1 Discussion 8

Today half of the discussion was on linear transformations. The second half was spent answering your questions about the exam. I'm not going to type up the second half.

Example 1 Let $E : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the line spanned by $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

a) Is E linear?

solution: We need to verify two equations:

i) $E(\overrightarrow{x} + \overrightarrow{y}) = E\overrightarrow{x} + E\overrightarrow{y}$ for all $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$

ii) $E(c\overrightarrow{x}) = c(E\overrightarrow{x})$ for all $c \in \mathbb{R}$ and $\overrightarrow{x} \in \mathbb{R}^2$.

To get started, we need to remember the formula for projection. In our case it will be:

$$E\overrightarrow{x} = \frac{\overrightarrow{x} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Now we verify i). Let $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$, then:

$$E(\overrightarrow{x} + \overrightarrow{y}) = \frac{(\overrightarrow{x} + \overrightarrow{y}) \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \left(\frac{\overrightarrow{x} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} + \frac{\overrightarrow{y} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} \right) \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= \frac{\overrightarrow{x} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \frac{\overrightarrow{y} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= E\overrightarrow{x} + E\overrightarrow{y}.$$

Now for ii). Let $\overrightarrow{x}$ and $c \in \mathbb{R}$.

$$E(c\overrightarrow{x}) = \frac{(\overrightarrow{cx}) \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= c\frac{\overrightarrow{x} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\|^2} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= c(E\overrightarrow{x}).$$

We have verified that E is linear.

b) What is the '2×2' matrix, $[M]_E$ that represents E?

solution: As many of you now know by heart, in general the formula for $[M]_E$ is

$$\begin{bmatrix} | & | \\ E\overrightarrow{e_1} & E\overrightarrow{e_2} \\ | & | \end{bmatrix}$$

Now we must go about our business finding out what the actual values of $E\overrightarrow{e_1}$ and $E\overrightarrow{e_2}$ are. Plugging $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ into the projection formula tells us that:

$$E\overrightarrow{e_1} = \frac{\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}3\\-1\end{bmatrix}}{\left|\left|\begin{bmatrix}3\\-1\end{bmatrix}\right|\right|^2}\begin{bmatrix}3\\-1\end{bmatrix} = \frac{3}{10}\begin{bmatrix}3\\-1\end{bmatrix}$$

$$E\overrightarrow{e_2} = \frac{\begin{bmatrix}0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\-1\end{bmatrix}}{\left|\left|\begin{bmatrix}3\\-1\end{bmatrix}\right|\right|^2}\begin{bmatrix}3\\-1\end{bmatrix} = -\frac{1}{10}\begin{bmatrix}3\\-1\end{bmatrix}$$

so

$$E = \frac{1}{10}\begin{bmatrix}9 & -3\\-3 & 1\end{bmatrix}$$

c) For later evil purposes, I'm going to ask whether or not the nullspace of $[M]_E$ is trivial.

solution: Clearly it is not because the columns of $[M]_E$ are linearly dependent. While we're at it, we might as well row reduce and find a basis. It row reduces to

$$\begin{bmatrix}1 & -\frac{1}{3}\\0 & 0\end{bmatrix}$$

$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\frac{1}{3}x_2\\x_2\end{bmatrix}$. So $\{\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}\}$ is a basis for the nullspace of $[M]_E$.

Example 2

Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be counter clockwise rotation by 90˙ about the z-axis. (In this problem we are orienting the x,y, and z axis according to the right-hand rule, and we assume that $\overrightarrow{e_1}$ lies in the x-axis, $\overrightarrow{e_2}$ in the y-axis, and $\overrightarrow{e_3}$ in the z).

a) What is the '3×3' matrix $[M]_F$ that represents F? (I think this problem is actually a duplicate from a previous discussion, so it may seem familiar)

solution: Again, as per the usual, we know that abstractly the formula for $[M]_F$ is given by

$$\begin{bmatrix} | & | & | \\ F\overrightarrow{e_1} & F\overrightarrow{e_2} & F\overrightarrow{e_3} \\ | & | & | \end{bmatrix}$$

