Hey there boys and girls. Today we covered subspaces and linear transformations. Warning, I left my notes at home, so this is just what I remember covering on Thursday.

# 1 Subspaces

**Definition 1.1.** *Let $V$ be a subset of $\mathbb{R}^n$. $V$ is a subspace if:*
*1) $\vec{0} \in V$*
*2) if $\vec{x}, \vec{y} \in V$ then $\vec{x} + \vec{y} \in V$ (closed under addition)*
*3) if $\vec{x} \in V$ and $c \in \mathbb{R}$ then $c\vec{x} \in V$ (closed under scalar multiplication)*

In other words, to verify that a set $V$ is a subspace, we just need to check the three conditions above.

**Exercise 1.1.** *Let $A$ be a '$m \times n$' matrix. Show that the nullspace, $N(A)$, is a subspace of $\mathbb{R}^n$.*

**solution:** As a prelude, lets just be clear: $N(A) := \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$. When we want to show that a vector $\vec{y}$ is a member of a set $\{\cdots | \cdots\}$ we have to show that $\vec{y}$ satisfies the conditions to the right of the "|", which in this case is just the equation $A\vec{x} = \vec{0}$.

1) showing $\vec{0} \in N(A)$: We must show that $\vec{0}$ satisfies the equation $A\vec{x} = \vec{0}$. This is obviously true.

2) showing closure under addition: The condition for addition involves two vectors in $N(A)$, so we need to give ourselves two such vectors. Let $\vec{y}, \vec{z} \in N(A)$. We want to show $\vec{y} + \vec{z} \in N(A)$. This means we must show $A(\vec{y} + \vec{z}) = \vec{0}$ (i.e. $\vec{y} + \vec{z}$ satisfies the equation to the right of the "|".

$$A(\vec{y} + \vec{z})$$

$$= A\vec{y} + A\vec{z}$$
$$= \vec{0} + \vec{0}$$
$$= \vec{0}$$

So yes, $\vec{y} + \vec{z}$ satisfies the equation $A\vec{x} = \vec{0}$ which means that $\vec{y} + \vec{z} \in N(A)$. So $N(A)$ is closed under addition. The key/tricky step is going from the second to the third line where we use the fact that $\vec{x} \in N(A)$ and $\vec{y} \in N(A)$.

3) Similar in feel to 2), let $\vec{y} \in N(A)$ and let $c \in \mathbb{R}$. Then we want to show that $c\vec{y}$ satisfies the defining condition for being in $N(A)$, namely that $A(c\vec{y}) = \vec{0}$. Well,

$$A(c\vec{y}) = cA\vec{y}$$

and now since $\vec{y} \in N(A)$

$$= c\vec{0}$$
$$= \vec{0}.$$

From this we conclude $c\vec{y} \in N(A)$. We are done showing $N(A)$ is a subspace.

**Exercise 1.2.** *Show that* $V := \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = 3\vec{x} \}$ *is a subspace.*

**solution** Ok, this is very similar to the previous problem, so we'll do it with less explanation. Remember though that the defining condition for $\vec{v}$ to be an element of $V$ is for $\vec{v}$ to satisfy the conditions/equations to the right of the "|". Now we verify the three criteria for a subspace:

1) $A\vec{0} = \vec{0} = 3\vec{0}$. So $\vec{0} \in V$.

2) Let $\vec{v}, \vec{w} \in V$. Then $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = 3\vec{v} + 3\vec{w} = 3(\vec{v} + \vec{w})$. So $V$ is closed under addition.

3) Let $\vec{v} \in V$ and let $c \in \mathbb{R}$. Then $A(c\vec{v}) = cA\vec{v} = c(3\vec{v} = 3(c\vec{v})$. So $c\vec{v} \in V$. So $V$ is closed under scalar multiplication.

**Subsets of $\mathbb{R}^n$ which are not subspaces:** Here are some examples:

1) The set of integers - is not a subset of $\mathbb{R}$ because it is not closed under scalar multiplication.

2) The solution space to $A\vec{x} = \vec{b}$ is not a subspace of $\mathbb{R}^n$ if and only if $\vec{b} \neq \vec{0}$. As a very small subexample, Let $A$ be the $1 \times 1$ matrix $[1]$, and let $\vec{b} = \begin{bmatrix} 3 \end{bmatrix}$. Then the solution space to $[1]\vec{x} = \begin{bmatrix} 3 \end{bmatrix}$ is exactly $\{x = \begin{bmatrix} 3 \end{bmatrix}\}$. This space fails all criteria!

3)The union of the first and third quadrants of $\mathbb{R}^2$ is not a subspace because it is not closed under addition.

4) The set of solutions to the equation $3x + 4y - z = 8$. This is actually a subcase of the second non-example.

OK, enough of this stuff.

# 2 Linear Transformations

**Definition 2.1.** *A map/function* $T : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear transformation if it satisfies the following criteria:*

*1) For any* $\vec{x}, \vec{y} \in \mathbb{R}^n$, $T(\vec{x} + \vec{y}) = T\vec{x} + T\vec{y}$.

