# 1 Determinants

When solving problems involving determinants keep the following facts in mind:

1) Expand along rows (or columns) with lots of zeros...okay, that's more of a suggestion than a fact.

2) If the rows (or columns) of a matrix are linearly dependent, then the determinant of a matrix is zero.

3) If $A$ and $B$ are square matrices then $\det(AB) = \det(A)\det(B)$.

**Exercise 1.1.** *Let*

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 4 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 3 & 0 & 4 & 0 \\ 4 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 7 & 4 & 6 \\ -1 & -7 & 2 & 4 \\ -3 & -21 & 3 & 0 \\ 3 & 21 & 1 & 2 \end{bmatrix}$$

*What are the determinants of $A, B, C,$ and $D$?*

**Solution:** i) For $A$ you can compute the determinant by hand, but it is quicker to notice that the first row + the second row = third row, so the row vectors of $A$ are linearly dependent. This implies that not all the rows will have pivots in them, so the matrix $A$ is not invertible. Thus $\det A$=0.

ii) OK, it seems you are all on to my usual tricks (I'll have to start covering my trails better). This is a classic "You Tarzan Me Jane" moment (sit back for a second and revel in it...) The determinant is only defined for square matrices. $B$ is not square, so $\det B$ is nonsensical.

iii) I got $\det C$=1. The point of iii) is that when you use the recursive definition of the determinant to calculate it, you can choose which column or row to expand along. It saves you time to expand along a row or column with as many zeros as possible. Also, as a brief note, it is easy to forget about the signs when expanding. The way to remember plus or minus is to imagine the matrix as being a checkerboard with black and white squares, a black square at the upper left. The black squares are "plus" and the white squares are "minus". If you want a formula, expanding using the $ij$th component should come along with an additional sign of $(-1)^{i+j}$. Ok, enough chit chat. In problem iii), the second column of $C$ is looking extra juicy with all those zeros, so lets expand along the second column.

$$C = \begin{vmatrix} 2 & 0 & 0 & 1 \\ 3 & 0 & 4 & 0 \\ 4 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 \end{vmatrix}$$

$$= -0 \begin{vmatrix} & | & \\ & sumthing & \\ & | & \end{vmatrix} + 0 \begin{vmatrix} & | & \\ & sumthing & \\ & | & \end{vmatrix} - 1 \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 0 \\ 5 & 1 & 2 \end{vmatrix} + 0 \begin{vmatrix} & | & \\ & sumthing & \\ & | & \end{vmatrix}$$

$$= - \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 0 \\ 5 & 1 & 2 \end{vmatrix}$$

Now the first row looks good to expand along.

$$= -2 \left( 2 \begin{vmatrix} 4 & 0 \\ 1 & 2 \end{vmatrix} - 0 \left| sumthing \right| + \begin{vmatrix} 3 & 4 \\ 5 & 1 \end{vmatrix} \right)$$

$$= -(2(8-0) + (3-20)) = -(16-17) = 1$$

iv) Again, like in i) $\det(D) = 0$. Here the columns are evidently linearly dependent. The second row is seven times the first row.

Again, the important ideas here are

· To compute the determinant, expand along a row or column that has lots of zeros. It will make computations faster.

· If the rows or columns of a matrix are linearly dependent, then we know the determinant is zero. Oftentimes it is worthwhile to quickly scan the matrix to see if there are any obvious linear dependencies.

**Exercise 1.2.** *Let*

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

*and*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

*What is* $\det(CA^{-1}C^{-1})$*?*

**Solution:** OK, in theory you could actually compute $C^{-1}$, multiply out $CAC^{-1}$, and then take the determinant. The better thing to know is "determinants are invariant under conjugation". This is overblown language for saying that:

$$\det(A^{-1}) = \det(CAC^{-1})$$

This is true because

$$\det(CA^{-1}C^{-1}) = \det(C)\det(A^{-1})\det(C^{-1})$$

$$= \det(C)\det(C^{-1})\det(A^{-1}) = \det(CC^{-1})\det(A^{-1}) = \det(Id)\det(A^{-1})$$

$$= \det(A^{-1}).$$

From this rule, it is clear then that $\det(CA^{-1}C^{-1}) = det(A)^{-1} = \frac{1}{\det(A)} = \frac{1}{24}$. Oh yes, and why is $\det(A^{-1}) = \frac{1}{\det(A)}$? That is because

$$1 = \det(Id) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

and then dividing both sides by $\det(A)$.

