1 Eigenvectors continued

The eigenvector train continues on and stops for no one.

OK, as most of you figured out, the eigenvalues of a matrix are the solutions of the equation

$$\det(A - \lambda I) = 0$$

and the eigenspace E_{λ_i} is the space $N(A - \lambda_i I)$.

Let A be an n×n matrix. The characteristic polynomial is $\det(A - \lambda I)$. We usually call this polynomial the Here are some relevent/useful facts:

· A is diagonalizable if and only if:

1) $p(\lambda)$ splits into linear factors.
2) The sum of the dimensions of all of the eigenspaces equals n.

· Suppose that the characteristic polynomial is equal to $p(\lambda) = \pm(\lambda - \alpha_1)^{n_1} \cdots (\lambda - \alpha_k)^{n_k}$. Don't be afraid by this expression. This says the characteristic polynomail splits and its eigenvalues are $\alpha_1, ..., \alpha_k$. Then the dimension of each eigenspace E_{α_i} is bounded:

$$1 \leq \dim E_{\alpha_i} \leq n_i$$

. As $\sum n_i = n$, we see that condition 2) is equivalent to saying that $E_{\alpha_i} = n_i$ for each i.

· If there are n distinct eigenvalues, then A is has an eigenbasis. (This follows from the previous two statements)

· The product of two diagonalizable matrices is NOT necessarily diagonalizable.

example

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

are both diagonalizable (both have 2 distinct eigenvalues, so we can use the previous fact to conclude this).

However their product $AB = C$

$$C = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$$

which is not diagonlizable. Its char. polynomial is $(\lambda - 3)^2$, so its only eigenvalue is 3. But $C - 3I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ and the dimension of its nullspace is only 1.

· Not all matrices have characteristic polynomials that split (this is a sign they don't have very many eigenvalues). An extreme example of this is the rotation matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Its characteristic polynomial is $\lambda^2 + 1$ which has no real solutions.

· We say that A is similar to B if there exists an invertible matrix C such that $A = CBC^{-1}$. Similar matrices have the same eigenvalues, and the dimensions of their eigenspaces are the same. Careful here though. The eigenspaces are not the same spaces. They are related in by the fact that if $\overrightarrow{x}$ is an eigenvector for B with associated eigenvalue λ, then $C\overrightarrow{x}$ is an eigenvector for A with associated eigenvalue λ. This is because:

$$A(C\overrightarrow{x} = CBC^{-1}(C\overrightarrow{x})$$

$$= CBI_n\overrightarrow{x}$$

$$= CB\overrightarrow{x}$$

$$= C(\lambda\overrightarrow{x}) = \lambda(C\overrightarrow{x})$$

In fact this shows the eigenvalues are the same, and since C is invertible, if we think for an eensy bit this computation also shows why the dimensions of the eigenspaces are the same.

· It is sometimes useful to have examples of matrices whose eigenspaces are dimensionally small compared to the number of times it factors in the matrix's characteristic polynomial.

For instance, suppose the char. poly of A is $(\lambda - 2)^3$. Find a matrix A whose eigenspace E_2 is three dimensional, two dimensional, one dimensional (and these are the only options).

Here are some respective examples:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

I'm sure you notice the pattern here. Every time we add a one along the super diagonal, we cut down the dimension of E_2 by one.

Computations with eigenvectors

Example 1 Let $A = \begin{bmatrix} 5 & 3 & -30 \\ 0 & 2 & 0 \\ 1 & 1 & -8 \end{bmatrix}$. Find the eigenvalues and eigenspaces of A. Is A diagonalizable?

solution:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 3 & -30 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & -8-\lambda \end{vmatrix}$$

$$= -(\lambda - 2)^2(\lambda + 5)$$

So the eigenvalues of A are 2 and -5. Before finding the eigenspaces E_2 and E_{-5}, we can use the characteristic polynomial to say a lot about the dimensions of these spaces:

1) $1 \leq \dim E_2 \leq 2$

2) $1 \leq \dim E_{-5} \leq 1$ so the dimension of E_{-5} must be 1.

Alright, lets be good boys and girls and find the eigenspaces now.

E_2 : Here we solve for the nullspace of $A - 2I$.

$$\left[\begin{array}{ccc|c} 3 & 3 & -30 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -10 & 0 \end{array}\right]$$

row reduces to

$$\left[\begin{array}{ccc|c} 1 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + 10x_3 \\ x_2 \\ x_3 \end{bmatrix}$. From this we conclude that $\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for E_2.

E_{-5} : Here we solve for the nullspace of $A + 5I$.

$$\left[\begin{array}{ccc|c} 10 & 3 & -30 & 0 \\ 0 & 7 & 0 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right]$$

row reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. From this we conclude that $\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis for E_{-5}.

Since $\dim E_2 = 2$, $\dim E_{-5} = 1$, and 2+1=3, we realize that A has an eigenbasis/is diagonalizable.

Example This is more an exercise in abstraction than anything else. Let $P =$ space of quadratic polynomials in one variable (In other words the elements of P are polynomials of the form $ax^2 + bx + c$). Let Φ be the map

$$\Phi : P \rightarrow P$$

$$ax^2 + bx + c \mapsto cx^2 + bx + a$$

In otherwords, Φ flips the order of the coefficients of the quadratic polynomial. Φ is linear (it is a good exercise to prove this. Let me know if you're having problems).
i) What is the matrix $[\Phi]$ representing Φ?
ii) What are the eigenvalues and eigenspaces of Φ?

solution

i) There is an obvious choice of coordinates here. The vector $ax^2 + bx + c$ should correspond to the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. To find $[\Phi]$ we should evaluate Φ on the standard basis vectors.

$$\Phi(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \Phi(1x^2 + 0x + 0) = 1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Phi(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \Phi(0x^2 + 1x + 0) = x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Phi(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \Phi(0x^2 + 0x + 1) = x^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

And so

$$[\Phi] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

ii) First we solve for the eigenvalues.

$$0 = \det([\Phi] - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix}$$

expand along the second row

$$= (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= -(\lambda-1)(\lambda^2-1) = -(\lambda-1)^2(\lambda+1)$$

. So the eigenvalues are 1 and -1.

As in the previous exercise, before finding the eigenspaces E_1 and E_{-1}, we can use the characteristic polynomial to say a lot about the dimensions of these spaces:
1) $1 \leq \dim E_1 \leq 2$

2) $1 \leq \dim E_{-1} \leq 1$ so the dimension of E_{-1} must be 1.

E_1 : Here we solve for the nullspace of $A - I$.

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

row reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. From this we conclude that $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis for E_1.

E_{-1} : Here we solve for the nullspace of $A + 5I$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

row reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solutions are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. From this we conclude that $\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis for E_{-1}.

Φ is diagonalizable we've found an eigenbasis.

As a side note, lets try to interpret what we found in terms of polynomials.

The basis vectors for E_1, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$ correspond to the polynomials x and x^2+1. Definitely they are fixed under Φ. More generally any polynomial in E_1 is a linear combination of these two polynomials. i.e $= a(x) + b(x^2+1) = ax^2 + bx + a$. These are precisely the kinds of polynomials that are unchanged when we flip the order of the coefficients.

The basis vector for E_{-1}, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ corresponds to the polynomial $-x^2+1$. So any polynomial in the -1 eigenspace is a multiple of this one i.e. it is of the form $-ax^2+a$. When we swap the order of its coeffecients we get ax^2-a which is precisely equal to -1$\times(-ax^2+a)$. This jives with the fact that these polynomials should be eigenvectors with associated eigenvalue -1.