1 Change Of Basis

This week we were forcibly exposed to change of basis. The main hurdle to understanding change of basis is philosophical. At this point it might be useful to reread the first article I posted, "What are vectors?". Or don't. Either way, remember that "fork" is a four letter word that *refers* to a metal eating utensil with three or four tines, but isn't the actual physical object itself. "Tragedy" is a word used to describe any social gathering yours truly attends, but it is not the actual feelings of confusion, discomfort, and loss or the hot tears. And mind you, there are other equally valid ways to refer to these objects and situations. For instance, French people call say "la fourchette" and "la tragedie". These are different words, but they refer to the same objects as their English counterparts. Similarly, we have been using n-tuples of numbers eg. $\begin{bmatrix} 1 \\ 43 \\ 9 \end{bmatrix}$ to label vectors. The vectors themselves are directed line segments in $\mathbb{R}^n$ up to translational equivalence; they are NOT the n-tuples of numbers. We shall see that any basis, $\beta = \{\overrightarrow{v}_1, ..., \overrightarrow{v}_n\}$ gives us a way to equate vectors with n-tuples of numbers. As there are many different basis we can choose, there are many different ways to assign numbers to vectors. You might ask at this point how is it then that we've been using the n-tuples of numbers up to this point. Turns out we've been secretly using the standard basis this whole time without telling you (so for you conspiracy theorists, this is more evidence that what you're first taught is either a lie or a partial truth). But you know, we had to start somewhere, and since we're ethnocentric, fascist bastards we might as well say that before learning Hungarian you've got to learn English (standard basis) first and you don't get to choose, in fact we didn't even let you know there was a choice at the time.

But why do we need to go to this bother? Why shouldn't we just stick with the standard basis all the time. Why should we learn other languages? Some people refuse, and they're called Republicans. More generally why should we learn anything at all? Well, that's a question that is too hard for me to answer. But I do know that change of basis problems are on the second midterm. Also, some problems in linear algebra become conceptually simpler if we're working in a different basis. In particular, analyzing linear transformations is rather easy if we can work in an *eigenbasis*. Continuing this overripe analogy, Ichi the Killer is better if you can speak Japanese, Superbad (which half of you listed as your favorite movie) is funnier if you know English, and Chalino Sanchez is harder to appreciate if you don't speak Spanish.

In conclusion:

$$\text{vectors} \Leftrightarrow \text{objects}$$

$$\text{n-tuples of numbers} \Leftrightarrow \text{words}$$

$$\text{choice of basis} \Leftrightarrow \text{a language}$$

$$\text{change of basis matrix} \Leftrightarrow \text{an x to y dictionary}$$

Now for the math.

How does a basis equate n-tuples of numbers to vectors

Suppose we are given a basis for $\mathbb{R}^n$, $\beta = \{\overrightarrow{v}_1, ..., \overrightarrow{v}_n\}$, and also we have been presented with a vector $\overrightarrow{w} \in \mathbb{R}^n$. Here is how we associate an n-tuple of real numbers to $\overrightarrow{w}$.

As $\{\overrightarrow{v}_1, ..., \overrightarrow{v}_n\}$ span $\mathbb{R}^n$ we know that $\overrightarrow{w}$ can be expressed as a linear combination, $\overrightarrow{w} = c_1\overrightarrow{v}_1 + \cdots + \overrightarrow{v}_n$. And since $\{\overrightarrow{v}_1, ..., \overrightarrow{v}_n\}$ is linearly independent, this expression is unique (try to prove this!). Since $\overrightarrow{w}$ is equal to a unique linear combination of the $\overrightarrow{v}_i$'s, we might as well replace $\overrightarrow{w}$ by the coefficients $(c_1, ..., c_n)$. Thus we write $[\overrightarrow{w}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ or in words "the coordinates of the vector $\overrightarrow{w}$ in the basis β are $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$".