By the way, another nice fact to know is:

· The determinant of a diagonal matrix is equal to the product of the diagonal entries. More generally, if the matrix is upper (lower) triangular (meaning all the components below (above) the diagonal are zero), its determinant is also equal to the product of the diagonal entries. eg. without much thought,

$$\begin{vmatrix} 2 & 103 & 4 & -3 \\ 0 & 1 & 74 & -87 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & -5 \end{vmatrix} = 2 \times 1 \times -3 \times -5 = 30$$

Sarkis and Rui helped us to see why this is true, because you can row reduce an upper (lower) triangular matrix to a diagonal matrix only using the row operation "add a multiple of a row to a different row" and those kind of row operations don't change the determinant.

**Exercise 1.3.** *In these problems we asked ourselves if we could find a matrix A that satisfy the given criteria:*

*i) A is a '2×2' matrix, det(A)=3,* $A\vec{e}_2 = 3A\vec{e}_1$.

*ii) A is a '2×2' matrix, det(A)=3,* $\vec{e}_2 = 3A\vec{e}_1$ *(Afternoon section: I screwed the proverbial pooch and instead wrote* $\vec{e}_2 = A\vec{e}_1$*, so your notes will be slightly different).*

*iii) A is a '4×4'.* $A\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, *and* $A\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

**Solution:** i) Such a matrix can't exist. The condition $A\vec{e}_2 = 3A\vec{e}_1$ tells us that the columns of $A$ are linearly dependent, so $A$ isn't invertible (there are a number of valid ways to say this). So $\det(A)$ must equal 0. On the other hand, we stipulated that $\det(A)$=3. Truuuuble.

ii) Many possible answers exist. The condition that $\vec{e}_2 = 3A\vec{e}_1$ tells us that $A\vec{e}_1 = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}$. But $A\vec{e}_1$ is the first row of $A$. So $A = \begin{bmatrix} 0 & a \\ \frac{1}{3} & b \end{bmatrix}$ for some $a, b \in \mathbb{R}$. The condition on the determinant of $A$ says that $0(b) - \frac{1}{3}a = 3 \Rightarrow a = -9$. So any matrix of the form $\begin{bmatrix} 0 & -9 \\ \frac{1}{3} & * \end{bmatrix}$ satisfies our criteria.

iii) Such a matrix can't exist. As Michelle pointed out,

$$A \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

so evidently $A$ is not one-to-one $\leftrightarrow$ $A$ has a non-trivial nullspace $A$ is not invertible and $\det(A)$ must be equal 0. As before, we stipulated that $\det(A)$=3 so we are at an impasse. Truuuuble.

## 2 How linear transformations distort volumes/areas

**Fact 1:** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose $S$ is some region in $\mathbb{R}^2$. Then

$$\text{Area}(T(S)) = |\det(T)|\text{Area}(S)$$

This is a formula that actually holds in higher dimensions if we replace 'Area' with 'Volume'. Really it is a theorem from calculus which is proved using the slightly more preliminary fact which we did not prove in section, but here it is anyway.

**Fact 2:** Let $\{\vec{e_1}, \vec{e_2}\}$ be the standard basis. Then the area of the parallelogram whose sides are $T\vec{e_1}$ and $T\vec{e_2}$ is $|\det T|$.

I'm not going to type up the proof, because a perfectly good version may be found in your beloved textbooks on page 119. We did do a sample problem, which is more dramatic on the blackboard. Without pictures, the problem seems a bit underwhelming:(

**Exercise 2.1.** *Let $S$ be the intersection of the unit disk with the upper right quadrant of $\mathbb{R}^2$. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T\vec{e}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $T\vec{e}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. What is the area of $T(S)$, (which visually on the blackboard* *looks like a weird curvy subset of a parallelogram, whose are you wouldn't expect you could easily measure).*

**Solution:** surprise, surprise, we use the formula $\text{Area}(T(S)) = |\det(T)|\text{Area}(S)$. The area of $S$ is $\frac{1}{4}$ the area of the unit disk. So $\text{Area}(S) = \frac{1}{4} \times \pi \times (1)^2 = \frac{\pi}{4}$. The matrix representing $T$ is

$$\begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$$

and $\det(T) = 1(2) - 5(4) = -18$. So $\text{Area}(T(S)) = |-18|\frac{\pi}{4} = \frac{9\pi}{2}$. Remember a fairy dies every time you find that the area of $T(S)$ is a negative number. Don't forget the absolute value.

