1 Discussion 4

Today we covered:

· Distance from a point to a line, distance from a point to the plane.
· Matrix Multiplication
· Nullspace of a matrix

Shortest distance from a point to a line

Here are the steps for finding the distance from a point to a line:

1) Find <u>*any*</u> vector from the line to the point. Call it $\vec{a}$.

2) Find a vector lying in the line, lets call it $\vec{b}$. Project $\vec{a}$ onto $\vec{b}$ using the formula $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$. Lets call this vector $\vec{x}$.

3) The vector representing the shortest line from the point to the line is then $\vec{a}-\vec{x}$.

4) Its length is $|\vec{a}-\vec{x}|$.

Example 1.1. *Find the shortest distance from the point (2,5,-3) to the line parametrized by* $\begin{bmatrix}2\\3\\1\end{bmatrix}+s\begin{bmatrix}1\\0\\1\end{bmatrix}$.

1) $\vec{a}=\begin{bmatrix}2-2\\5-3\\-3-1\end{bmatrix}=\begin{bmatrix}0\\2\\-4\end{bmatrix}$.

2) $\vec{x}=\dfrac{\begin{bmatrix}0\\2\\-4\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}}{\left|\begin{bmatrix}1\\0\\1\end{bmatrix}\right|^2}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{-4}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}-2\\0\\-2\end{bmatrix}$.

3) $\vec{a}-\vec{x}=\begin{bmatrix}0\\2\\-4\end{bmatrix}-\begin{bmatrix}-2\\0\\-2\end{bmatrix}=\begin{bmatrix}2\\2\\-2\end{bmatrix}$.

4) Then shortest distance is then $\left|\begin{bmatrix}2\\2\\-2\end{bmatrix}\right|=\sqrt{4+4+4}=3\sqrt{2}$.

Shortest distance from a point to a plane

Here are the steps for finding the distance from a point to a plane. The key difference from before is now we want a normal vector to the plane and we don't do any subtractions.

1) Find <u>any</u> vector from the line to the point. Call it $\vec{a}$.

2) Find a normal vector for the plane, call it $\vec{n}$.

3) Project $\vec{a}$ onto $\vec{n}$ using the formula $\frac{\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}$. This is the vector representing the shortest line segment from the point to the plane.

4) Its length is the absolute value of $\frac{\vec{a}\cdot\vec{n}}{|\vec{n}|}$.

Example 1.2. *Find the distance between the point (2,1,7) and the plane given by the equation* $3x + 4(y-7) = 0$

solution: 1) $\vec{a} = \begin{bmatrix} 2-0 \\ 1-7 \\ 7-0 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix}$. Aside: as Rui pointed out, the last component of this vector is arbitrary, because we should think of the equation

$$3x + 4(y-7)$$

as really being

$$3x + 4(y-7) + 0(z-*),$$

so $\vec{a} = \begin{bmatrix} 2-0 \\ 1-7 \\ 7-* \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ * \end{bmatrix}$.

2) $\vec{n} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$.

3/4) The distance is the absolute value of $\frac{\begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}}{\left|\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}\right|} = \frac{6-24+0}{\sqrt{9+16}} = |\frac{-18}{5}| = \frac{18}{5}$.

Nullspace of a matrix A

· The nullspace of a matrix A is the space of solutions to the equation $A\vec{x} = \vec{0}$.

· If the number of columns is bigger than the number of rows then there is an infinite number of solutions. (This is because the number of pivot columns is at most equal to the number of rows, which by our assumption is less than the number of columns. So at least one of the columns must be a free column).

· There is ALWAYS a solution to $A\vec{x} = \vec{0}$, namely $\vec{x} = \vec{0}$.

· Next time we will solve $A\vec{x} = \vec{b}$, where the $\vec{b}$ isn't necessarily zero. Then the two previous observations aren't true!!

· to find the nullspace of A, rewrite the equation $A\vec{x} = \vec{0}$ as an augmented matrix $[A|\vec{0}]$ and solve using the row-reduction methods we've been practicing.

Example 1.3. *Find the nullspace for* $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 3 & 2 \end{bmatrix}$.

solution:

$$A = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right]$$

multiply row 1 by $\frac{1}{2}$.

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 4 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right]$$

add -4 $\times$ row 1 to row 2.

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 3 & 2 & 0 \end{array}\right]$$

add -3 $\times$ row 1 to row 2.

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right]$$

swap row 2 and row 3.

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

multiply row 2 by 2.