Example 1: Suppose $\beta = \{\overrightarrow{v}_1, ..., \overrightarrow{v}_4\}$ is a basis for $\mathbb{R}^4$. What are the β-coordinates for the vector $-\overrightarrow{v}_2 + 3\overrightarrow{v}_4$?

solution: the answer is $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}$. If you don't see why, read the previous paragraph again.

Example 2: What vector's β-coordinates are $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$? How bout $\begin{bmatrix} -1 \\ 43 \\ 2 \\ 0 \end{bmatrix}$

solution: The vector $\overrightarrow{v}_3$ and $-\overrightarrow{v}_1 + 43\overrightarrow{v}_2 + 2\overrightarrow{v}_3$ respectively.

Changing coordinates

To change coordinates between the coordinates given by a basis $\beta = \{\overrightarrow{w}_1, ..., \overrightarrow{w}_n\}$ and a basis γ, we need to know the coordinates of the elements of β with respect to new basis $\gamma : \{[\overrightarrow{w}_1]_\gamma, ..., [\overrightarrow{w}_n]_\gamma\}$.

Then the matrix

$$C = \begin{bmatrix} | & & | \\ [\overrightarrow{w}_1]_\gamma & \cdots & [\overrightarrow{w}_n]_\gamma \\ | & & | \end{bmatrix}$$

changes β coordinates to γ coordinates. Lets see why this is true. Since $\overrightarrow{w}_i = 0\overrightarrow{w}_1 + \cdots + 1\overrightarrow{w}_i + \cdots + 0\overrightarrow{w}_n$, $[\overrightarrow{w}_i]_\beta = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, where the only nonzero entry is a 1 in the ith slot. Then

$$C[\overrightarrow{w}_i]_\beta$$

$$= \begin{bmatrix} | & & | \\ [\overrightarrow{w}_1]_\gamma & \cdots & [\overrightarrow{w}_n]_\gamma \\ | & & | \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= i\text{th column of C}$$

$$= [w_i]_\gamma.$$

Thus multiplying the n-tuple of numbers which represents a vector in β-coordinates by C gives us the n-tuple of numbers which represents that same vector but in γ-coordinates.

Example 3 Let $\beta = \{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ be a basis for $\mathbb{R}^3$. Let γ be the basis $\{\overrightarrow{v}_2 + \overrightarrow{v}_3, -\overrightarrow{v}_1 + 2\overrightarrow{v}_2, \overrightarrow{v}_1 + \overrightarrow{v}_3\}$. Suppose $[\overrightarrow{r}]_\beta = \begin{bmatrix} -7 \\ 10 \\ 1 \end{bmatrix}$. What is $[\overrightarrow{r}]_\gamma$?

solution OK, we need a change of basis matrix from β coordinates to γ coordinates. We might hope that we know the β basis vectors' γ-coordinates. But we don't. Instead, we have been given the β coordinates of the γ basis vectors. They are $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ respectively. Thus the matrix

$$C = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

is the change of basis matrix that takes γ coordinates and produces β coordinates. This is the opposite direction from what we want, so C^{-1} will be the change of basis matrix we want.

The answer is then $C^{-1} \begin{bmatrix} -7 \\ 10 \\ 1 \end{bmatrix}$.

Just this once we will explicitly show how to find C^{-1}. We row reduce this augmented double matrix where initially the left matrix is C and the right matrix is I. Because C is invertible, it row reduces to the identity. The matrix on the right that started as I is transformed into C^{-1}.

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

swap rows 1 and 2,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

subtract row 1 from row 3,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & -1 & 1 \end{array}\right]$$

scale the second row by -1

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 & -1 & 1 \end{array}\right]$$

add 2×row 2 to row 3,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right]$$

scale the third row by -1,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

add row three to row two,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

add -2× row two to row one,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

.

So

$$C^{-1} = \begin{bmatrix} -2 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

and

$$C^{-1} \begin{bmatrix} -7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}$$

is our answer.

