

1 Matrix-vector multiplication, matrix-vector equations

Let A be a matrix and \vec{x} .

· Here is how we define matrix-vector multiplication: $A\vec{x}$ is a vector whose length is equal to the number of rows of A . The i th component of $A\vec{x}$ is defined to be the i th row of A dotted with \vec{x} . (Technically we need to turn the i th row of A into a column-vector before dotting with \vec{x} . We do this by changing “left to right” to “top to bottom”).

· It only makes sense to multiply \vec{x} by A if the number of columns of A is equal to the length of \vec{x} .

· There is another way to define matrix-vector multiplication. It gives the same

answer, and is sometimes more convenient to use. Lets write out $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

We can think of any matrix with m rows and n columns as n vectors $\vec{a}_1, \dots, \vec{a}_n$ of length m stacked next to each other like so:

$$A = [\vec{a}_1 | \vec{a}_2 | \dots | \vec{a}_n]$$

Then we define $A\vec{x}$ by the formula

$$x_1\vec{a}_1 + \dots + x_n\vec{a}_n$$

Example 1.1. Compute $\begin{bmatrix} 4 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ using both definitions of matrix-vector multiplication. Show you get the same answer both ways.

solution: Lets use the first definition. Since the matrix is a 2×3 we expect the answer to be a vector of length 2. The first component is equal to the first row of A , $\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ dotted with $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ which yields $4 - 2 - 2 = 0$. Similarly, the

second component is equal to the second row of A , $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ dotted with $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

which yields $1 + 1 + 2 = 4$. So $A\vec{x} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

Now lets use the second definition. We split A up into its column vectors: $\vec{a}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and $\vec{a}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Then the formula says $A\vec{x} =$

$$1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

The main hurdle in a lot of the problems you will see is knowing how to converting the question into the correct matrix equation to solve. As a general rule of thumb, if the question involves finding the set of vectors normal to a collection of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ then you want to use the first way of thinking about matrix multiplication. You want to solve the equation

$$\begin{bmatrix} - & \vec{v}_1^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{bmatrix} \vec{x} = \vec{0}.$$

On the otherhand, if you want to see if the vector \vec{x} is in the span of a collection of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$, then you want to solve the equation

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \vec{y} = \vec{x}.$$

If you want to check if the vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent then you want to check that the only solution to

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \vec{y} = \vec{0}$$

is $\vec{y} = \vec{0}$.

Example 1.2. Find the set of vectors \vec{x} such that $\vec{x} \cdot \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = 2$ and $\vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1$.

Here we are going to want to use the first way of thinking of matrix-vector multiplication. The corresponding equation we'd like to solve is

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -3 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Lets solve.

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 2 & -3 & -1 & | & 2 \end{bmatrix}$$

add $-2 \times$ row 1 to row 2,

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & -3 & 3 & | & 0 \end{bmatrix}$$

scale row 2 by $-\frac{1}{3}$,

$$\begin{bmatrix} 1 & 0 & 1 & | & 1 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$$

We're in RRef form. There are two pivots, and their corresponding row equations are

$$x_1 + x_3 = 1$$

$$x_2 - x_3 = 0$$

Now we write the pivots in terms of the free variables

$$x_1 = 1 - x_3$$

$$x_2 = x_3$$

So the set of all desired vectors \vec{x} are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ for all } x_3 \in \mathbb{R}.$$