1 Limits

We say the limit of a function $f : \mathbb{R}^n \to \mathbb{R}$ exists at $p \in \mathbb{R}^n$ and is equal to c if: when I want to be guaranteed that the values of f are arbitrarily close to c, I can consider f restricted to a domain that consists of an arbitrarily small open ball around p. Lets take a look at the ways in which a limit will FAIL to exist.

As a first example, take
Example 1)

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \sin(\frac{1}{x})$$

It does NOT have a limit as $x \to 0$. There are points arbitrarily close to 0 on which f assumes the value ± 1 (any $x = \frac{2}{\pi n}$ where n is an integer). On the other hand there are points arbitrarily close to 0 on which f assumes the value 0 (any value $x = \frac{1}{\pi n}$ where n is a non-zero integer).

There are other times when a limit DNE.
Example 2)

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \frac{1}{x}$$

approaches both $-\infty$ and ∞ as $x \to 0$. We say the limit does not exist in these situations.

More open to interpretation is the function
Example 3)

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \frac{1}{x^2}$$

limits to ∞ when $x \to 0$. Depending on how much you want to bequeath ∞ the qualities of a number, you might actually say the limit exists here and equals ∞. I'm often tempted to do this, but in this class we will again say the limit DNE.

Example 4) Next, consider piecewise jumping function

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto 0 \text{ if } x < 0$$

$$x \mapsto 1 \text{ if } x \geq 0$$

If we restrict our attention to the domain $[0, \infty)$, then yes the limit exists and equals 1 since f is constantly 1. We say that the limit at 0 "from the right" is equal to 1. Similarly if we restrict our attention to the domain $(-\infty, 0)$, then yes the limit of f exists and equals 0. We say that the limit from the left is equal to 0. Since the limits from the left and right disagree though, the actual limit of f DNE.

When dealing with multivariable functions, this kind of obstruction to having a limit exist becomes much more interesting. Instead of being able to approach the origin from just the right or left, we are able to approach along many different directions. We need to take this into account whenever trying to decide if a limit exists or not.

2 Continuity

$f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\vec{p}\in\mathbb{R}$ if the limit of f at $\vec{p}$ is equal to $f(\vec{p})$. Equivalently, some bastards would say, f is continuous at $\vec{p}$ if

$$\lim_{\vec{x}\to\vec{p}} f(\vec{x}) = f(\vec{p})$$

If the limit of f at $\vec{p}$ does not exists, then f cannot be continuous there.

Othertimes, the limit exists but does not equal to $f(\vec{p})$. As an easy example of this, take

Example 5)

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto 0 \text{ if } (x, y)\neq(4, 3)$$

$$(x, y) \mapsto -7 \text{ if } (x, y) = (4, 3)$$

In this case the limit of f at (4,3) equals 0, but $f((4, 3))$ equals -7 which is inherently not equal to 0.

Sometimes the limit of f at $\vec{p}$ will exist, but a value of f at $\vec{p}$ has not yet been chosen. Then we can extend f to a function that is defined on $\vec{p}$ and is continuous at $\vec{p}$ by DECLARING that $f(\vec{p}) = \lim_{\vec{x}\to\vec{p}} f(\vec{x})$. Then the definition of continuity will be satisfied.

Some of the most important facts about continuous functions concern how to build continuous functions out of simpler ones: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Then

1) $f + g$ is continuous
2) fg is continuous
3) $f\circ g$ and $g\circ f$ are continuous
4) Wherever $g(x)\neq 0$, $\frac{f}{g}$ is continuous.

Last Note Technically, if you are asked to find the limit of f at $\vec{p}$ exists, it would circular logic to simply assert that the limit of f at $\vec{p}$ is $f((\vec{p}))$. However, because we never went through the whole rigamarole of how to show basic functions such as x, x^2, sinθ have good limits and are continuous, we will assume that they are. By using the 4) facts about continuous functions this lets us conclude that lots of functions that are built out of the simpler functions (which in this class we assume are continuous) must be continuous. Since we are "assured" that these functions are continuous, it is fair to calculate the limit of f at $\vec{p}$ by evaluating f at $\vec{p}$. I will call this situation: "Proof by assuredness".

3 In Practice

Showing the limit exists

In practice we have about four ways (that I can think of at least) to show that a limit exists. They are:

· Factor f and cancel off troublesome terms, leaving something easier

· Use polar coordinates

· Suicide Squeeze Theorem

· Proof by reduction to "obviously continuous" function building blocks, and "Proof by assuredness". Simply assert that f is continuous at p, so the limit is equal to $f(p)$. CAREFUL! This is close to circular reasoning because saying that the limit at p is equal to $f(p)$ *is* the definition of continuity. Probably you're going to want to say that f is a product/sum/composition/quotient of non-zero continuous functions, where is it really obvious that these simpler functions are continuous (this is probably acceptable to do in this class when the simpler functions are polynomials, sin, cos, e^x).

