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^m$ I will write this vector as $\left[\begin{array}{c} \vec{v} \\ \hline \\ \vec{w} \end{array}\right]$.

OK, the parametrization for the plane tangent to the graph of f at $(\vec{p}, f(\vec{p}))$ is given by:

$$\left[\begin{array}{c} \vec{p} \\ \hline \\ f(\vec{p}) \end{array}\right] + t_1\vec{v}_1 + \cdots + t_n\vec{v}_n$$

where $\vec{v}_i$ is the ith column of the matrix

$$\left[\begin{array}{c} Id_{n\times n} \\ \hline \\ D_{\vec{p}}f \end{array}\right].$$

Example 1

Let

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x, y) \mapsto (xy, y^4 + x)$$

Find the plane tangent to the graph of f at (2,-1).

solution

We will need:

1) f((2,-1))=(-2,3)

2) $D_{(2,-1)}f$

$$= \begin{bmatrix} \frac{\partial}{\partial x}(xy) & \frac{\partial}{\partial y}(xy) \\ \frac{\partial}{\partial x}(y^4 + x) & \frac{\partial}{\partial y}(y^4 + x) \end{bmatrix}_{(-2,3)}$$

$$= \begin{bmatrix} y & x \\ 1 & 4y^3 \end{bmatrix}_{(-2,3)} = \begin{bmatrix} -1 & 2 \\ 1 & -4 \end{bmatrix}$$

OK, then

$$\left[\begin{array}{c} \vec{p} \\ \hline \\ f(\vec{p}) \end{array}\right] = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

and the vectors $\vec{v}_1, \vec{v}_2$ will be the columns of the matrix obtained by stacking $Id_{2\times 2}$ on $D_{(2,-1)}f$ which is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \\ 1 & -4 \end{bmatrix}$$

So the plane P is parametrized as

$$\begin{bmatrix} 2 \\ -1 \\ -2 \\ 3 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

When f is REAL VALUED

For this section alone

$$f : \mathbb{R}^n \to \mathbb{R}$$

Then there is a trick to express the plane tangent to the graph of f as the tangent plane to a level set of a different function g. (Since the plane must be a subset of domain $\times$ codomain $= \mathbb{R}^{n+1}$ we know that whatever g must be, it must be real valued and its domain must be $\mathbb{R}^{n+1}$ because the tangent planes to level curves are subsets of the domain.

OK, enough with the suspense.... Define

$$g : \mathbb{R}^{n+1} \to \mathbb{R}$$

$$(x_1, ..., x_n, t) \mapsto f(x_1, ..., x_n) - t$$

Notice that the level set $g = 0$ consists of all points in $\mathbb{R}^{n+1}$ where the last coordinate is equal to $f(x_1, ..., x_n)$. In other words it is the set of points of the form $(\vec{x}, f(\vec{x}))$. But this is exactly what the graph of f is!

OK, now we can use our formula for the tangent plane to a level curve to find that the plane tangent to the graph of f at $(\vec{p}, f(\vec{p})$ is:

$$0 = \nabla_{\vec{p}, f(\vec{p})} g \cdot \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ t \end{bmatrix} - \begin{bmatrix} p_1 \\ \vdots \\ p_n \\ f(\vec{p}) \end{bmatrix} \right)$$

Since $g(\vec{x}, t) := f(\vec{x}) - t$,

$$\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \text{ for all } 1 \leq i \leq n$$

and

$$\frac{\partial g}{\partial t} = -1$$

So in components the equation for the plane becomes

$$0 = \frac{\partial f}{\partial x_1}\vec{p}(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}\vec{p}(x_n - p_n) - (t - f(\vec{p}))$$

or in a form that looks more familiar:

$$t - f(\vec{p}) = \frac{\partial f}{\partial x_1}\vec{p}(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}\vec{p}(x_n - p_n)$$

Again, remember this formula only works when the function f is REAL VALUED. Also it is an equation for a n-dimensional plane that is a subset of $\mathbb{R}^{n+1}$, and it is NOT the equation for a plane tangent to a level curve of f. (There is an extra variable involved here, and it IS the level curve of the function of g).