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Now that A has been row reduced, we write out the real-variable equations associated to each row.

$$x_1 + 0x_2 = 0$$

$$0x_1 + 1x_2 = 0$$

$$0x_1 + 0x_2 = 0$$

simplifying to:

$$x_1 = 0$$

$$x_2 = 0$$

All solutions are of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So $\vec{0}$ is the only solution.

Example 1.4. *Find the nullspace for* $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 3 & 2 \end{bmatrix}$.

solution:

$$A = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \\ -3 & -6 & 0 \end{array}\right]$$

add -2 × row 1 to row 2.

$$A = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ -3 & -6 & 0 \end{array}\right]$$

add 3 × row 1 to row 3.

$$A = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

A is now in RRef. We write out the real-variable equations associated to each row and express each pivot variable in terms of the free variables.

$$x_1 + 2x_2 = 0$$

$$0x_1 + 0x_2 = 0$$

$$0x_1 + 0x_2 = 0$$

giving

$$x_1 = -2x_2$$

. So the nullspace is parametrized as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Example 6: Find the nullspace for $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

solution:
Again, we row-reduce the augmented matrix $[A|\vec{0}]$.

add -1×row 1 to row 2.

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & -1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

multiply row 2 by -1.

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

add -1×row 2 to row 3.

$$A = \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right]$$

add -2×row 2 to row 3.

$$A = \left[\begin{array}{cccc|c} 1 & 0 & -3 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right]$$

multiply row 3 by -$\frac{1}{3}$.

$$A = \left[\begin{array}{cccc|c} 1 & 0 & -3 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

add -3×row 3 to row 2.

$$A = \left[\begin{array}{cccc|c} 1 & 0 & -3 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

add 3×row 3 to row 1.

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

A is now in RRef. We write out the real-variable equations associated to each row and express each pivot variable (x_1, x_2, x_3) in terms of the free variables (x_4).

$$x_1 = 0$$

$$x_2 + x_4 + 0$$

$$x_3 - x_4 = 0$$

yielding

$$x_1 = 0$$
$$x_2 = -x_4$$
$$x_3 = x_4$$

.

The nullspace is parametrized by: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_4 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Example 1.5. *Find the space of all vectors normal to the plane spanned by* $\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 2 \\ 4 \\ 1 \end{bmatrix}$.

solution: This space is in fact equal to the space of all vectors normal to the individual vectors $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix}$ (why? basically linearity of dot product, but see me if you still don't see why). So we're looking to find the space of all $\vec{x}$ such that $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \cdot \vec{x} = 0$ and $\begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix} \cdot \vec{x} = 0$. Now remember the first formula for matrix multiplication, and see that these two dot product conditions on $\vec{x}$ are precisely the same as asking $\vec{x}$ to satisfy the equation

$$\begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 2 & 4 & 2 \end{bmatrix} \vec{x} = \vec{0}$$

which is a nullspace problem. Lets do it.

$$\left[\begin{array}{cccc|c} 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 4 & 2 & 0 \end{array}\right]$$

swap rows to get a pivot.

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right]$$

multiply row 2 by $\frac{1}{2}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

add -2×row 2 to row 1.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

A is now in RRef. We write out the real-variable equations associated to each row and express each pivot variable (x_1, x_2) in terms of the free variables (x_3, x_4).

$$x_1 + 3x_3 + x_4 = 0$$
$$x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 0$$

yielding

$$x_1 = -3x_3 - x_4$$
$$x_2 = -\frac{1}{2}x_3 - \frac{1}{2}x_4$$

The nullspace is parametrized by: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 - x_4 \\ -\frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -\frac{1}{2}x_3 \\ x_3 \\ 0 \end{bmatrix} +$ $\begin{bmatrix} -x_4 \\ -\frac{1}{2}x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$. This is the space is the desired solution.

If you're still not comfortable RRef-ing and using the RRef form to find a parametrization for the solution space (and by comfortable, I mean bordering on being able to do it in your sleep), this is a moderate sign for concern. Practice, and/or come by my office.

Extra Exercise 1 Find the distance between the point (1,-1,0) and the plane parametrized by $\begin{bmatrix} 2 \\ -5 \\ 5 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$. (Hint: use the method outlined above...well, we need a normal vector, so your first step should be to find a normal vector.)

Extra Exercise 2 Find the distance between the point (1,-2,-1,0) and the space parametrized by $z_1 \begin{bmatrix} 1 \\ 5 \\ 1 \\ 0 \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} + z_3 \begin{bmatrix} 0 \\ 4 \\ 0 \\ 4 \end{bmatrix}$.