Example 4 Let $\beta = \{ \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$. What are the β-coordinates for the vector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

solution The only tricky thing here is to realize that one of the two basis being used here is the standard basis. (Whenever we refer to vectors without mentioning a basis, it is implied that we were using standard basis. To be pedantic we could rewrite the β as $\{4\overrightarrow{e}_1 + 3\overrightarrow{e}_2, \overrightarrow{e}_1 + 2\overrightarrow{e}_2\}$). Importantly though, we know the standard basis coordinates for the elements of the β basis. So

$$C = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

is the change of basis from β coordinates to standard basis coordinates. Again, we want to change standard basis coordinates into β coordinates, so we're more interested in $C^{-1} =$

$$= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

The β coordinates of $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ are

$$\frac{1}{5}\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 \\ -7 \end{bmatrix}$$

Example 5 Let $\beta = \{\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3\}$. Let $\gamma = \{3\overrightarrow{u}_2, \overrightarrow{u}_3, 8\overrightarrow{u}_1 - 7\overrightarrow{u}_3\}$.

solution Let $[\overrightarrow{s}]_\gamma = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$. What is $[\overrightarrow{s}]_\beta$. There are two ways of doing this, but secretly they are the same.

1) Change of Basis: We know the β-coordinates of the elements of γ. So the matrix

$$C = \begin{bmatrix} 0 & 0 & 8 \\ 3 & 0 & 0 \\ 0 & 1 & -7 \end{bmatrix}$$

changes γ coordinates to β coordinates. The answer is:

$$C[\overrightarrow{s}]_\gamma = \begin{bmatrix} 0 & 0 & 8 \\ 3 & 0 & 0 \\ 0 & 1 & -7 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24 \\ 12 \\ -23 \end{bmatrix}$$

2) Directly,

$$\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}_\gamma = [4(3\overrightarrow{u}_2) - 2(\overrightarrow{u}_3 + 3(8\overrightarrow{u}_1 - 7\overrightarrow{u}_3)]_\gamma$$

group in terms of $\overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{u}_3$

$$= 24\overrightarrow{u}_1 + 12\overrightarrow{u}_2 + (-2 - 21)\overrightarrow{u}_3 = 24\overrightarrow{u}_1 + 12\overrightarrow{u}_2 - 23\overrightarrow{u}_3$$

which in β coordinates is clearly $\begin{bmatrix} 24 \\ 12 \\ -23 \end{bmatrix}$.

Example 6 Let $\beta = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\}$ and let $\gamma = \{\begin{bmatrix} -4 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \end{bmatrix}\}$. Let $[\overrightarrow{r}]_\gamma = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

i) What is $[\overrightarrow{r}]_\beta$?

ii) What is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ in β-coordinates?

solution i) OK, we are given the standard basis coordinates for the elements of β. So the matrix

$$B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

is the change of basis matrix that takes β coordinates and produces standard basis coordinates.

Likewise, we are given the standard basis coordinates for the elements of γ. So the matrix

$$C = \begin{bmatrix} -4 & -1 \\ 2 & -6 \end{bmatrix}$$

is the change of basis matrix that takes γ coordinates and produces standard basis coordinates.

We want $\gamma \Rightarrow \beta$, but we have the change of basis matrices involving standard basis coordinates, we should pass through standard basis coordinates i.e. $\gamma \Rightarrow$ standard basis $\Rightarrow \beta$. So first multiply by the change of basis matrix C and then by B^{-1}. In otherwords $B^{-1}C$.

A number of you wondered why it isn't CB^{-1}. Remember that matrices are functions, and functions compose from right to left. So $B^{-1}C$ is the matrix multiplication that first multiplies by C and then by B^{-1}, which is what we want.