Showing the limit does not exist

Again there are a couple ways:

· If the limit along a path approaching p DNE, then the limit exists

· Show that although the limits along some paths approaching p exist, the value of the limit is dependent on which path you take. (This is the higher dimensional analogue of saying the limit from the right exists and the limit from the left exists, but they don't agree).

· As last recourse, the most general way to find the limit DNE is to find a sequence of points $x_1, x_2, ...$ that converge to p, but the sequence $f(x_1), f(x_2), ...$ doesn't converge to anything (e.g. the sequence 0,1,0,1,... doesn't converge to anything).

Note: The second way tends to come up in a lot of the math51 problems, and because they are simply designed it tends to be the case that checking the limit of f along the vertical and horizontal lines (holding the x and y coordinate constant respectively) will provide you limits that disagree. IN GENERAL THIS IS NOT ALWAYS GOING TO HELP. Sometimes the limits along the x and y-axis will agree, but taking the limit along a different path to p will disagree! So just be open to this possibility.

Knowing when to show the limit DNE, and when to show it exists

The process for showing a limit exists is different from showing it DNE. If you start down the wrong decision tree, you'll be fucked. With experience you will begin get an intuition for when a limit should exist and when it shouldn't. BUT, each f is different. If you're trying to prove a limit equals c, say, and you're having a really hard time showing this, then maybe the problem is that the limit DNE.

4 Exercises

Exercise 1) Is

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto \frac{3x - y}{(x-4)\sin(\frac{\pi}{2} + y)}$$

continuous at (3,0)?

solution: Alright. I'm going to say it is continuous. Why? Well, it is the quotient of two continuous functions, $3x - y$ and $(x-4)\sin(\frac{\pi}{2} + y)$, so the only reason I'd have trouble is if the denominator function were zero at (3,0), but $(3-4)\sin(\frac{\pi}{2} + 0) = -1 \neq 0$. So f is continuous at (3,0) and I will then assert that the limit at (3,0) must equal $f((3,0)) = \frac{9}{-1} = -9$.

Exercise 2)
Is

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto \frac{4x^3y - 2}{x^4 + y^4}$$

continuous?

solution: To get some idea of whether the limit of f exists at (0,0), the first thing I'd do is try to evaluate the limit of the numerator and denominator individually. The limit of the numerator is -2 whereas the limit of the denominator is 0. Dividing -2 by 0 gives some sort of infiinity, so it is clear from this that the limit DNE.

Exercise 3)
Is

$$f : \mathbb{R}^2 \to \mathbb{R}$$
$$(x, y) \mapsto \frac{4x^3y}{x^4 + y^4}$$

continuous?

solution: When we evaluate the limits of the numerator and the denominator we get 0 in both cases. That doesn't tell us anything. Also both the polynomials are homogeneous of order 4 (every term has order 4), AND, there is no kind of obvious factoring that we can do....Hmm, in instances like these my instinct, is to guess then that the limit along any particular path exists, however, the limit depends on which path we take. To verify this the two common tricks are to use polar coordinates and see that the limit as $r \to 0$ depends on θ or to substitute $y = cx$ and find that the limit as $x \to 0$ depends on c. Both ways are very similar.

Approaching (0,0) along the line $y = cx$ the limit is:

$$\lim_{x\to 0}\frac{x^3cx}{x^4+(cx)^4} = \lim_{x\to 0}\frac{x^4c}{x^4(1+c^4)}$$

$$= \lim_{x\to 0}\frac{c}{1+c^4} = \frac{c}{1+c^4}$$

So this limit depends on c i.e. which line we took to approach (0,0). So the limit DNE.

Exercise 4)

Find a value of c for which f is continuous (or say why no such c exists), where

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto \frac{xy}{\sqrt{4x^2+y^2}} \text{ if } (x, y) \neq (0, 0)$$

$$(x, y) \mapsto c \text{ if } (x, y) = (0, 0)$$

solution: $xy, 4x^2 + y^2$, and $\sqrt{\ }$ are obviously all continuous functions. By the composition rule $\sqrt{4x^2+y^2}$ is continuous. Also since $\sqrt{4x^2+y^2}$ is nonzero away from (0,0) by the quotient rule for continuous functions, $\frac{xy}{\sqrt{4x^2+y^2}}$ is continuous away from the origin. So the only point in question is (0,0). By definition, f is continuous at (0,0) if:

$$\lim_{(x,y)\to(0,0)} f = f((0, 0))$$

The righthand side is defined to be c, so f will be continuous if the limit on the lefthand side exists and we set c equal to it.

My immediate guess is that the limit exists and is zero. Why? well, loosely speaking, the numerator has order 2 and the denominator has order 1. So lets try to prove the limit equals zero.

Also I'm tempted to use polar coordinates because I see something like x^2+y^2 in the denominator. In fact since $4x^2+y^2 > x^2+y^2$ then $\left|\frac{xy}{\sqrt{4x^2+y^2}}\right| < \left|\frac{xy}{\sqrt{x^2+y^2}}\right|$. Thus if I can show the limit of $\frac{xy}{\sqrt{x^2+y^2}}$ is zero at (0,0) then I'm done by the squeeze theorem.

