1 Tangents Planes To Level Curves

For this discussion, f is a REAL-VALUED function:

$$f : \mathbb{R}^n \to \mathbb{R}.$$

Let $c \in \mathbb{R}$. A level curve of f at c is the set of solutions to the equation $f(x_1, ..., x_n) = c$. If $\vec{p} \in \mathbb{R}^n$ and $\vec{p}$ lies on the curve (i.e. $f(\vec{p}) = c$) we talk about the tangent plane to this curve at $\vec{p}$. Intuitively, since this plane is tangent to the set on which f is constant, the plane consists of all vectors pointing in directions along which f DOESN'T change. The direction along which f changes the most is not surprisingly the normal direction to this plane. We know this vector is $\bigtriangledown_{\vec{p}} f$. So the equation for this tangent plane to the level curve $f(x_1, ..., x_n) = c$ at $\vec{p}$ is:

$$0 = \bigtriangledown_{\vec{p}} \cdot (\vec{x} - \vec{p})$$

or written out in components

$$0 = \frac{\partial f}{\partial x_1}(\vec{p})(x_1 - p_1) + \cdots + \frac{\partial f}{\partial x_n}(\vec{p})(x_n - p_n)$$

Remember!:
· This plane is an n-1 dimensional affine plane that is a subset of the domain ($\mathbb{R}^n$) and contains the point $\vec{p}$.
· = At some points on the level curve, a tangent plane may not exist! These are the points at which $\bigtriangledown_{\vec{p}} f = \vec{0}$

Example 1

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(x, y) \mapsto (x^2 + xy)$$

Find the tangent to the level curve, $f(x, y) = 12$, at the point (4,-7).
solution First find $\bigtriangledown_{(4,-7)} f$:

$$= \begin{bmatrix} 2x + y \\ x \end{bmatrix}_{(4,-7)} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

So the equation for the plane is

$$0 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 4 \\ -7 \end{bmatrix} \right)$$

or...

$$0 = (x - 4) + 4(y + 7)$$

Example 2

$$f : \mathbb{R}^3 \to \mathbb{R}$$

$$(x, y, z) \mapsto (z^4 - x^2 - y^2)$$

Find the tangent to the level curve, $f(x, y) = 0$, at the point (0,0,0).

solution First find $\bigtriangledown_{(0,0,0)} f$:

$$= \left.\begin{bmatrix} -2x \\ -2y + y \\ 4z^3 \end{bmatrix}\right|_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $\bigtriangledown_{(0,0,0)} f = \vec{0}$ we conclude there is no tangent plane to the level curve at (0,0,0).