# 3 Inverses

**Exercise 3.1.** *Suppose $\{\vec{u}, \vec{v}, \vec{w}\}$ are linearly independent vectors in $\mathbb{R}^n$ and suppose $A$ is a $n \times n$ matrix. Also assume that for two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ satisfies $A(\vec{x}) = 3\vec{v} + \vec{w}$ and $A(\vec{x} + 2\vec{y}) = \vec{v} - \vec{u}$. Simplify $A^{-1}(-2\vec{u} + 5\vec{v} + \vec{w})$.*

**Solution:** There are two main steps to this problem. 1) Write $-2\vec{u}+5\vec{v}+\vec{w}$ as a linear combination of $3\vec{v} + \vec{w}$ and $\vec{v} - \vec{u}$. 2) Use this information and the linearity of $A^{-1}$ to finish the problem.

Let begin. We want to find $c_1, c_2 \in \mathbb{R}$ satisfying

$$-2\vec{u} + 5\vec{v} + \vec{w} = c_1(3\vec{v} + \vec{w}) + c_2(\vec{v} - \vec{u}).$$

Regroup the RHS in terms of $\vec{u}, \vec{v}, \vec{w}$

$$= -c_2\vec{u} + (3c_1 + c_2)\vec{v} + c_1\vec{w}$$

now since $\{\vec{u}, \vec{v}, \vec{w}\}$ are linearly independent, two linear combinations of these vectors are equal if and only if the coefficients are the same. This tells us that

$$-2 = c_2$$

$$5 = 3c_1 + c_2$$

$$1 = c_1$$

It is patently clear that $c_1 = 1$ and $c_2 = 2$ is a solution, but more generally you would/should solve this system using the matrix equation

$$\left[\begin{array}{cc|c} 0 & -2 & -2 \\ 3 & 1 & 5 \\ 1 & 0 & 1 \end{array}\right].$$

Okay now for the second step.

$$A^{-1}(-2\vec{u} + 5\vec{v} + \vec{w}) = A^{-1}((3\vec{v} + \vec{w}) + 2(\vec{v} - \vec{u}))$$

which we got by substituting the equation from the first step. Now using the linearity of $A^{-1}$ we get

$$= A^{-1}(3\vec{v} + \vec{w}) + 2A^{-1}(\vec{v} - \vec{u}).$$

Since $A(\vec{x}) = 3\vec{v} + \vec{w}$, we know that $\vec{x} = A^{-1}(3\vec{v} + \vec{w})$. Likewise we know that $\vec{x} + 2\vec{y} = A^{-1}(\vec{v} - \vec{u})$. Substituting this in yields,

$$= \vec{x} + 2(\vec{x} + 2\vec{y}) = 3\vec{x} + 4\vec{y}.$$

**Exercise 3.2.** *Keep the variables the same as in the previous problem. What is $A^{-1}(\frac{1}{2})$? What is $A^{-1}(-2\vec{u} + 4\vec{v} + \vec{w})$?*

**Solution:** Ok, the first problem is nonsense. $\frac{1}{2}$ isn't a vector. For the second problem, we need to express $-2\vec{u} + 4\vec{v} + \vec{w}$ as a linear combination of $3\vec{v} + \vec{w}$ and $\vec{v} - \vec{u}$. Following the steps from the previous example we end up wanting to solve the matrix equation

$$\left[\begin{array}{cc|c} 3 & 1 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right]$$

This row reduces to

$$\left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right].$$

From the bottom row we conclude there is no solution. Thus we cannot express $-2\vec{u} + 4\vec{v} + \vec{w}$ as a linear combination of $3\vec{v} + \vec{w}$ and $\vec{v} - \vec{u}$. There is not enough information to solve this problem.

OK, one last exercise before the Rubiks Cube break.

**Exercise 3.3.** *Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear transformation and $\{\vec{x}_1, ..., \vec{x}_n\}$ is linearly independent. Show $\{T\vec{x}_1, ..., T\vec{x}_n\}$ is linearly independent.*

**Solution:** We want to show that the only solution to $c_1 T\vec{x}_1 + \cdots + c_n T\vec{x}_n = \vec{0}$ is for all $c_i = 0$.

So suppose

$$c_1 T\vec{x}_1 + \cdots + c_n T\vec{x}_n = \vec{0}.$$

Because $T$ is linear we may factor it out,

$$T(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n) = \vec{0}.$$

Since $T$ is one-to-one, there is at most one vector $\vec{v}$ such that $T\vec{v} = \vec{0}$. But obviously $T\vec{0} = \vec{0}$ so $\vec{v}$ must equal $\vec{0}$ (or put another way, the kernel of a one-to-one linear transformation is always $\{\vec{0}\}$. So we conclude

$$c_1\vec{x}_1 + \cdots + c_n\vec{x}_n = \vec{0}$$

Ah, but now we remember that $\{\vec{x}_1, ..., \vec{x}_n\}$ is linearly independent so the equation above has only one solution: all the $c_i$'s must be zero. That is what we needed to show.