*2) For any* $\vec{x} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$, $T(c\vec{x}) = cT\vec{x}$

**Exercise 2.1.** *Verify that* $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (-2y + x, x)$ *is a linear transformation.*

Note: For now when we talk about linear transformations often time we go back and forth between the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and the point $(x, y, z)$. Consult the notes from the first week if you want some explanation about vectors vs. points (or email me).

**solution:** We need to check that $T$ satisfies the two conditions for being linear. 1) Let $\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. Then

$$T((x_1, y_1, z_1) + (x_2, y_2, z_2))$$

$$= T((x_1 + x_2, y_1 + y_2, z_1 + z_2))$$

use the definition of $T$,

$$= (-2(y_1 + y_2) + (x_1 + x_2), x_1 + x_2)$$

separate the variables in terms of "1"'s and "2"'s,

$$= (-2y_1 + x_1, x_1) + (-2y_2 + x_2, x_2)$$

and using the definition of $T$ we recognize this to be

$$= T((x_1, y_1, z_1)) + T((x_2, y_2, z_2))$$

**Example 2.1.** *$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (xy, y)$ is NOT linear.*

For instance it fails to preserve the second property: Let $c = 2$ and $(x, y) = (1, 1)$. Then $T(2(1, 1)) = T((2, 2)) = (4, 2)$. On the otherhand $2T((1, 1)) = 2(1, 1) = (2, 2)$. If $T$ were linear then these two 2-tuples would have to be equal, which they are clearly not.

Note: If you are given a formula for $T$ then you will quickly be able to identify linear functions. They are those functions whose components looks like something of the form $(..., ax + by + cz + c \ldots + dw, ...)$. Anything else is NOT linear. Even the constant function $T(x) = 2$ is NOT linear (for instance $T(1 + 1) = T(2) = 2$ but on the otherhand $T(1) + T(1) = 2 + 2 = 4$).

# 3 Exam Problems

Okay, we took a detour from linear transformations to go over two problems that have more of a midterm flavor.

**Exercise 3.1.** *For what values of $x, y \in \mathbb{R}$ does the matrix*

$$A := \begin{bmatrix} x & y \\ x^2 & y^2 \end{bmatrix}$$

*have $\dim N(A) = 0$. For what values of $x, y$ is $\dim N(A) = 1, = 2$?*

**solution:** Remember that the dimension of the nullspace is equal to the number of free columns of $A$. This suggests we should try row-reducing $A$. Lets get going.

Immediately we are confronted with two cases, $x = 0$ and $x \neq 0$.

**Case 1:** $x = 0$ Here the matrix $A$ reduces to

$$\begin{bmatrix} 0 & y \\ 0 & y^2 \end{bmatrix}.$$

Clearly the first column is free, so we move onto the second column. Again we have two options. If $y = 0$ the matrix is the zero matrix, which has $\dim N(A) =$

2. Otherwise, if $y \neq 0$, then it is clear the second column becomes a pivot column and thus $\dim N(A) = 1$.

**Case 2:** $x \neq 0$ Here we begin by row-reducing the matrix. The first column is evidently a pivot column. After one step we get

$$\begin{bmatrix} x & y \\ 0 & y^2 - xy \end{bmatrix}.$$

Here we can see that the second column will be a free column precisely when

$$y^2 - xy = 0.$$

This equation factors as $y(y - x) = 0$. Thus the second column is free if $y = 0$ or if $x = y$ (notice that we already assumed that $x \neq = 0$ here, so $x = y$ implies neither $x$ nor $y$ is 0).

OK, now we're ready to summarize our results. $\dim N(A) = 2$ when $x = y = 0$. $\dim N(A) = 1$ when $x \neq 0$ and either $x = y$ or $y = 0$ Finally $\dim N(A) = 0$ when neither $x$ nor $y$ are zero and $x \neq y$.

Now it is time for a problem which is admittedly hard. It was supposed to be a lesson it trying to use some of your cleverness and theory you've acquired instead of blindly plugging and chugging. Keep this in mind especially when on an exam you are given a huge matrix which would take 30 minutes to row-reduce....there is probably another way.

**Exercise 3.2.** *Let $A$ be the following matrix:*

$$A := \begin{bmatrix} 1 & -13 & 2 & 0 & 5 & 9 & -6 \\ 2 & 4 & 11 & 0 & 1 & 1 & -1 \\ 3 & -0 & 3 & 0 & 3 & 2 & -4 \\ 0 & -2 & -5 & 4 & 5 & -2 & -10 \\ 0 & -1 & 3 & 5 & 7 & 2 & 8 \\ 0 & 7 & 7 & 6 & 0 & 1 & 1 \end{bmatrix}$$

*which you are told row-reduces to*

$$RRef(A) := \begin{bmatrix} 1 & 3 & 0 & 7 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Find all solutions to the equation*

$$A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The key to this problem is to remember that the set of solutions of the equation $A\vec{x} = \vec{b}$ is given by translating the set of solutions of the equation $A\vec{x} = \vec{0}$ by $\vec{x}_p$ where $\vec{x}_p$ is *some* solution, $A\vec{x}_p = \vec{b}$. So we need two ingredients. We need to find $N(A)$ and we need to find *a* solution to $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Since we are given $RRef(A)$ it is easy to do the first part.

