1 Reduced Row Echelon Matrices

Definition A matrix is in Reduced Row Echelon Form if:
1) All its rows look like: $\begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \end{bmatrix}$. ("*" stands for any real number). In other words, if reading from left to right, the first nonzero entry is a 1. We call the leading 1 a *pivot*
2) # of leading zeros in each row <u>*strictly*</u> increases from top to bottom.
3) The components of a column containing a pivot are zero (except for the pivot itself). We refer to such a column as a *pivot column.* If a column isn't a pivot column, then we call it a *free* column.

For example:

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is row reduced.

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

is too.

$$\begin{bmatrix} 1 & 3 & 0 & 8 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

is NOT because the fourth column violates rule 3).

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

is NOT because the size of the second row is not strictly bigger than the first row, violating rule 2).

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 9 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

is NOT because the second row violates rule 1).

2 Why row-reduced matrices are our friends

Suppose we have an augmented matrix, $[A|\overrightarrow{b}]$, and luckily for us A is row-reduced. Then there is an easy procedure to find a *parametrization* for the solution space of the equation $A\overrightarrow{x} = \overrightarrow{b}$. Here is how the procedure works:

Example 1

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

0) The non-augmented part of the matrix has 5 columns, so our solutions are of the form $\begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$.
1) First we locate the pivots. There are only two of them here, the 1's in the 2nd and 4th column. Because of this, we'll call x_2 and x_4 the pivot variables, and the rest, x_1, x_3, and x_5, will be called free variables.
2) Next we write out the real-variable equations associated to the rows containing the pivots. In this case that means the first and second rows. We get

$$x_2 + 2x_3 + x_5 = 3$$

$$x_4 + 2x_5 = 0$$

(Notice that each equation contains exactly one pivot variable. The remaining are free. Furthermore, the index of the pivot variable is the lowest index appearing in the equation, and the coefficient in front of the pivot variable is 1.)
3) Solve each of the above equations for its lone pivot variable by subtracting the free variables to the other side.

$$x_2 = 3 - 2x_3 - x_5$$

$$x_4 = -2x_5$$

4) Subsitute these equations to find a parametrization of the space of solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3 - 2x_3 - x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix}$$

And split according to variables and constants to easily see the parametrization

$$= \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Key Observations/Facts:

1- Notice that the solution space is parametrized by 3 vectors, and that there were 2 pivots. 3+2=5, which is the number of columns in the non-augmented portion of the matrix. We will see later that this is not a coincidence.

2- Sometimes there is no solution. This happens IF AND ONLY IF there is a row in $[A|\overrightarrow{b}]$ that looks like $[00\cdots0|4(\neq 0)]$. Remember, each row of an augmented matrix corresponds to a real-valued equation, and a row like the one above translates to $0 = 0x_1 + \cdots + 0x_n = 4$ which clearly has no solution.

3- If the number of pivots is equal to the number of columns (since we're assuming A is row-reduced, this actually means A is the identity matrix), there is exactly ONE solution, and it obviously $\overrightarrow{b}$.

4- If the number of pivots is less than the number of columns, and there are no rows like $[00\cdots0|4(\neq 0)]$, then there is an infinite number of solutions. (Try to use observation 1 to explain why).

In summary: It is easy to find all solutions to the equation $[A|\overrightarrow{b}]$ when A is in Reduced Row Echelon Form.

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Now we're NOT going to assume A is in reduced row echelon form. So suppose we're given any old matrix A and asked to find the solution space for $[A|\overrightarrow{b}]$. Well, we might try to see if we can convert the problem to a related problem, $[RRef(A)|\overrightarrow{b'}]$, but where now $RRef(A)$ is row reduced. By "related" we mean that the solution space of $[A|\overrightarrow{b}]$ equals the solutions space of $[RRef(A)|\overrightarrow{b'}]$.

Surprise, surprise, there is a way, and the process goes by the name "row-reducing" or "putting into row-echelon form" or whatever. I'm getting a little sleepy to explain how to row-reduce, but hopefully you're well on your way to figuring that out on your owns. Of course, let me if you need help.

Let me take this opportunity though to point out that until you row-reduce A, it is nigh impossible to tell much about the number of solutions (ie. number of pivots) or if any solutions exist at all. Yes, if there is a row in $[A|\overrightarrow{b}]$ that looks like $[00\cdots0|4(\neq 0)]$, there are no solutions. BUT, it is possible that at first none of the rows resemble $[00\cdots0|4(\neq 0)]$, and yet there are still no solutions. As an easy example:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \end{array}\right]$$

row reduces to

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

and now we see there are no solutions, because $[00\cdots0|4(\neq 0)]$ IS row reduced, and we make "observation no. 2".