Alright, lets use polar coordinates on $\frac{xy}{\sqrt{x^2+y^2}}$. So we substitute $x = r\cos\theta$ and $y = r\sin\theta$:

$$\frac{r^2\cos\theta\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2}}$$

$$= r\cos\theta\sin\theta$$

Alright, now we look at the limit as $r \to 0$ and θ is allowed to vary within $[0, 2\pi)$. Again, here I'm interested in using a squeeze theorem because no matter what value θ takes, $-1 \leq \cos\theta \sin\theta \leq 1$. So for all $r \in [0, \infty)$ and $\theta \in [0, 2\pi)$ we have

$$-r \leq r\cos\theta\sin\theta \leq r.$$

Thus by the squeeze theorem:

$$\lim_{r\to 0} - r \leq \lim_{r\to 0} r\cos\theta\sin\theta \leq \lim_{r\to 0} r,$$

$$0 \leq \lim_{r\to 0} r\cos\theta\sin\theta \leq 0$$

which implies

$$\lim_{r\to 0} r\cos\theta\sin\theta = 0$$

Ok, so we're done by the squeeze theorem (used as second time!) as stated above. To write it out formally, since

$$-\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{xy}{\sqrt{4x^2+y^2}} \leq \left|\frac{xy}{\sqrt{x^2+y^2}}\right|$$

and both the limits of the bounding functions exist and equal 0 (what we verified above) we know by the squeezing theorem that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{4x^2+y^2}} = 0$$

So, to make f continuous at (0,0) we should set $c = 0$.

Exercise 5)
i) Is

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto \frac{xy + 3x - 3y - 9}{x - 3} + 3x \text{ if } x \neq 3$$

$$(x, y) \mapsto x^2 \text{ if } x = 3$$

continuous at (3,0)? ii) In general where is f continuous?

solution: This is a piecewise defined function that is continuous when we restrict to each piece. The placed where f may not be not be continuous are those points that are on the boundary of more than one piece of definition. To be continuous at these places, the functions on each piece must be continuous, and their values must agree on the overlap.

For i) we're asked if f is continuous at (3,0). (3,0) is definitely on the boundary of both the regions $\{(x, y)|x = 3\}$ and $\{(x, y)|x\neq\}$, so we need to the check that the limit of f on the first region as (x,y)→(3,0) is the same as the limit of f on the second region as (x,y)→(3,0) and that $f((3, 0))$ is equal to both these limits.

First off

$$f((3,0)) = 3^2 = 9$$

Now lets see if the limit exists as (x,y)→(3,0) for f defined on the region $\{(x,y)|x = 3\}$. Well, on that region, $f((x,y)) = x^2$ so we can just appeal to our "proof by assuredness" and claim that since f is continuous, the limit may be found by evaluating f at (3,0).

So really the only hard thing we need to check is whether

$$\lim_{(x,y)\to(3,0)} \frac{xy + 3x - 3y - 9}{x - 3} + 3x = 9$$

The first thing I do when I see a limit such as that one, is to break it up into easier pieces and get those out of the way:

$$\lim_{(x,y)\to(3,0)} \frac{xy + 3x - 3y - 9}{x - 3} + 3x = \lim_{(x,y)\to(3,0)} \frac{xy + 3x - 3y - 9}{x - 3} + \lim_{(x,y)\to(3,0)} 3x$$

$$= \lim_{(x,y)\to(3,0)} \frac{xy + 3x - 3y - 9}{x - 3} + 9$$

OK, now we're in a bit of a pickle, because the denominator, $x - 3$ evaluates to 0 at (3,0). We might try to factor $(x - 3)$ out of $xy + 3x - 3y - 9$. We try and succeed: $xy + 3x - 3y - 9 = (x - 3)(y + 3)$. Now we're ready to go:

$$= \lim_{(x,y)\to(3,0)} \frac{(x - 3)(y + 3)}{(x - 3)} + 9 = \lim_{(x,y)\to(3,0)} (y + 3) + 9 = 12$$

OK, so the limit of f along this region (12) does not equal the limit of f along the other region (9). So f is NOT continuous at (3,0).

ii) Where is f continuous? Well definitely in those places that aren't on the border of the regions. So definitely all points (x, y) where $x \neq 3$. When $x = 3$ though we need to verify that the limits of f given the two different definitions of f agree. So we're solving

$$\lim_{(x,y)\to(3,y_0)} \frac{xy + 3x - 3y - 9}{x - 3} + 3x = \lim_{(x,y)\to(3,y_0)} x^2$$

The limit on the right is easy to evaluate. It is equal to 9, regardless of what y_0. As for the limit on the left we repeat the factoring argument as above so conclude it is equal to

$$\lim_{(x,y)\to(3,y_0)} (y + 3) + 9$$

$$= y_0 + 12.$$

Thus we wish to solve

$$y_0 + 12 = 9 \Rightarrow y_0 = -3$$

So f is continuous at all points (x, y) such that $x \neq 3$ and also the point (3,-3).