OK, now it is just plug and chug.

$$\begin{aligned} B^{-1}C \begin{bmatrix} 0 \\ 3 \end{bmatrix} &= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} -6 & 16 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 48 \\ -21 \end{bmatrix} \end{aligned}$$

So $[\overrightarrow{r}]_\beta = \begin{bmatrix} \frac{48}{5} \\ -\frac{21}{5} \end{bmatrix}$.

ii) We know the standard basis coordinates here. We just need to multiply by the change of basis matrix taking standard to β-coordinates, which is B^{-1}.

The answer is

$$\frac{1}{5}\begin{bmatrix}2 & -3\\1 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}\frac{-7}{5}\\\frac{-1}{5}\end{bmatrix}$$

Okay, last one.

Example 7 Suppose $\alpha = \{\overrightarrow{w}_1, \overrightarrow{w}_2, \overrightarrow{w}_3\}$ is a basis for $\mathbb{R}^3$.

Let $\beta = \{\overrightarrow{w}_1, 5\overrightarrow{w}_2 + 4\overrightarrow{w}_3, \overrightarrow{w}_2 + \overrightarrow{w}_3\}$ and $\gamma = \{-\overrightarrow{w}_2, \overrightarrow{w}_2 + \overrightarrow{w}_3, 2\overrightarrow{w}_1 + \overrightarrow{w}_2\}$.

i) $[\overrightarrow{r}]_\beta = \begin{bmatrix}4\\-1\\-3\end{bmatrix}$ What is $[\overrightarrow{r}]_\alpha$?

ii) $[\overrightarrow{r}]_\gamma = \begin{bmatrix}2\\1\\1\end{bmatrix}$ What is $[\overrightarrow{r}]_\beta$?

iii) What is $\begin{bmatrix}2\\1\\3\end{bmatrix}$ in γ coordinates?

iv) What is $2(-\overrightarrow{w}_2) + 4(\overrightarrow{w}_2 + \overrightarrow{w}_3) - 2(2\overrightarrow{w}_1 + \overrightarrow{w}_2)$ in γ coordinates? in α coordinates?

v) What is $4\overrightarrow{w}_1 - 3\overrightarrow{w}_2$ in α coordinates? in γ coordinates?

solution Using the same kind of reason as before we see the matrix

$$B = \begin{bmatrix}1 & 0 & 0\\0 & 5 & 1\\0 & 4 & 1\end{bmatrix}$$

is the change of basis matrix from β coordinates to α coordinates.

$$C = \begin{bmatrix}0 & 0 & 2\\-1 & 1 & 1\\0 & 1 & 0\end{bmatrix}$$

is the change of basis matrix from γ coordinates to α coordinates.

i)

$$[\overrightarrow{r}]_\alpha = B[\overrightarrow{r}]_\beta = \begin{bmatrix}1 & 0 & 0\\0 & 5 & 1\\0 & 4 & 1\end{bmatrix}\begin{bmatrix}4\\-1\\-3\end{bmatrix}$$

$$= \begin{bmatrix}4\\-8\\1\end{bmatrix}$$

ii) We need to go from γ coordinates to β coordinates, but it is clear that given what we know, we'll have to pass through α coordinates: γ coord. $\Rightarrow\alpha$ coord. $\Rightarrow$ β coord. . Thus we'll need to use the change of basis matrix

$$B^{-1}C$$

(Remember matrices are functions, and so they compose from left to right.)

The answer is

$$
\begin{aligned}
&B^{-1}C[\overrightarrow{r}]_\gamma \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 2 \\ -1 & 0 & 1 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}
\end{aligned}
$$

iii) Trick question. We cannot answer it because we have no clue what any of our γ vectors are in standard coordinates, so there is no way to find a change of basis matrix.

iv) In γ coordinates the vector is: $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ almost by definition! For α coordinates: we multiply $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$ by the γ-to-α change of basis matrix which is simply C. So the answer is

$$\begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

v) In α coordinate the vector is: $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ by definition! For γ coordinates we must multiply $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ by the α-to-γ change of basis matrix which is C^{-1}. So the answer is:

$$
\begin{aligned}
&C^{-1} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -1 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.
\end{aligned}
$$