First we use the rows containing the pivot variables to write the pivot variables in terms of the free variables:

$$x_1 + 3x_2 + 7x_4 - 3x_7 = 0$$

$$x_3 - 2x_5 - x_7 = 0$$

$$x_6 + 7x_7 = 0$$

yields

$$x_1 = -3x_2 - 7x_4 + 3x_7$$

$$x_3 = 2x_5 + x_7$$

$$x_6 = -7x_7$$

Thus the null space of $A$ is parametrized by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} -3x_2 - 7x_4 + 3x_7 \\ x_2 \\ 2x_5 + x_7 \\ x_4 \\ x_5 \\ -7x_7 \\ x_7 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ -7 \\ 1 \end{bmatrix}$$

for all $x_2, x_4, x_5, x_6 \in \mathbb{R}$.

Now we notice that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ appears as the first column of the matrix $A$. Also remember the important fact that the standard basis vector $\vec{e}_i$ (you know, the one whose's components are all zero except for the $i^{th}$ component which is 1) enjoys the property that

$$A\vec{e}_i = \text{ the } i^{th} \text{ column of } A.$$

So in the case when $i = 1$, we see that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Aha, we've found a particular solution to our equation $A\vec{x} = \vec{b}$. It is $\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Thus all set of all solutions to the equation are parametrized by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ -7 \\ 1 \end{bmatrix}$$

for all $x_2, x_4, x_5, x_6 \in \mathbb{R}$.

Michelle asked why this fact that the solutions to the equation $A\vec{x} = \vec{b}$ are given by adding one particular solution, $\vec{x}_p$, to the vector in $N(A)$. I gave a proof, but it was a bit hurried, and I'm not sure all of you followed. Here it is again.

**Exercise 3.3.** *Let $A$ be an $m \times n$ matrix, and $\vec{b} \in \mathbb{R}^m$. Let $V = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{b} \}$. Suppose $\vec{x}_p$ is one particular solution to the equation $A\vec{x} = \vec{b}$. Then $V$ is equal to the set of vectors of the form $\vec{x}_p + \vec{y}$ where $y \in N(A)$.*

**solution:** For convenience, lets give the set of vectors of the form $\vec{x}_p + \vec{y}$ with $y \in N(A)$ a name. Lets call this set $\vec{x}_p + N(A)$. We wish to show that $V = \vec{x}_p + N(A)$. In mathematics when we want to show that two sets are equal the standard approach is to show that each set is a subset of the other (or in other words, that every element in the first set also belongs to the second set, and conversely that every element in the second set belongs in the first). Math language for "is a subset of" is the symbol $\subset$. Thus we must prove two statements: $V \subset \vec{x}_p + N(A)$ and $\vec{x}_p + N(A) \subset V$.

Lets prove $\vec{x}_p + N(A) \subset V$ first. Suppose we have an element $\vec{x} \in \vec{x}_p + N(A)$. This means that $\vec{x} = \vec{x}_p + \vec{y}$ for some $\vec{y} \in N(A)$. We want to show that $\vec{x} \in ; V$ or by definition of $V$, that $A\vec{x} = \vec{b}$. Well, lets see:

$$A\vec{x} = A(\vec{x}_p + \vec{y}) = A\vec{x}_p + A\vec{y}.$$

$$= \vec{b} + A\vec{y}$$

because by hypothesis $A\vec{x}_p = \vec{b}$. Now, we remember that $\vec{y}$ is in $N(A)$ so the expression reduces to

$$= \vec{b} = \vec{0}$$

which is what we wanted to show. Thus $\vec{x} \in V$ and we've shown that $\vec{x}_p + N(A) \subset V$.

Now to finish we must prove $V \subset \vec{x}_p + N(A)$. Let $\vec{x} \in V$. We need to show that $\vec{x} \in \vec{x}_p + N(A)$ or in other words that there is some $\vec{y} \in N(A)$ such that $\vec{x} = \vec{x}_p + \vec{y}$. Notice that an equivalent statment is showing that $\vec{x} - \vec{x}_p$ is an element of $N(A)$. To show this we need to verify that $A(\vec{x} - \vec{x}_p) = \vec{0}$. Well:

$$A(\vec{x} - \vec{x}_p) = A\vec{x} - A\vec{x}_p$$

and now by hypothesis both $A\vec{x}$ and $A\vec{x}_p$ are equal to $\vec{b}$, so the equation becomes

$$= \vec{b} - \vec{b} = \vec{0}$$

So $\vec{x} - \vec{x}_p$ is in the nullspace of $A$. This implies that $x \in \vec{x}_p + N(A)$. Thus $V \subset \vec{x}_p + N(A)$. This completes the proof.

As usual, if you're unclear about something, send me an email, or visit my office. Enjoy the weekend.