If we put on our visualization goggles (drawing a picture helps here...bust out your pastels), we see that F sends the positive x-axis to the positive y-axis. So $F\overrightarrow{e_1} = \overrightarrow{e_2}$. We see that F sends the positive y-axis to the negative x-axis. So $F\overrightarrow{e_2} = -\overrightarrow{e_1}$, and lastly we see that z-axis just chills, so $F\overrightarrow{e_3} = \overrightarrow{e_3}$. Plugging in this newfound information tells us

$$[M]_F = \begin{bmatrix} | & | & | \\ \overrightarrow{e_2} & -\overrightarrow{e_2} & \overrightarrow{e_3} \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) As in the previous example, for nefarious purposes, I ask whether or not the nullspace of $[M]_F$ is trivial.

solution: This time the answer is yes. There are a number of (very much related) ways to justify this. For one, it is clear the columns of $[M]_F$ are linearly independent. Or, you can check that after row reducing there are no free columns.

Philosophomosizing So here is where I got on my soapbox, drank my Hemlock etc. Suppose we're curious little cats and we think to ourselves, "hmmm, is there any linear transformation, G, that *undoes* F?" (or in lingua mathematicae if $F\overrightarrow{x} = \overrightarrow{y}$, then $G\overrightarrow{y} = \overrightarrow{x}$). Well, in class most of you quickly pointed out that obviously the answer for this particular F was a resounding YES, namely define G to be *clockwise* rotation by 90˙ along the z-axis. It must be then that $F \circ G = G \circ F = I_3$. We even confirmed this using matrices since we know that the matrix $[M]_{G\circ F}$ that represents $G \circ F$ is equal to $[M]_G[M]_F$. Using almost identical reasoning as for F we find that $G\overrightarrow{e_1} = -\overrightarrow{e_2}$, $G\overrightarrow{e_2} = \overrightarrow{e_1}$, and $G\overrightarrow{e_3}\overrightarrow{e_3}$. Then

$$[M]_G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and we find

$$[M]_F[M]_G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ the identity matrix!}$$

Now we ask ourselves, "is there an inverse for E?" It turns out there isn't. Suppose D were an inverse. Remember what it means for a map D to be an inverse of E: that if $E\overrightarrow{x} = \overrightarrow{y}$ then $\overrightarrow{x} = D\overrightarrow{y}$. Now we run into problems because we remember that since the nullspace has a nonzero vector, $\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$,

$E(\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}) = \overrightarrow{0}$. But also $E\overrightarrow{0} = \overrightarrow{0}$. If we try to find an inverse, D, we would conclude that D would have to map $\overrightarrow{0}$ to $\overrightarrow{0}$ AND also map $\overrightarrow{0}$ to $\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$. This means we're stuck defining D. The problem was that E mapped two vectors to the same vector. It turns (a good little exercise) that this happens if and only if the nullspace of $[M]_E$ is trivial. Now we formulate a rule:

Generally, **Observation -** If F is invertible then $[M]_F$ must have a trivial nullspace.

Exercise: Let A be a 'm×n' matrix. Then the map $A : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if $N(A) = \{\overrightarrow{0}\}$. (see me if this is too hard for you).

Example 3 Can a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ have an inverse?

solution: No. The matrix $[T]_A$ representing T has more columns than rows, so it must have at least one free column. As the number of free columns is equal to the dimension of the nullspace, we conclude the nullspace is nontrivial and hence T cannot be invertible (by **Observation**). This is an excellent spot to use the rank-nullity theorem. The argument is: $rk(T) + dimN([M]_T) = 4$. Now since $rk(T) = dimC([T]_A)$ and the columnspace of $C(A)$ is a subspace of $\mathbb{R}^2$ (which has dimension 2), we see that $rk(T) \leq 2$. Then

$$dimN(T) = 4 - rk(T) \geq 2$$

So actually we find that the nullspace has dimension greater than or equal to 2 (and less than or equal to 4).

A number of people pointed out that we know T can't be invertible because invertible matrices must be 'n×n'. This is definitely true, and I don't want to take anything away from the validity of this point! However today we were "starting from scratch". I was trying to pretend we didn't know this fact about square matrices and was working towards actually showing why it is true...not sure how well that plan worked:)